A Similarity Transformation and the Decay Mode Solutions for Three-Dimensional Cylindrical Kadomtsev-Petviashvili Equation

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Abstract: In this paper, a similarity transformation between the solutions of three-dimensional cylindrical Kadomtsev-Petviashvili equation and the solutions of three-dimensional Kadomtsev-Petviashvili equation with constant coefficients is firstly derived, and the corresponding constraint conditions for the coefficients of three-dimensional cylindrical Kadomtsev-Petviashvili equations are obtained. Then the exact solutions of the three-dimensional cylindrical Kadomtsev-Petviashvili equation are expressed by the similarity transformation and the solutions of the three-dimensional Kadomtsev-Petviashvili equation with constant coefficients. Lastly, four special three-dimensional cylindrical Kadomtsev-Petviashvili equations are studied, especially, the decay mode solutions of these three-dimensional cylindrical Kadomtsev-Petviashvili equations are obtained.

Key words: The three-dimensional cylindrical Kadomtsev-Petviashvili equation; The three-dimensional Kadomtsev-Petviashvili equation; Similarity transformation; Decay mode solution

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1. Introduction

The cylindrical Kadomtsev-Petviashvili equation (CKP) in the form
\[(u_t + 6uu_x + u_{xxx})_x + \frac{1}{2t}u_x + \frac{3\alpha^2}{t^2}u_{yy} = 0,\]  
(1.1)
was introduced by Johnson\cite{1-2} to describe surface wave in a shallow incompressible fluid. The CKP (1.1) for magnetized plasmas with pressure effects and transverse perturbations in cylindrical geometry was also derived by using the small amplitude perturbation expansion method \cite{3}. And Eq. (1.1) is a (2+1)-dimensional generalization of the cylindrical KdV equation (CKdV)\cite{4-5}
\[u_t + 6uu_x + u_{xxx} + \frac{1}{2t}u = 0.\]  
(1.2)

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Due to the importance and wide application, CKP (1.1) has been paid attention by many researchers in mathematical physics. For instance, In [6], Klein et al. have shown that the Lax pair corresponding KP and CKP equation are gauge equivalent, and some class of solutions (Such as horseshoe-like-front solutions, lump solutions and rational solutions) were obtained by using Darboux transformation approach. In [7], DENG has shown that the decay mode solution for CKP (1.1) can be obtained by Bäcklund transformation and Hirota’s method.

In [8-9], the reductive perturbation method is employed to derive a three-dimensional cylindrical Kadomtsev-Petviashvili (3D-CKP) equation

\[
\frac{\partial}{\partial r} \left( \frac{\partial u}{\partial t} + Au \frac{\partial u}{\partial r} + B \frac{\partial^3 u}{\partial r^3} \right) + \frac{1}{2t} \frac{\partial u}{\partial r} + \frac{C}{2\lambda^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\lambda D \partial^2 u}{2 \partial z^2} = 0, \tag{1.3}
\]

which is used to describe the non-planar ion-acoustic waves in positive-negative ion plasmas with stationary dust particles, the generalized expansion method is used to solve analytically the evolution equation, and a train of well-separated bell-shaped periodic pulses which can change to solitary pulses at certain conditions are obtained. In this paper, the 3D-CKP (1.3) will be considered.

The paper is organized as follows: In Section 2, we derive a similarity transformation \[^{10-13}\] and the constraint condition for 3D-CKP (1.3); In Section 3, some special types of 3D-CKP are studied, and the decay mode solutions are derived.

2. The Derivation of Similarity Transformation

Suppose that the exact solutions of Eq. (1.3) are in the forms

\[
u(r, z, \theta, t) = \rho(t) W(r, z, \theta, T), \quad T = T(t), \tag{2.1}\]

where \(\rho(t), T(t)\) are determined later, and \(W(r, z, \theta, T)\) satisfy three dimensional Kadomtsev-Petviashvili (3D-KP) equation with constant coefficients

\[
\frac{\partial}{\partial r} \left( \frac{\partial W}{\partial T} + 6W \frac{\partial W}{\partial r} + \frac{\partial^3 W}{\partial r^3} \right) + c \frac{\partial^2 W}{\partial \theta^2} + d \frac{\partial^2 W}{\partial z^2} = 0, \tag{2.2}\]

where \(c, d\) are constants. From (2.1) we have

\[
u_t = \rho W_T + \rho' W, \quad u_{rt} = \rho W_{TT} + \rho' W_T, \tag{2.3}\]

\[
\frac{\partial}{\partial r} \left( Au \frac{\partial u}{\partial r} + B \frac{\partial^3 u}{\partial r^3} \right) = \rho^2 \frac{\partial}{\partial r} \left( AW \frac{\partial W}{\partial r} \right), \quad \frac{\partial}{\partial r} \left( B \frac{\partial^3 u}{\partial r^3} \right) = \rho \frac{\partial}{\partial r} \left( B \frac{\partial^3 W}{\partial r^3} \right), \tag{2.4}\]

\[
1 + \frac{C}{2t} \frac{\partial^2 u}{\partial \theta^2} + \frac{\lambda D \partial^2 u}{\partial z^2} = \frac{C}{2t} \frac{\partial^2 W}{\partial \theta^2} + \frac{\lambda D \partial^2 W}{\partial z^2}. \tag{2.5}\]

Substituting (2.1), (2.3)-(2.5) into the left hand side of system (1.3) and considering (2.2) simultaneously yields

\[
\rho T' \left( \frac{\partial}{\partial r} \left( \frac{\partial W}{\partial T} + 6W \frac{\partial W}{\partial r} + \frac{\partial^3 W}{\partial r^3} \right) + c \frac{\partial^2 W}{\partial \theta^2} + d \frac{\partial^2 W}{\partial z^2} \right) + \frac{\partial}{\partial r} \left( \rho(A \rho - 6T') \frac{\partial W}{\partial r} \right)
\]

\[
= \frac{\partial}{\partial r} \left( \rho(B - T') \frac{\partial^3 W}{\partial r^3} \right) + \rho \left( \frac{C}{2\lambda^2} - T' c \right) \frac{\partial^2 W}{\partial \theta^2} + \rho \left( \frac{\lambda D}{2} - T' d \right) \frac{\partial^2 W}{\partial z^2} + \left( \rho' + \frac{1}{2t} \rho \right) \frac{\partial W}{\partial r}. \tag{2.6}\]

From (2.6), we obtain a set of partial differential equations

\[A \rho - 6T' = 0, \quad B - T' = 0, \quad \frac{C}{2\lambda^2} - T' c = 0, \quad \frac{\lambda D}{2} - T' d = 0, \quad \rho' + \frac{1}{2t} \rho = 0. \tag{2.7}\]
Solving these ODEs (2.7), we find the following expressions

\[ T = \int B \, dr, \quad \rho = t^{-\frac{1}{2}}, \quad A = 6Bt^{\frac{1}{2}}, \quad C = 2\lambda t^2 B, \quad D = \frac{2dB}{\lambda}. \]  

\[ (2.8) \]

Then 3D-CKP (1.3) is rewritten as

\[ \frac{\partial}{\partial r} \left( \frac{\partial u}{\partial t} + 6B t^\frac{1}{2} \frac{\partial u}{\partial r} + B \frac{\partial^3 u}{\partial r^3} \right) + \frac{1}{2t} \frac{\partial u}{\partial r} + \epsilon B \frac{\partial^2 u}{\partial \theta^2} + dB \frac{\partial^2 u}{\partial z^2} = 0, \]  

\[ (2.9) \]

and similarity transformation (2.1) is rewritten as

\[ u(r, z, \theta, t) = t^{-\frac{1}{2}} W(r, z, \theta, T), \quad T = \int B \, dr. \]  

\[ (2.10) \]

Using the similarity transformation (2.10) and the solutions of 3D-KP (2.2), we can easily obtain the solutions of 3D-CKP (2.9).

3. Some Special Type of 3D-CKP (2.9) and the Decay Mode Solutions

In this section, we consider some special type of 3D-CKP (2.9).

1) Cylindrical KdV equation with variable coefficients and the decay mode solutions. Setting \( c = d = 0 \) in Eq. (2.9) yields a cylindrical KdV equation with variable coefficients

\[ \frac{\partial u}{\partial t} + 6B t^\frac{1}{2} \frac{\partial u}{\partial r} + B \frac{\partial^3 u}{\partial r^3} = 0, \]  

\[ (3.1) \]

and 3D-KP (2.2) becomes a KdV equation with constant coefficients

\[ \frac{\partial W}{\partial T} + 6W \frac{\partial W}{\partial r} + \frac{\partial^3 W}{\partial r^3} = 0. \]  

\[ (3.2) \]

From [14], some special soliton solutions of KdV (3.2) are listed as follows.

One soliton:

\[ W = \frac{k^2}{2} \text{sech}^2 \frac{k_1 r + \omega_1 T}{2}, \quad \omega_1 = -k_1^3. \]  

\[ (3.3) \]

Two soliton:

\[ W = 2 \left[ \ln \left(1 + e^{\xi_1} + e^{\xi_2} + e^{\xi_1 + \xi_2 + A_{12}} \right) \right]_{xx}, \]  

\[ (3.4) \]

where \( \xi_j = k_j r + \omega_j T, \omega_j = -k_j^3, j = 1, 2 \), \( e^{A_{12}} = \left(\frac{k_j - k_l}{k_j + k_l}\right)^2 \) \( j < l, l \in \{1, 2, 3\} \).

Three soliton:

\[ W = 2 \left[ \ln \left(1 + e^{\xi_1} + e^{\xi_2} + e^{\xi_3} + e^{\xi_1 + \xi_2 + A_{12}} + e^{\xi_1 + \xi_3 + A_{13}} + e^{\xi_2 + \xi_3 + A_{23}} + e^{\xi_1 + \xi_2 + \xi_3 + A_{123}} \right) \right]_{xx}, \]  

\[ (3.5) \]

where \( \xi_j = k_j r + \omega_j T, \omega_j = -k_j^3, j = 1, 2, 3 \), \( e^{A_{12}} = \left(\frac{k_j - k_l}{k_j + k_l}\right)^2 \) \( j < l, l \in \{1, 2, 3\} \).

Using the similarity transformation (2.10) and the solutions above, the decay mode soliton solutions of cylindrical KdV equation with variable coefficients (3.1) are obtained as follows.

One decay mode soliton:

\[ u(r, z, \theta, t) = \frac{k^2}{2} t^{-\frac{1}{2}} \text{sech}^2 \frac{k_1 r + \omega_1 T}{2}, \quad T = \int B \, dr, \quad \omega_1 = -k_1^3. \]  

\[ (3.6) \]

Two decay mode soliton:

\[ u(r, z, \theta, t) = 2t^{-\frac{1}{2}} \left[ \ln \left(1 + e^{\xi_1} + e^{\xi_2} + e^{\xi_1 + \xi_2 + A_{12}} \right) \right]_{xx}, \]  

\[ (3.7) \]

where \( \xi_j = k_j r + \omega_j T, \omega_j = -k_j^3, j = 1, 2 \), \( e^{A_{12}} = \left(\frac{k_j - k_l}{k_j + k_l}\right)^2 \) \( j < l, l \in \{1, 2, 3\} \).

Three decay mode soliton:

\[ u(r, z, \theta, t) = 2t^{-\frac{1}{2}} \left[ \ln \left(1 + e^{\xi_1} + e^{\xi_2} + e^{\xi_3} + e^{\xi_1 + \xi_2 + A_{12}} + e^{\xi_1 + \xi_3 + A_{13}} + e^{\xi_2 + \xi_3 + A_{23}} \right) \right]_{xx}. \]
where \( \xi_j = k_j r + \omega_j T, T = \int B \mathrm{d} \tau, \omega_j = -k_j^2 (j = 1, 2, 3), e^{A_{ij}} = \left( \frac{k_j - k_i}{k_j + k_i} \right)^2 (j < l, l = 1, 2, 3).

**Note 3.1** The decay mode multi-soliton solutions of Eq. (3.1) can be obtained using the results in [14].

2) Two-dimensional cylindrical Kadomtsev-Petviashvili (2D-CKP) equation and the decay mode solutions.

Setting \( c = 3\sigma^2, d = 0 \) in Eq. (2.9) yields a two dimensional cylindrical Kadomtsev-Petviashvili (2D-CKP) equation

\[
\frac{\partial}{\partial r} \left( \frac{\partial u}{\partial t} + 6B t \frac{\partial u}{\partial r} + B \frac{\partial^3 u}{\partial r^3} \right) + \frac{1}{2l} \frac{\partial u}{\partial r} + 3\sigma^2 B \frac{\partial^2 u}{\partial \theta^2} = 0,
\]

and 3D-KP (2.2) becomes Kadomtsev-Petviashvili equation with constant coefficients

\[
\frac{\partial}{\partial r} \left( \frac{\partial W}{\partial T} + 6W \frac{\partial W}{\partial r} + \frac{\partial^3 W}{\partial r^3} \right) + 3\sigma^2 \frac{\partial^2 W}{\partial \theta^2} = 0.
\]

From [14], soliton solutions of KP (3.10) are listed as follows.

One-line soliton:

\[
W = \frac{k_1^2}{2} \text{sech}^2 \frac{k_1(r + \omega_1 T + p_1 \theta)}{2}, \quad \omega_1 = -k_1^2 - 3\sigma^2 p_1^2.
\]

Two-line soliton:

\[
W = 2 \left[ \ln \left( 1 + e^{\xi_1} + e^{\xi_2} + e^{\xi_1 + \xi_2 + A_{12}} \right) \right]_{xx},
\]

where \( \xi_j = k_j(r + \omega_j T + p_j \theta), \omega_j = -k_j^2 - 3\sigma^2 p_j^2 (j = 1, 2), e^{A_{12}} = \frac{(k_1 - k_2)^2 - \sigma^2 (p_1 - p_2)^2}{(k_1 + k_2)^2 - \sigma^2 (p_1 + p_2)^2}.

Using the similarity transformation (2.10), the decay mode soliton solutions of two-dimensional cylindrical Kadomtsev-Petviashvili (2D-CKP) equation (3.9) are obtained as follows.

One-line decay mode soliton:

\[
u(r, z, \theta, t) = t^{-\frac{1}{2}} \frac{k_1^2}{2} \text{sech}^2 \frac{k_1(r + \omega_1 T + p_1 \theta)}{2},
\]

where \( T = \int B \mathrm{d} \tau, \omega_1 = -k_1^2 - 3\sigma^2 p_1^2.

Two-line decay mode soliton:

\[
u(r, z, \theta, t) = 2t^{-\frac{1}{2}} \left[ \ln \left( 1 + e^{\xi_1} + e^{\xi_2} + e^{\xi_1 + \xi_2 + A_{12}} \right) \right]_{xx},
\]

where

\[
\xi_j = k_j(r + \omega_j T + p_j \theta), T = \int B \mathrm{d} \tau, \omega_j = -k_j^2 - 3\sigma^2 p_j^2 (j = 1, 2),
\]

\[
e^{A_{12}} = \frac{(k_1 - k_2)^2 - \sigma^2 (p_1 - p_2)^2}{(k_1 + k_2)^2 - \sigma^2 (p_1 + p_2)^2}.
\]

**Note 3.2** The decay mode multi-soliton solutions of Eq. (3.1) can be obtained using the results in [14]. Here we omit it for simplicity.

3) First special three-dimensional cylindrical Kadomtsev-Petviashvili equation and the decay mode solutions.

Setting \( B=\text{const} \) in Eq. (2.9) yields a three dimensional cylindrical Kadomtsev-Petviashvili (3D-CKP) equation

\[
\frac{\partial}{\partial r} \left( \frac{\partial u}{\partial t} + 6B t \frac{\partial u}{\partial r} + B \frac{\partial^3 u}{\partial r^3} \right) + \frac{1}{2l} \frac{\partial u}{\partial r} + cB \frac{\partial^2 u}{\partial \theta^2} + dB \frac{\partial^2 u}{\partial z^2} = 0,
\]

where

\[
e^{A_{ij}} = \left( \frac{k_i - k_j}{k_i + k_j} \right)^2 (j < l, l = 1, 2, 3).
\]
and 3D-KP (2.2) leads
\[
\frac{\partial}{\partial r} \left( \frac{\partial W}{\partial r} + 6W \frac{\partial W}{\partial r} + \frac{\partial W}{\partial r^2} \right) + c \frac{\partial^2 W}{\partial \theta^2} + d \frac{\partial^2 W}{\partial z^2} = 0. \tag{3.16}
\]

The similarity transformation (2.10) is rewritten as
\[
u(r, z, \theta, t) = t^{-\frac{1}{2}} W(r, z, \theta, T), T = Bt. \tag{3.17}
\]

Using the similarity transformation (3.17) and the travelling wave solutions of Eq. (3.16), the decay mode travelling wave solutions of Eq. (3.15) can obtained as follows.

When \( n + cm^2 + dl^2 < 0 \),
\[
u_1 = -\frac{1}{2} t^{-\frac{1}{2}} (n + cm^2 + dl^2) \text{sech}^2 \left( \frac{1}{2} \sqrt{- (n + cm^2 + dl^2)} \xi \right), \tag{3.18}
\]
\[
u_2 = \frac{1}{2} t^{-\frac{1}{2}} (n + cm^2 + dl^2) \text{csch}^2 \left( \frac{1}{2} \sqrt{- (n + cm^2 + dl^2)} \xi \right). \tag{3.19}
\]

When \( n + cm^2 + dl^2 > 0 \),
\[
u_3 = -\frac{1}{2} t^{-\frac{1}{2}} (n + cm^2 + dl^2) - \frac{1}{2} t^{-\frac{1}{2}} (n + cm^2 + dl^2) \tan^2 \left( \frac{1}{2} \sqrt{n + cm^2 + dl^2} \right), \tag{3.20}
\]
\[
u_4 = -\frac{1}{2} t^{-\frac{1}{2}} (n + cm^2 + dl^2) - \frac{1}{2} t^{-\frac{1}{2}} (n + cm^2 + dl^2) \cot^2 \left( \frac{1}{2} \sqrt{n + cm^2 + dl^2} \right). \tag{3.21}
\]

When \( n + cm^2 + dl^2 = 0 \),
\[
u_5 = -\frac{1}{2} t^{-\frac{1}{2}} (n + cm^2 + dl^2) - 2t^{-\frac{1}{2}} \left( \frac{C_2}{C_1 + C_2} \right)^2, \tag{3.22}
\]
where \( \xi = r + lz - \frac{bC}{2cM} \), \( C_1, C_2, l, m, n \) are constants.

4) Second special three-dimensional cylindrical Kadomtsev-Petviashvili equation and the decay mode solutions.

Setting \( B = \frac{C}{\sqrt{t}} \) in Eq. (2.9) yields a three-dimensional cylindrical Kadomtsev-Petviashvili (3D-CKP) equation
\[
\frac{\partial}{\partial r} \left( \frac{\partial u}{\partial r} + \frac{3C}{cM^2} \frac{\partial u}{\partial r} + \frac{C}{2cM^2} \frac{\partial^2 u}{\partial r^2} \right) + \frac{1}{2t} \frac{\partial u}{\partial r} + \frac{C}{2cM^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{dC}{2cM^2} \frac{\partial^2 u}{\partial z^2} = 0. \tag{3.23}
\]

The similarity transformation (2.10) is rewritten as
\[
u(r, z, \theta, t) = t^{-\frac{1}{2}} W(r, z, \theta, T), T = \int^1 \frac{bC}{2cM^2} d\tau. \tag{3.24}
\]

If \( b, C, c, \lambda \) are constants, \( T = -\frac{bC}{2cM} \).

Using the similarity transformation (3.24) and the travelling wave solutions of Eq. (3.16), the decay mode travelling wave solutions of Eq. (3.23) can obtained as follows.

When \( n + cm^2 + dl^2 < 0 \),
\[
u_1 = -\frac{1}{2} t^{-\frac{1}{2}} (n + cm^2 + dl^2) \text{sech}^2 \left( \frac{1}{2} \sqrt{- (n + cm^2 + dl^2)} \xi \right), \tag{3.25}
\]
\[
u_2 = \frac{1}{2} t^{-\frac{1}{2}} (n + cm^2 + dl^2) \text{csch}^2 \left( \frac{1}{2} \sqrt{- (n + cm^2 + dl^2)} \xi \right). \tag{3.26}
\]

When \( n + cm^2 + dl^2 > 0 \),
\[
u_3 = -\frac{1}{2} t^{-\frac{1}{2}} (n + cm^2 + dl^2) - \frac{1}{2} t^{-\frac{1}{2}} (n + cm^2 + dl^2) \tan^2 \left( \frac{1}{2} \sqrt{n + cm^2 + dl^2} \right), \tag{3.27}
\]
\[
u_4 = -\frac{1}{2} t^{-\frac{1}{2}} (n + cm^2 + dl^2) - \frac{1}{2} t^{-\frac{1}{2}} (n + cm^2 + dl^2) \cot^2 \left( \frac{1}{2} \sqrt{n + cm^2 + dl^2} \right). \tag{3.28}
\]

When \( n + cm^2 + dl^2 = 0 \),
\[
u_5 = -\frac{1}{2} t^{-\frac{1}{2}} (n + cm^2 + dl^2) - 2t^{-\frac{1}{2}} \left( \frac{C_2}{C_1 + C_2} \right)^2, \tag{3.29}
\]
where \( \xi = r + lz - \frac{bC}{2cM} \), \( C_1, C_2, l, m, n \) are constants.

**Note 3.3** Except Eqs. (3.15) and (3.23), obviously, there exists other type of 3D-CKP (2.9).
相似变换及三维柱KP方程的衰减解

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摘要：本文首先导出三维柱KP方程与三维常系数KP方程解之间的相似变换及三维柱KP方程的系数所满足的约束条件；借助于该相似变换及三维常系数KP方程的解，得到三维柱KP方程的解；最后，讨论四个特殊的三维柱KP方程，特别地，得到这些三维柱KP方程的衰减解。

关键词：三维柱KP方程; 三维KP方程; 相似变换; 衰减解

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