Almost Sure Convergence of Weighted Sums for Extended Negatively Dependent Random Variables Under Sub-Liner Expectations

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Abstract: In this thesis, we discuss almost sure convergence of weighted sums for END random variables with the condition of $\mathbb{CV}(|X|^p) < \infty$, even $\mathbb{E}(|X|^p) \leq \mathbb{CV}(|X|^p)$, $0 < p \leq 2$ for sub-linear expectations space. As an application, our result extends the corresponding results of SILVA(2015) relative to the classical probability. In addition, the result of our paper is the extension of the almost sure convergence of weighted sums under the sub-linear expectation space.

Key words: Weighted sum; Sub-linear expectation; Almost sure convergence; END random variables

1. Introduction

As is well known that limit theorems exert a enormous function on probability limit theory and mathematical statistics. It becomes a vital theoretical tool to study the phenomenon of randomness or uncertainty. In fact, probability measures and linear additivity themselves do not account for many of the uncertainties in statistics, measures of risk, mathematical economics and super-hedging in finance.\[4.6-7.8-11]\ In order to solve this problem, PENG\[10-12]\ established a new system of nonlinear expectations theory from a completely new perspective. This theory does not start from the classical probability space but directly define the independence of random variables from the non-linear expectation space, and proves the theoretical basis of nonlinear expectation with PDE.

The sub-linear expectation created a lot of fascinating properties which are different from those of the linear expectations. So the limit theorems of the sub-linear expectations have been received a lot of attention and research recently. A great deal of useful results have been established by many researchers. For instance, PENG\[10-11]\ established the basic...
properties, denoted the relevant definition of notation and distribution and a new central limit theorem in the nonlinear expectation space. CHEN\cite{3} studied three variety of strong laws of larger numbers under the feasible condition of $\tilde{E}(|X|^{1+\alpha}) < \infty$ for independent identically distributed random variables. ZHANG\cite{17-19} found the the laws of the iterated logarithm for negatively dependent random variables in a successive non-linear expectation. He also established Kolmogorov’s exponential inequalities and the rosenthal’s inequalities which are the powerful tools for future research in the sub-linear expectation.

Almost sure convergence is one of the most important issues in limit theorems. In sub-linear expectations, due to the uncertainty of expectation and capacity, the strong convergence is essentially different from the ordinary probability space. The study of strong convergence for sub-linear expectations is much more complex and difficult. As a result of the extensive utility of weighted sums in financial statistics, many scholars put emphasis on its properties. Numerous studies on almost sure convergence of weighted sums can be found in probability space. For example, CHOI and SUNG\cite{1} gained the diverse conditions on $\{a_{n}\}$ and $\{X_{n}\}$ under which $\sum_{i=1}^{n} a_{ni} X_{i}$ converges to zero almost surely. CHEN and GAN\cite{2} established the strong laws of large numbers, laws of the single logarithm and Chover’s laws of the iterated logarithm (LIL) for weighted sums of i.i.d. random variables under appropriate conditions concerning both the distribution and the weights. TEICHER\cite{14} researched a Marchinkiewicz-Zygmund type strong law for the special case $a_{ni} = a_{i}/ \left( \sum_{j=1}^{n} a_{j}^{p} \right)^{1/p}$, $0 < p < 2$. XU and YU\cite{15} obtained the law of the single logarithm for NA random variables with the condition of $\max_{1 \leq i \leq n} |a_{ni}| = O \left( \frac{1}{n^{\gamma} (\log n)^{1-\gamma}} \right)$. WU\cite{16} acquired a strong limit theorem for weighted sums of ND random variables. The prime motivation of the article is to gain the almost sure convergence for weighted sums of END random variables with the condition of $C_{V}(|X|^{p}) < \infty$, even $\tilde{E}(|X|^{p}) \leq C_{V}(|X|^{p})$, $0 < p \leq 2$ under sub-linear expectations. In addition, our result extends the corresponding results of SILVA\cite{5} relative to the classical probability.

2. Preliminaries

The study of this article utilizes the symbols and framework which are established by PENG\cite{10-12}. He established the definitions of sub-linear expectation $\tilde{E}$, subadditivity of capacities $V$, Choquet integrals/expectations $(C_{V}, C_{v})$ and some of the properties, and so on. As a result, we can omit these various definitions and relevant properties.

Under the sub-linear expectation space, CHEN\cite{3} and ZHANG\cite{17-18} researched such a kind of convergence, but they don’t have a specific summary. So, we summarize the definition of almost surely as follows.

**Definition 2.1** A sequence of random variables $\{X_{n}; n \geq 1\}$ is named to converge to $X$ almost surely $V$ (a.s. $V$), denoted by $X_{n} \xrightarrow{V} X$ a.s. $V$ as $n \to \infty$, if $V(X_{n} \to X) = 0$.

$V$ can be substituted for $\forall$ and $v$ severally. By $v(A) \leq V(A)$ and $v(A) + V(A^{c}) = 1$ for any $A \in \mathcal{F}$, it is obvious that $X_{n} \xrightarrow{V} X$ a.s. $\forall$ implies $X_{n} \xrightarrow{a.s. v} X$, but $X_{n} \xrightarrow{a.s. v}$ does not imply $X_{n} \xrightarrow{V} X$. Further

$$X_{n} \xrightarrow{V} X \quad a.s. \quad \forall \iff v(X_{n} \to X) = 1 \iff \forall(|X_{n} - X| \geq \varepsilon, i.o.) = 0 \quad \text{for} \quad \forall \varepsilon > 0,$$

and

$$X_{n} \xrightarrow{V} X \quad a.s. \quad v \iff v(X_{n} \to X) = 0 \iff \forall(X_{n} \to X) = 1.$$
Remark 2.1 In probability space, it’s commonly known that \(X_n \to X\) a.s. \(\iff\ P(X_n \to X) = 1\). However, in the sub-linear expectation space, the formula \(\mathbb{V}(A) + \mathbb{V}(A^c) = 1\) is no longer valid, which implies \(\mathbb{V}(X_n \to X) = 1 \iff \mathbb{V}(X_n \to X) = 0\). In fact, we have \(\mathbb{V}(X_n \to X) = 0 \implies \mathbb{V}(X_n \to X) = 1\), but \(\mathbb{V}(X_n \to X) = 1 \implies \mathbb{V}(X_n \to X) = 0\). Therefore, we can’t define \(X_n \to X\) a.s. \(\forall\) with \(\mathbb{V}(X_n \to X) = 1\).

Definition 2.2[13] A couple of \((\mathbb{V}, \nu)\) of capacities under the sub-linear space \((\Omega, \mathcal{H}, \hat{\mathbb{E}})\) is indicated by
\[
\mathbb{V}(A) := \inf\{\hat{\mathbb{E}}\xi; I(A) \leq \xi, \xi \in \mathcal{H}\}, \quad \nu(A) := 1 - \mathbb{V}(A^c), \quad \forall A \in \mathcal{F},
\]
where \(A^c\) is the supplement series of \(A\). By definitions of \(\mathbb{V}\) and \(\nu\), it is distinct that \(\mathbb{V}\) is sub-additive, and
\[
\nu(A) \leq \mathbb{V}(A), \quad \forall A \in \mathcal{F}; \quad \mathbb{V}(A) = \hat{\mathbb{E}}(I(A)), \quad \nu(A) = \hat{\mathbb{E}}(I(A)), \quad \text{if} \quad I(A) \in \mathcal{H},
\]
\[
\hat{\mathbb{E}}(I(1 - \mathbb{V}(A) \leq \hat{\mathbb{E}} g), \quad \hat{\mathbb{E}} f \leq \mathbb{V}(A) \leq \hat{\mathbb{E}} g, \quad \hat{\mathbb{E}} f \leq \nu(A) \leq \hat{\mathbb{E}} g, \quad \text{if} \quad f \leq I(A) \leq g, \quad f, g \in \mathcal{H}. \quad (2.1)
\]

Definition 2.3[13] (Identical distribution) Assume that \(X_1\) and \(X_2\) are two \(n\)-dimensional random vectors established separately in sub-linear expectation spaces \((\Omega_1, \mathcal{H}_1, \hat{\mathbb{E}}_1)\) and \((\Omega_2, \mathcal{H}_2, \hat{\mathbb{E}}_2)\). They are named identically distributed, denoted by \(X_1 \overset{d}{=} X_2\) if
\[
\hat{\mathbb{E}}_1(\varphi(X_1)) = \hat{\mathbb{E}}_2(\varphi(X_2)), \quad \forall \varphi \in C_{\text{Lip}}(\mathbb{R}^n),
\]
whenever the sub-expectations are limited. A sequence \(\{X_n; n \geq 1\}\) of random variables is referred to as identically distributed, for each \(i \geq 1\) if \(X_i \overset{d}{=} X_1\).

Definition 2.4[17] (Extended negative dependence) A sequence of random variables \(\{X_n; n \geq 1\}\) is called upper (resp. lower) extended negatively dependent if there is some leading constant \(K \geq 1\) such that
\[
\hat{\mathbb{E}}\left(\prod_{i=1}^{n} \varphi_i(X_i)\right) \leq K \prod_{i=1}^{n} \hat{\mathbb{E}}(\varphi_i(X_i)), \quad \forall n \geq 2,
\]
whenever the sub-expectations are finite and the non-negative functions \(\varphi_i(x) \in C_{\text{Lip}}(\mathbb{R}^n), i = 1, 2, \ldots,\), are all non-decreasing (resp. all non-increasing). They are named extended negatively dependent if they are both upper extended negatively dependent and lower extended negatively dependent.

We can easily see that, if \(\{X_n; n \geq 1\}\) is a sequence of independent random variables and \(f_1(x), f_2(x), \ldots \in C_{\text{Lip}}(\mathbb{R}^n)\), then \(\{f_n(X_n); n \geq 1\}\) is also a sequence of independent random variables with \(K = 1\); if \(\{X_n; n \geq 1\}\) is a sequence of upper (resp. lower) extended negatively dependent random variables and \(f_1(x), f_2(x), \ldots \in C_{\text{Lip}}(\mathbb{R}^n)\) are all non-decreasing (resp. all non-increasing) functions, then \(\{f_k(X_k); k \geq 1\}\) is also a sequence of upper (resp. lower) extended negatively dependent random variables.

By the definition, it is visible that, if \(Y\) is independent to \(X\), then \(Y\) is extended negatively dependent to \(X\). Example 1.6 in [18] indicates that the reverse is not true.

Example 2.1[18] Assume that \(\mathcal{P}\) is a family of probability measures defined on \((\Omega, \mathcal{F})\). For any random variable \(\xi\), we indicate the upper expectation by \(\hat{\mathbb{E}}(\xi) = \sup_{Q \in \mathcal{P}} \mathbb{E}_Q(\xi)\). Then \(\hat{\mathbb{E}}(\cdot)\) is a sub-linear expectation. Moreover, if \(X\) and \(Y\) are independent under each \(Q \in \mathcal{P}\), then \(Y\) is extended negatively dependent to \(X\) under \(\hat{\mathbb{E}}\). In reality,
\[
\hat{\mathbb{E}}(\varphi_1(X)\varphi_2(Y)) = \sup_{Q \in \mathcal{P}} \mathbb{E}_Q(\varphi_1(X)\varphi_2(Y)) = \sup_{Q \in \mathcal{P}} \mathbb{E}_Q(\varphi_1(X))\mathbb{E}_Q(\varphi_2(Y))
\]
\[ \leq \sup_{Q \in \mathcal{P}} \mathbb{E}_Q(\varphi_1(X)) \sup_{Q \in \mathcal{P}} \mathbb{E}_Q(\varphi_2(Y)) = \tilde{\mathbb{E}}(\varphi_1(X))\tilde{\mathbb{E}}(\varphi_2(Y)), \]

whenever the sub-expectations are finite and \( \varphi_1(X) \geq 0, \varphi_2(Y) \geq 0. \)

However, \( Y \) may be not independent to \( X. \)

### 3. Results and Discussions

In this section, we discuss our primary result and some useful lemmas. Next, we have a convention, let \( \{X_n; n \geq 1\} \) be a sequence of random variables in \((\Omega, \mathcal{H}, \tilde{\mathbb{E}})\), and \( S_n = \sum_{i=1}^{n} X_i. \)

The symbol \( c \) is on behalf of a generic positive constant that may be different in various place. Let \( a_n \ll b_n \) mean that there exists a constant \( c > 0 \) such that \( a_n \leq cb_n \) for adequately large \( n, \) and \( I(\cdot) \) denotes an indicator function. \( \log x \) is a mark of \( \ln(\max(x, e)) \), where \( \ln \) is the natural logarithm.

For the proof of our theorem, we declare the following two lemmas.

**Lemma 3.1** (Borel-Cantelli’s Lemma, Lemma 3.9 in [18]) \( \{A_n; n \geq 1\} \) is a sequence of events in \( \mathcal{F}. \) Suppose that \( V \) is a countably sub-additive capacity. If \( \sum_{n=1}^{\infty} V(A_n) < \infty, \)

then \( V(A_n; \text{i.o.}) = 0, \) where \( \{A_n; \text{i.o.}\} = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m. \)

**Lemma 3.2** (Theorem 3.1 in [17]) \( \{X_n; n \geq 1\} \) is a sequence of upper extended negatively dependent random variables in \((\Omega, \mathcal{H}, \tilde{\mathbb{E}}), \tilde{\mathbb{E}}X_k \leq 0, k = 1, \ldots, n, \) and there exists a constant \( K > 0. \) Then for all \( x, y > 0, \)

\[ \mathbb{V}(S_n \geq x) \leq \mathbb{V}
\left(\max_{k \leq n} X_k \geq y\right) + K \exp\left\{-\frac{x^2}{2(xy + B_n)}\left(1 + \frac{2}{3} \ln(1 + \frac{xy}{B_n})\right)\right\}, \]

where \( B_n = \sum_{k=1}^{n} \tilde{\mathbb{E}}(X_k^2). \)

The following theorem is the result of the main discussion in this article.

**Theorem 3.1** Suppose that \( 0 < p \leq 2, \) and \( \mathbb{V} \) is countably sub-additive. Let \( \{X_n; n \geq 1\} \) be a sequence of upper END random variables under sub-linear expectations. There exists a r.v. \( X \) and a constant \( c > 0 \) satisfying

\[ \tilde{\mathbb{E}}(h(X_n)) \leq c\tilde{\mathbb{E}}(h(X)) \quad \text{for all} \quad n \geq 1, 0 \leq h \in C_{1, \text{lip}}(\mathbb{R}), \] (3.1)

\[ C_V(|X|^p) < \infty. \] (3.2)

If \( p \geq 1 \) further assume that

\[ \tilde{\mathbb{E}}(|X|^p) \leq C_V(|X|^p). \] (3.3)

Let \( \{a_{nk}; 1 \leq k \leq n, n \geq 1\} \) be an array of real positive numbers such that

\[ \max_{1 \leq k \leq n} a_{nk} = O\left(\frac{1}{n^{1/p} \log n}\right), \quad n \to \infty. \] (3.4)

Then

\[ \limsup_{n \to \infty} \sum_{k=1}^{n} a_{nk}(X_k - c_k) \leq 0 \quad \text{a.s.} \quad \forall, \] (3.5)

where \( c_k = 0 \) if \( 0 < p < 1 \) and \( c_k = \tilde{\mathbb{E}}X_k \) if \( p \geq 1. \)

Further, if \( \{X_n; n \geq 1\} \) is lower extended negatively dependent, then

\[ \liminf_{n \to \infty} \sum_{k=1}^{n} a_{nk}(X_k - \tilde{c}_k) \geq 0 \quad \text{a.s.} \quad \forall, \] (3.6)

where \( \tilde{c}_k = 0 \) if \( 0 < p < 1 \) and \( \tilde{c}_k = \mathbb{E}X_k \) if \( p \geq 1. \)
In particular, if \( \{X_n; n \geq 1\} \) is extended negatively dependent and \( \hat{E}X_k = \hat{\varepsilon}X_k \) for \( 1 \leq p \leq 2 \), then
\[
\lim_{n \to \infty} \sum_{k=1}^{n} a_{nk}(X_k - c_k) = 0 \quad \text{a.s.} \quad \forall.
\]

(3.7)

**Remark 3.1** Theorem 3.1 generalizes SILVA’s result from probability space to sub-linear expectation space, which improves SILVA’s conclusion to some extent. Secondly, in sub-linear expectations, ZHANG\(^{[17]}\) studied partial sums, and we study the weighted sums in Theorem 3.1. Because \( a_{nk} = 1 \) is a condition that does not satisfy the condition (3.4) of Theorem 3.1, the above theorem is different from the strong number theorem studied by ZHANG.

**Remark 3.2** According to Definition 2.4, in the sub-linear expectation, the END sequence is a very broad dependent sequence. If \( K = 1 \), then the END sequence is an extended independent sequence. If \( K = 1, n = 2 \), then the END sequence is an ND sequence. Therefore, in the sub-linear expectation, Theorem 3.1 is still valid for extended independent sequences and ND sequences.

4. Proof

**Proof of Theorem 3.1** Without loss of generality, for \( p \geq 1 \), we can suppose that \( \hat{E}X_k = 0 \). Obviously, \( CV(|X|^p) < \infty \) is equivalent to \( CV(|X|^p/c^p) < \infty \) for any \( c > 0 \). We have
\[
CV(|X|^p/c^p) = \int_0^\infty V(|X|^p/c^p > x)dx = \int_0^\infty V(|X|^p > c^px)dx = \int_0^\infty V(|X| > cx^{1/p})dx.
\]

Note that
\[
CV(|X|^p/c^p) = \int_0^\infty V(|X| > cx^{1/p})dx < \infty \iff \sum_{n=1}^\infty V(|X| > cn^{1/p}) < \infty.
\]

Therefore, (3.2) is equivalent to for any \( c > 0 \),
\[
\sum_{n=1}^\infty V(|X| > cn^{1/p}) < \infty. \tag{4.1}
\]

Note that
\[
\sum_{n=1}^\infty V(|X| > cn^{1/p}) \geq \sum_{k=1}^\infty \sum_{2^{k-1} \leq n < 2^k} V(|X| > cn^{1/p}) \geq \sum_{k=1}^\infty \sum_{2^{k-1} \leq n < 2^k} V(|X| > c2^{k/p}) = 2^{-1} \sum_{k=1}^\infty 2^k V(|X| > c2^{k/p}).
\]

Hence, by (4.1), it indicates that
\[
\sum_{k=1}^\infty 2^k V(|X| > c \cdot 2^{k/p}) < \infty, \quad \forall c > 0. \tag{4.2}
\]
For upper extended negatively dependent random variables \( \{X_n; n \geq 1\} \), in order to ensure that the truncated random variables are also upper extended negatively dependent, we need that truncated functions belong to \( C_{\text{Lip}} \) and are non-decreasing. Let \( f_c(x) = -cI(x < -c) + xI(|x| \leq c) + cI(x > c) \), for any \( 1 \leq k \leq n, n \geq 1 \),

\[
Y_k := f_{\frac{k}{16}}(X_k) = -\frac{\varepsilon}{16} n^{1/p} I(X_k < -\frac{\varepsilon}{16} n^{1/p}) + X_k I(|X_k| \leq \frac{\varepsilon}{16} n^{1/p}) + \frac{\varepsilon}{16} n^{1/p} I(X_k > \frac{\varepsilon}{16} n^{1/p}),
\]

and

\[
Z_k := X_k - Y_k = (X_k + \frac{\varepsilon}{16} n^{1/p}) I(X_k < -\frac{\varepsilon}{16} n^{1/p}) + (X_k - \frac{\varepsilon}{16} n^{1/p}) I(X_k > \frac{\varepsilon}{16} n^{1/p}),
\]

and

\[
Y := f_{\frac{n}{16}}(X), \quad Z := X - Y.
\]

Then \( \{Y_k; 1 \leq k \leq n, n \geq 1\} \) is also a sequence of upper extended negatively dependent random variables by \( f_c(x) \in C_{\text{Lip}} \) and \( f_c(x) \) being non-decreasing.

Note that

\[
\sum_{k=1}^{n} a_{nk} X_k = \sum_{k=1}^{n} a_{nk} Z_k + \sum_{k=1}^{n} a_{nk} (Y_k - \bar{E}Y_k) + \sum_{k=1}^{n} a_{nk} \bar{E}Y_k
\]

\[
:= I_1 + I_2 + I_3.
\]

Thus, to prove (3.5), it suffices to verify that

\[
\limsup_{n \to \infty} I_i \leq 0 \quad \text{a.s.} \quad V \quad i = 1, 2 \quad \text{and} \quad \lim_{n \to \infty} I_3 = 0. \tag{4.3}
\]

It should be pointed out that (3.1) does not imply \( \mathbb{V}(h(X_n) \in A) \leq \mathbb{V}(h(X_1) \in A) \). Therefore, in the calculation of \( \mathbb{V}(f(X_n) \in A) \), we need to convert \( \mathbb{V} \) to \( \bar{E} \) by (2.1). On the other hand, in the probability space, there is an equality: \( E(\{|X| \leq a\}) = P(X \leq a) \), however, in the sub-linear expectation space, \( \bar{E} \) is defined through continuous functions in \( C_{\text{Lip}} \) and the indicator function \( I(|X| \leq a) \) is not continuous. Therefore, the expression \( \bar{E}I(|X| \leq a) \) does not exist. This needs to modify the indicator function by functions in \( C_{\text{Lip}} \). To this end, we define the function \( g(x) \in C_{\text{Lip}} \) as follows.

For \( 0 < \mu < 1 \), let \( g(x) \in C_{\text{Lip}}(\mathbb{R}) \) such that \( 0 \leq g(x) \leq 1 \) for all \( x \), \( g(x) = 1 \) if \( |x| \leq \mu \), and \( g(x) = 0 \) if \( |x| > 1 \). Then

\[
I(|x| \leq \mu) \leq g(|x|) \leq I(|x| \leq 1), \quad I(|x| > 1) \leq 1 - g(x) \leq I(|x| > \mu).
\]  \tag{4.4}

First, we prove that \( \limsup I_i \leq 0 \) a.s. \( \mathbb{V} \). By (2.1), (3.1), (4.1), (4.4), we can obtain,

\[
\sum_{k=1}^{\infty} \mathbb{V}(Z_k \neq 0) \leq \sum_{k=1}^{\infty} \mathbb{V}(|X_k| > \frac{\varepsilon}{16} k^{1/p})
\]

\[
\leq \sum_{k=1}^{\infty} \bar{E} \left( 1 - g \left( \frac{X_k}{16 k^{1/p}} \right) \right)
\]

\[
\leq \sum_{k=1}^{\infty} \bar{E} \left( 1 - g \left( \frac{X}{16 k^{1/p}} \right) \right)
\]

\[
\leq \sum_{k=1}^{\infty} \mathbb{V}(|X| > \mu \frac{\varepsilon}{16} k^{1/p})
\]

\[
< \infty.
\]

Thus, \( \mathbb{V}(Z_k \neq 0, \text{i.o.}) = 0 \) follows from the Borel-Cantelli’s lemma (Lemma 3.1) and \( \mathbb{V} \) being countably sub-additive. It follows that from (3.4)

\[
|I_1| = \left| \sum_{k=1}^{n} a_{nk} Z_k \right| \leq \max_{1 \leq k \leq n} a_{nk} \sum_{k=1}^{n} |Z_k|
\]
\[
\leq c \frac{1}{n^{1/p} \log n} \sum_{k=1}^{n} |Z_k| \to 0 \quad \text{a.s.} \quad \forall.
\]  

(4.5)

Now, we prove that \( |I_3| \to 0 \), as \( n \to \infty \). For any \( r > 0 \), by the \( c_r \) inequality and (4.4), we have
\[
|Y_k|^r \ll |X_k|^r I(|X_k| \leq \frac{\varepsilon}{16} n^{1/p}) + \frac{\varepsilon}{16} n^{r/p} I(|X_k| > \frac{\varepsilon}{16} n^{1/p})
\]
\[
\leq |X_k|^r g\left(\frac{\mu X_k}{16 n^{1/p}}\right) + \frac{\varepsilon}{16} n^{r/p} (1 - g\left(\frac{X_k}{16} n^{1/p}\right)).
\]

Thus,
\[
\hat{E}(|Y_k|^r) \leq \hat{E}\left[|X_k|^r g\left(\frac{\mu X_k}{16 n^{1/p}}\right)\right] + \frac{\varepsilon}{16} n^{r/p} \hat{E}\left(1 - g\left(\frac{X_k}{16} n^{1/p}\right)\right)
\]
\[
\ll \hat{E}\left[|X|^r g\left(\frac{\mu X}{16 n^{1/p}}\right)\right] + \frac{\varepsilon}{16} n^{r/p} \hat{E}\left(1 - g\left(\frac{X}{16} n^{1/p}\right)\right)
\]
\[
\leq \hat{E}\left[|X|^r g\left(\frac{\mu X}{16 n^{1/p}}\right)\right] + \frac{\varepsilon}{16} n^{r/p} \hat{V}(|X| > \mu \cdot \frac{\varepsilon}{16} n^{1/p})
\]

(4.6)

holds from (3.1) and (4.4).

Case 1 \( 0 < p < 1 \). By (3.4) and (4.6), we can imply that
\[
|I_3| \leq \sum_{k=1}^{n} \max_{1 \leq k \leq n} a_n \hat{E}(|Y_k|)
\]
\[
\ll \frac{n}{n^{1/p} \ln n} \hat{E}\left[|X|^r g\left(\frac{\mu X}{16 n^{1/p}}\right)\right] + \frac{\varepsilon}{16} n^{1/p} \hat{V}(|X| > \mu \cdot \frac{\varepsilon}{16} n^{1/p})
\]
\[
\ll \frac{1}{n^{1/p-1} \ln n} \hat{E}\left[|X|^r g\left(\frac{\mu X}{16 n^{1/p}}\right)\right] + \frac{n}{\ln n} \hat{V}(|X| > \mu n^{1/p})
\]
\[
:= I_{31} + I_{32}.
\]

By (4.1) and \( \hat{V}(|X| > \mu n^{1/p}) \downarrow \), so we get \( \frac{n}{\ln n} \hat{V}(|X| > \mu n^{1/p}) \to 0 \), as \( n \to \infty \).

It follows that
\[
I_{32} \to 0, \quad n \to \infty.
\]

(4.7)

Next, we estimate \( I_{31} \). Let \( g_j(x) \in C_L^{1, \text{Lip}}(\mathbb{R}), j \geq 1 \) such that \( 0 \leq g_j(x) \leq 1 \) for all \( x \), \( g_j\left(\frac{X}{2^{j/p}}\right) = 1 \) if \( 2^{(j-1)/p} < |x| \leq 2^j/p \), and \( g_j\left(\frac{X}{2^{j/p}}\right) = 0 \) if \( |x| \leq \mu 2^{(j-1)/p} \) or \( |x| > (1 + \mu)2^j/p \). Then
\[
g_j\left(\frac{X}{2^{j/p}}\right) \leq I(\mu 2^{(j-1)/p} < |X| \leq (1+\mu)2^j/p), \quad |X|^r g\left(\frac{X}{2^{k/p}}\right) \leq 1 + \sum_{j=1}^{k} |X|^r g_j\left(\frac{X}{2^{j/p}}\right) \forall r > 0.
\]

(4.8)

For every \( n \), there exists a constant \( k \) such that \( 2^{k-1} \leq n < 2^k \), thus, by (4.8), \( g(x) \downarrow \), we get
\[
|I_{31}| \leq \frac{1}{2^{(k-1)(1/p-1)} \ln 2^k} \hat{E}\left[|X|^r g\left(\frac{\mu X}{2^{k/p}}\right)\right]
\]
\[
\ll \frac{1}{2^{(k-1)(1/p-1)} \ln 2^k} \sum_{j=1}^{k} \hat{E}\left[|X|^r g_j\left(\frac{\mu X}{2^{j/p}}\right)\right]
\]
\[
\ll \frac{1}{2^{(k-1)(1/p-1)} \ln 2^k} \sum_{j=1}^{k} 2^{j/p} \hat{V}(|X| > 2^{(j-1)/p})
\]
\[
\leq \frac{1}{\ln 2^p} \sum_{j=1}^{k} \frac{2^{j/p}}{2^{(1/p-1)}} \mathbb{V}(|X| > 2^{1/p}2^{j/p})
\]
\[
\leq \frac{1}{\ln 2^p} \sum_{j=1}^{\infty} 2^j \mathbb{V}(|X| > c2^{j/p}).
\]

By (4.2), it follows that
\[
I_{31} \to 0, \quad n \to \infty. \tag{4.9}
\]

Combining (4.7) with (4.9), we get
\[
I_3 \to 0, \quad n \to \infty. \tag{4.10}
\]

Case 2 \quad p \geq 1.

By \(\hat{\mathbb{E}}X_k = 0\), combining (3.1) and (4.4), we have
\[
|\hat{\mathbb{E}}Y_k| = |\hat{\mathbb{E}}X_k - \hat{\mathbb{E}}Y_k| \leq \hat{\mathbb{E}}|X_k - Y_k|
\]
\[
= \hat{\mathbb{E}}[(X_k + \frac{\varepsilon}{16}n^{1/p})I(X_k < -\frac{\varepsilon}{16}n^{1/p}) + (X_k - \frac{\varepsilon}{16}n^{1/p})I(X_k > \frac{\varepsilon}{16}n^{1/p})]
\]
\[
\leq \hat{\mathbb{E}} \left[ |X_k| \left( 1 - g \left( \frac{X_k}{\frac{\varepsilon}{16}n^{1/p}} \right) \right) \left( 1 - g \left( \frac{X_k}{\frac{\varepsilon}{16}n^{1/p}} \right) \right) \right]
\]
\[
\leq cn^{-p/2} \mathbb{E}(|X|^p).
\]

Once again, using (3.2)-(3.4), we have
\[
|I_3| \leq \sum_{k=1}^{n} a_{nk} \hat{\mathbb{E}}Y_k \leq \max_{1 \leq k \leq n} a_{nk} \sum_{k=1}^{n} \hat{\mathbb{E}}Y_k
\]
\[
\leq c \frac{1}{\log n} \hat{\mathbb{E}}(|X|^p)
\]
\[
\leq c \frac{1}{\log n} C_Y(|X|^p) \to 0, \quad n \to \infty. \tag{4.11}
\]

By (4.10) and (4.11), we get
\[
I_3 \to 0, \quad n \to \infty. \tag{4.12}
\]

Finally, we estimate \(I_4\). Because \(\{a_{ni}(Y_i - \hat{\mathbb{E}}Y_i); 1 < i < n, n \geq 1\}\) is a sequence of upper extended negatively dependent random variables with \(\hat{\mathbb{E}}a_{ni}(Y_i - \hat{\mathbb{E}}Y_i) = 0\) and satisfies the conditions of Lemma 3.2. By the condition (3.4), \(a_{nk} \to 0\), without loss of generality, we can suppose that \(a_{nk} \leq 1\), and let \(x = \varepsilon, y = \frac{2x}{\ln \ln n}\), then \(|a_{nk}(Y_k - \hat{\mathbb{E}}Y_k)| \leq y\), for sequence \(\{a_{nk}(Y_k - \hat{\mathbb{E}}Y_k); 1 < k \leq n, n \geq 1\}\). By Lemma 3.2, we obtain for every \(\varepsilon > 0\),
\[
\mathbb{V} \left( \sum_{k=1}^{n} a_{nk}(Y_k - \hat{\mathbb{E}}Y_k) > \varepsilon \right)
\]
\[
\leq K \exp \left\{ -\frac{\varepsilon^2}{2(\varepsilon y + \sum_{k=1}^{n} \hat{\mathbb{E}}[a_{nk}^2(Y_k - \hat{\mathbb{E}}Y_k)^2])] \left( 1 + \frac{2}{3} \ln(1 + \frac{x y}{B_n}) \right) \right\}
\]
\[
\leq K \exp \left\{ -\frac{\varepsilon^2}{2(\varepsilon y + \sum_{k=1}^{n} \hat{\mathbb{E}}[a_{nk}^2(Y_k - \hat{\mathbb{E}}Y_k)^2])} \right\}
\]

Since \(\hat{\mathbb{E}}(Y_k - \hat{\mathbb{E}}Y_k)^2 \leq \hat{\mathbb{E}}(Y_k)^2 \leq \hat{\mathbb{E}}|Y_k|^p (\frac{\varepsilon}{16}n^{1/p})^{2-p} \frac{1}{\ln n} \sim o(1) \frac{1}{\ln n} \), we have
\[
\exp \left\{ -\frac{\varepsilon^2}{2(\varepsilon y + \sum_{k=1}^{n} \hat{\mathbb{E}}[a_{nk}^2(Y_k - \hat{\mathbb{E}}Y_k)^2])} \right\}
\]
≤ \exp \left\{ -\frac{\epsilon^2}{2 \left( \frac{2}{16 \ln n} + \frac{n}{n - \ln n} \hat{E}|Y|^{p} \left( \frac{n^{1/p}}{16} \right)^{2-p} \right)} \right\}
\leq \exp \left\{ -\frac{\epsilon^2}{2 \left( \frac{2}{8 \ln n} + \frac{2}{8 \ln n} \right)} \right\}
\leq \exp\{-2 \ln n\}.

So,
\sum_{k=1}^{\infty} \mathbb{V} \left( \sum_{k=1}^{n} a_{nk}(Y_{k} - \hat{E}Y_{k}) > \epsilon \right) \leq \sum_{k=1}^{\infty} \exp\{-2 \ln n\} \leq \sum_{k=1}^{\infty} \frac{c}{n^2} < \infty.

According to the Borel-Cantelli’s lemma and arbitrariness of \( \epsilon \), we have \( \mathbb{V}(\sum_{k=1}^{n} a_{nk}(Y_{k} - \hat{E}Y_{k}) > \epsilon; \text{i.o.}) = 0 \), that is to say,
\limsup_{n \to \infty} I_{2} \leq 0 \quad \text{a.s.} \quad \forall.
(4.13)
Together with (4.3), (4.5), (4.12) and (4.13), we can get (3.5).

If \( \{X_{n}; n \geq 1\} \) is lower extended negatively dependent, then \( \{-X_{n}; n \geq 1\} \) is upper extended negatively dependent and \( \{-X_{n}; n \geq 1\} \) satisfies the condition of Theorem 3.1. Letting \( \{-X_{n}; n \geq 1\} \) instead of \( \{X_{n}; n \geq 1\} \) in (3.5), we have
\limsup_{n \to \infty} \sum_{k=1}^{n} a_{nk}(-X_{k} - \hat{E}(-X_{k})) = -\liminf_{n \to \infty} \sum_{k=1}^{n} a_{nk}(X_{k} - \hat{\epsilon}(X_{k})) \leq 0,

namely,
\liminf_{n \to \infty} \sum_{k=1}^{n} a_{nk}(X_{k} - \hat{\epsilon}(X_{k})) \geq 0 \quad \text{a.s.} \quad \forall,

and we can get (3.6). Therefore, the proof of Theorem 3.1 is completed.

5. Conclusions
We have obtained the almost sure convergence of weighted sums under sub-linear expectations. We point out that the key tools to the proofs of the main results of SILVA\cite{5} are not suitable for our theorem. We propose the new method to prove our result. Moreover, the theorem of our paper not only extends the corresponding results of SILVA\cite{5} under sub-linear expectation space, but also is proved with the conditions of \( C_{\mathbb{V}}(|X|^{p}) < \infty \) and \( \max_{1 \leq k \leq n} a_{nk} = O\left(\frac{1}{n^{1/p} \log n}\right) \), further \( \hat{E}(|X|^{p}) \leq C_{\mathbb{V}}(|X|^{p}) < \infty \), \( 0 < p \leq 2 \) without assumption of identical distribution. Hence, studying the almost sure convergence of weighted sums for END random variables and its application under the sub-linear expectation space are of great interest.

References:
次线性期望空间下广义ND序列的加权和的几乎处处收敛

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摘要：本文研究了条件为$C_{V}(|X|^p) < \infty$, $\hat{E}(|X|^p) \leq C_{V}(|X|^p)$, $0 < p \leq 2$的次性期望空间下广义ND序列的加权和的几乎处处收敛。作为应用，我们的结果扩展了SILVA(2015)在概率空间下的相应结果。此外，本文的结果扩展了次性期望空间下加权和的几乎处处收敛。

关键词：加权和；次性期望；几乎处处收敛；END随机变量