Exponential Stability of Numerical Solutions for Neutral Stochastic Pantograph Differential Equations

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Abstract: This paper mainly studies the numerical stability of neutral stochastic pantograph differential equations (NSPDEs), with semimartingale convergence theorem. This paper establishes the new rules of linear and nonlinear NSPDEs. We will prove that under linear growth conditions, the Euler-Maruyama (EM) method can retain almost surely exponential stability of the NSPDEs, and the backward EM method can retain almost surely exponential stability for the nonlinear NSPDEs.

Key words: Neutral stochastic pantograph equation; Numerical stability; Almost surely exponential stability; The backward EM method

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1. Introduction

Numerical solutions of neutral stochastic differential delay equations (NSDDEs) have caught much attention\textsuperscript{[1-4]}, and there are many studies on its convergence\textsuperscript{[5-7]}. But the stability for NSDDEs is less studied, especially for neutral stochastic pantograph differential equations (NSPDEs), which is a special unbounded delay system.

The stability of numerical solutions for NSPDEs is inspired by [8], in the paper they got the conclusion that the backward EM method can reserve almost surely exponential stability of nonlinear stochastic delay differential equations. Subsequently, ZHOU\textsuperscript{[9]} studied stability of numerical solution to neutral stochastic functional differential equation, and ZHOU\textsuperscript{[10]} studied stability of numerical solution to stochastic pantograph differential equation.

In this paper, we will develop a new criterion on almost surely exponential stability of numerical solution to NSPDEs. This paper is divided into four sections. In Section 2, we will prove the existence and uniqueness of the global solution to NSPDEs, besides the almost surely exponential stability of the analytical solutions under some assumptions. Section 3 shows the EM method can reserve almost surely exponential stability for NSPDEs under...
linear growth condition. Section 4 proves the backward EM method preserves almost surely exponential stability with nonlinear NSPDEs.

2. The Global Solution

In this paper, we let $|x|$ be the Enulidean norm in $x \in \mathbb{R}^n$. If $D$ is a vector or matrix, $D^T$ denotes the transpose of $D$, $|D| = \sqrt{\text{trace}(D^T D)}$ is the trace norm of $D$, and $\|D\| = \sup\{|Dx| : |x| = 1\}$ is its operator norm. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}$, satisfying the usual condition which means the filtration is increasing and right continuous, and $\mathcal{F}_0$ contains all $P$-null sets. Let $w(t)$ be a $n$-dimensional scalar Brownian motion defined on the probability space.

Consider an $n$-dimensional neutral stochastic pantograph equation
\[ d[x(t) - u(x(qt))] = f(x(t), x(qt))dt + g(x(t), x(qt))dw(t), \] (2.1)
here $0 < q < 1$, on $t \leq 0$ with initial data $x_0 \in \mathbb{R}^n$, $u : C(\mathbb{R}^n) \rightarrow \mathbb{R}^n$ is a Borel measurable function, $f : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, and $g : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ are locally Lipschitz continuous. For the convenience, denote $\bar{x}(t) = x(t) - u(x(qt))$.

(C1) (The local lipschitz condition) There exists a constant $K_A > 0$ for each integer $A \geq 1$, such that
\[ |f(x_1, y_1) - f(x_2, y_2)| + |g(x_1, y_1) - g(x_2, y_2)| \leq K_A(|x_1 - x_2| + |y_1 - y_2|) \]
for all $x_k, y_k \in \mathbb{R}^n$, with $|x_k| \leq A(i = 1, 2)$.

(C2) (The polynomial growth conditions) There exist positive constants $a, a_1, a_2, a_3, a_4$, $b_1, b_2, b_3$ for all $x \in \mathbb{R}^n$, such that
\[ 2\langle \bar{x}(s), f(x(s), x(qs)) \rangle \leq -a_1|x(s)|^2 + a_2|x(qs)|^2 + a_3|x(qs)|^{a+2} - a_4|x(s)|^{a+2}, \] (2.2)
\[ |g(x(s), x(qs))|^2 \leq b_1|x(s)|^2 + b_2|x(qs)|^{a+2} + b_3|x(qs)|^{a+2}, \] (2.3)

(C3) (The contractive condition) There exists a constant $h(0 < h < 1)$, such that
\[ |u(x(qs)) - u(x(qt))|^2 \leq h|x(qs) - x(qt)|^2. \] (2.4)

Clearly, $s, t \geq 0$, moreover let $u(0) = 0$.

**Theorem 2.1** If (C1)-(C3) hold with $a_1 > b_1 + \frac{a_2}{q}, a_4 > b_2 + \frac{a_3 + b_3}{q}$, then for any initial data $x_0$, there exists almost surely a unique global solution $x(t)$ to Eq.(2.1) on $t \geq 0$.

**Proof** First, by the standing truncation technique (see [11]) under the condition (C1) to Eq.(2.1) for any initial data $x_0$, we get that there exists almost surely a unique maximal local strong solution on $0 < t < \beta_c$, and $\beta_c$ denotes the explosion time. Now, in order to prove that the solution is global, we need to show $\beta_c = \infty$ a.s.

Then we define a stopping time
\[ \beta_j = \inf\{t \in [0, \beta_c) : |\bar{x}(t)| \geq j\}, (j \in \mathbb{N}). \] (2.5)
Let $j_0 > 0$, satisfying $j_0 > |\bar{x}_0|$, each $j > j_0$. Obviously, $\beta_j$ is an increasing function with $j$. Thus, when $j \rightarrow \infty$, $\beta_j \rightarrow \beta_\infty \leq \beta_c$, a.s. If $\beta_\infty = \infty$ a.s. then $\beta_c = \infty$ a.s. So we only need to show $\beta_\infty = \infty$ a.s., that is, $P(\beta_j \leq t) \rightarrow 0 (j \rightarrow \infty, t > 0)$. Defining $V(x) = |x|^2$, by the Itô formula, we have
\[ V(\bar{x}(t \wedge \beta_j)) = V(\bar{x}(0)) + \int_0^{t \wedge \beta_j} \mathcal{L}V(x(s), x(qs))ds + M(t), \] (2.6)
where
\[ \mathcal{L}V(x(s), x(qs)) = 2\langle \bar{x}(s), f(x(s), x(qs)) \rangle + |g(x(s), x(qs))|^2. \]
\[ M(t) = \int_{0}^{t^{\wedge} \beta_j} V_{x} g(x(s), x(qs))dw(s). \]

Taking expectations at both sides of (2.6), we have
\[ \text{EV}(\bar{x}(t \wedge \beta_j)) = \text{EV}(\bar{x}(0)) + \mathbb{E} \int_{0}^{t^{\wedge} \beta_j} \mathcal{L}V(x(s), x(qs))ds. \] (2.7)

Applying (2.2)(2.3), we get that
\[ \mathcal{L}V(x(s), x(qs)) \leq (b_1 - a_1)|x(s)|^2 + (b_2 - a_4)|x(s)|^{\alpha + 2} + (a_3 + b_3)|x(qs)|^{\alpha + 2} + a_2|x(qs)|^2 \]
\[ = \frac{a_3 + b_3}{q}[q|x(qs)|^{\alpha + 2} - |x(s)|^{\alpha + 2}] + \frac{a_2}{q}[q|x(qs)|^2 - |x(s)|^2] \]
\[ - (a_1 - b_1 - \frac{a_2}{q})|x(s)|^2 - (a_4 - b_2 - \frac{a_3 + b_3}{q})|x(s)|^{\alpha + 2}. \] (2.8)

Denote
\[ J(x(s)) = (a_1 - b_1 - \frac{a_2}{q})|x(s)|^2 + (a_4 - b_2 - \frac{a_3 + b_3}{q})|x(s)|^{\alpha + 2}. \]

Because
\[ a_1 > b_1 + \frac{a_2}{q}, a_4 > b_2 + \frac{a_3 + b_3}{q}, \alpha > 0, \]

obviously, there exists a positive constant \( d_0 \) such that
\[ J(x(s)) \geq d_0|x(s)|^2. \] (2.9)

Substituting (2.8) and (2.9) into (2.7) yields
\[ \text{EV}(\bar{x}(t \wedge \beta_j)) \leq \text{EV}(\bar{x}(0)) - d_0\mathbb{E} \int_{0}^{t^{\wedge} \beta_j} |x(s)|^2ds + \frac{a_3 + b_3}{q} \mathbb{E} \int_{0}^{t^{\wedge} \beta_j} [q|x(qs)|^{\alpha + 2} - |x(s)|^{\alpha + 2}]ds \]
\[ + \frac{a_2}{q} \mathbb{E} \int_{0}^{t^{\wedge} \beta_j} [q|x(qs)|^2 - |x(s)|^2]ds. \] (2.10)

Since
\[ |\bar{x}(0)|^2 = |x(0) - u(x(0))|^2 \leq 2|x(0)|^2 + 2h|x(0)|^2 = 2(1 + h)|x_0|^2, \] (2.11)

by the nature of integral, we may get that
\[ \int_{0}^{t} [q|x(qs)|^{\alpha + 2} - |x(s)|^{\alpha + 2}]ds = \int_{0}^{qt} |x(s)|^{\alpha + 2}ds - \int_{0}^{t} |x(s)|^{\alpha + 2}ds = - \int_{qt}^{t} |x(s)|^{\alpha + 2}ds. \] (2.12)

Similarly,
\[ \int_{0}^{t} [q|x(qs)|^2 - |x(s)|^2]ds = - \int_{qt}^{t} |x(s)|^2ds. \] (2.13)

Substituting (2.11), (2.12) and (2.13) into (2.10), we get
\[ \text{EV}(\bar{x}(t \wedge \beta_j)) \leq 2(1 + h)\mathbb{E}|x_0|^2 - \frac{a_3 + b_3}{q} \mathbb{E} \int_{q(t^{\wedge} \beta_j)}^{t^{\wedge} \beta_j} |x(s)|^{\alpha + 2}ds \]
\[ - \frac{a_2}{q} \mathbb{E} \int_{q(t^{\wedge} \beta_j)}^{t^{\wedge} \beta_j} |x(s)|^2ds - d_0\mathbb{E} \int_{q(t^{\wedge} \beta_j)}^{t^{\wedge} \beta_j} |x(s)|^2ds \leq 2(1 + h)\mathbb{E}|x_0|^2. \] (2.14)

Since \( \mathbb{P}(\beta_j \leq t)V(\bar{x}(\beta_j)) \leq \text{EV}(\bar{x}(t \wedge \beta_j)) \), note that \|\bar{x}(\beta_j)\| = j, \text{EV}(\bar{x}(\beta_j)) = j^2 \), and let \( j \to \infty \), which implies \( \mathbb{P}(\beta_j \leq t) \leq \frac{2(1 + h)\mathbb{E}|x_0|^2}{j^2} \to 0 \) \( (j \to \infty) \). The proof is complete.

For stability, we need to strengthen the conditions.

(C4) (The polynomial growth conditions) Assume that there exist positive constants \( \alpha, a_1, a_2, a_3, a_4, b_1, b_2, b_3 \) such that
\[ 2(\ddot{x}(s), f(x(s), x(qs))) \leq -a_1|x(s)|^2 + a_2e^{-(1-q)x(s)|x(qs)|^2} \]
\[ + a_3 e^{--(1-\theta)c_s}|x(qs)|^{\alpha+2} - a_4 |x(s)|^{\alpha+2}, \]  
(2.15)

\[ |g(x(s), x(qs))|^2 \leq b_1 |x(s)|^2 + b_2 |x(s)|^{\alpha+2} + b_3 e^{--(1-\theta)c_s}|x(qs)|^{\alpha+2}, \]  
(2.16)

(C5) (The contractive condition) Assume any \( s, t \geq 0 \), satisfying

\[ |u(x(qs)) - u(x(qt))|^2 \leq q e^{--(1-\theta)c_s(s-t)} |x(qs) - x(qt)|^2. \]  
(2.17)

Moreover, let \( u(0) = 0 \).

**Theorem 2.2** Suppose that (C1), (C4) and (C5) hold with \( a_1 > b_1 + \frac{a_2}{q} \), \( a_4 > b_2 + \frac{a_3+b_3}{q} \), then for any initial data \( x_0 \), the solution \( x(t) \) to (2.1) is \( p \)th moment and almost surely exponentially stable, that is \( \limsup_{t \to \infty} \frac{1}{t} \log|x(t)| \leq -\frac{\varepsilon}{2} \) a.s., where \( \varepsilon < \frac{1}{2}(a_1 - b_1 - \frac{a_2}{q}) \).

**Proof** Obviously, the conditions (C4)(C5) are stronger than the conditions (C2)(C3), so there exists a unique global solution to Eq. (2.1). Define \( V(x) = |x|^2 \), for any \( \varepsilon > 0 \). Applying the Itô formula, we get

\[ e^{\varepsilon t} V(\tilde{x}(t)) = V(\tilde{x}(0)) + \int_0^t e^{\varepsilon s} [\mathcal{L}V(x(s), x(qs)) + \varepsilon V(\tilde{x}(s))] ds + M_1(t), \]  
(2.18)

where \( \mathcal{L}V(x(s), x(qs)) = 2(\tilde{x}(s), f(x(s), x(qs))) + |g(x(s), x(qs))|^2 \), and \( M_1(t) = \int_0^t e^{\varepsilon s} V(\tilde{x}(s)) g(x(s), x(qs)) ds \) is a real-valued continuous local martingale with \( M(0) = 0 \). Since

\[ |\tilde{x}(s)|^2 \leq 2|x(s)|^2 + 2|x(u)|^2 \leq 2|x(s)|^2 + 2q e^{-\varepsilon c_s} |x(qs)|^2, \]  
(2.19)

by (2.15),(2.16) and (2.19), we have

\[ \mathcal{L}V(x(s), x(qs)) + \varepsilon V(\tilde{x}(s)) \leq - (a_1 - b_1 - 2\varepsilon)|x(s)|^2 + (a_2 + 2\varepsilon q) e^{-\varepsilon c_s} |x(qs)|^2 \]
\[ + (a_3 + b_3) e^{-\varepsilon c_s} |x(s)|^{\alpha+2} + (b_2 - a_4)|x(s)|^{\alpha+2} \]
\[ \leq (2\varepsilon + \frac{a_2}{q}) [q e^{-\varepsilon c_s} |x(qs)|^2 - |x(s)|^2] + \frac{a_3 + b_3}{q} e^{-\varepsilon c_s} |x(qs)|^{\alpha+2} - |x(s)|^{\alpha+2} \]
\[ - (a_1 - b_1 - 4\varepsilon - \frac{a_2}{q}) |x(s)|^2 - (a_4 - b_2 - \frac{a_3+b_3}{q})|x(s)|^{\alpha+2}. \]  
(2.20)

Define \( I(x(s)) = (a_1 - b_1 - 4\varepsilon - \frac{a_2}{q}) |x(s)|^2 + (a_4 - b_2 - \frac{a_3+b_3}{q})|x(s)|^{\alpha+2} \). Since \( a_1 > b_1 + \frac{a_2}{q} \), \( a_4 > b_2 + \frac{a_3+b_3}{q} \), \( \alpha > 0 \), take \( \varepsilon < \frac{1}{4}(a_1 - b_1 - \frac{a_2}{q}) \). Obviously there exists a constant \( d_1 > 0 \), such that \( I(x(s)) \geq d_1 |x(s)|^2 \). Substituting \( I(x(s)) \geq d_1 |x(s)|^2 \) and (2.20) into (2.18), we get

\[ e^{\varepsilon t} V(\tilde{x}(t)) \leq V(\tilde{x}(0)) + (2\varepsilon + \frac{a_2}{q}) \int_0^t e^{\varepsilon s} [q e^{-\varepsilon c_s} |x(qs)|^2 - |x(s)|^2] ds \]
\[ + \frac{a_3 + b_3}{q} \int_0^t e^{\varepsilon s} [q e^{-\varepsilon c_s} |x(qs)|^{\alpha+2} - |x(s)|^{\alpha+2}] ds - d_1 \int_0^t e^{\varepsilon s} |x(s)|^2 ds + M_1(t). \]  
(2.21)

By the property of the integral, we get

\[ \int_0^t e^{\varepsilon s} [q e^{-\varepsilon c_s} |x(qs)|^{\alpha+2} - |x(s)|^{\alpha+2}] ds = \int_0^t [q e^{\varepsilon qs} |x(qs)|^{\alpha+2} - e^{\varepsilon s} |x(s)|^{\alpha+2}] ds \]
\[ = \int_0^q t e^{\varepsilon s} |x(s)|^{\alpha+2} ds - \int_0^t e^{\varepsilon s} |x(s)|^{\alpha+2} ds \]
\[ = - \int_q^t e^{\varepsilon s} |x(s)|^{\alpha+2} ds. \]  
(2.22)

Similarly,

\[ \int_0^t e^{\varepsilon s} [q e^{-\varepsilon c_s} |x(qs)|^2 - |x(s)|^2] ds = - \int_q^t e^{\varepsilon s} |x(s)|^2 ds. \]  
(2.23)
Substituting (2.22), (2.23) into (2.21) yields
\[ e^{\varepsilon t}V(\tilde{x}(t)) \leq V(\tilde{x}(0)) + M_1(t) - (2\varepsilon + \frac{a_2}{q}) \int_0^t e^{\varepsilon s}|x(s)|^2ds \]
\[ - \frac{a_3 + b_3}{q} \int_0^t e^{\varepsilon s}|x(s)|^2 + d_1 \int_0^t e^{\varepsilon s}|x(s)|^2 + \frac{b_2}{\sqrt{q}} \sup_{0 \leq s < r} e^{\varepsilon s}u(x(qs)) \leq \frac{1}{1 - \sqrt{q}} \sup_{0 \leq s} e^{\varepsilon s}|\tilde{x}(s)|^2 + \frac{1}{\sqrt{q}} \sup_{0 \leq s} e^{\varepsilon s}|u(x(qs))|^2. \]  
(2.26)

Applying the inequality
\[(x + y)^2 \leq \frac{x^2}{\delta} + \frac{y^2}{1 - \delta}, x, y > 0, 0 < \delta < 1,\]
moreover, choosing \(\delta = \sqrt{q}\), since \(0 < q < 1\), we get \(\delta = \sqrt{q} > q\). At the same time, we may get
\[ \sup_{0 \leq s} e^{\varepsilon s}|x(s)|^2 \leq \sup_{0 \leq s} e^{\varepsilon s}|\tilde{x}(s)|^2 + \sup_{0 \leq s} e^{\varepsilon s}|u(x(qs))|^2. \]  
(2.27)

Substituting (2.25) and (2.27) into (2.26), we have
\[ \sup_{0 \leq s} e^{\varepsilon s}|x(s)|^2 \leq \frac{d_2}{1 - \sqrt{q}} + \sqrt{q} \sup_{0 \leq s} e^{\varepsilon s}|x(s)|^2. \]  
(2.28)

So
\[ \lim_{t \to \infty} \sup_{t \geq 0} e^{\varepsilon t}|x(t)|^2 \leq \frac{d_2}{(1 - \sqrt{q})^2}, \]
which implies the desired result.

3. Exponential Stability of the EM Approximation

The section will show that the Euler-Maruyama method can reproduce the almost surely exponential stability under the linear growth condition, with the help of the discrete semi-martingale convergence theorem.

(C6) (The linear growth condition) Assume that there exist positive constants \(a, a_1, a_3, b_1, b_3\) such that
\[ 2(x(s) - u(x(qs)), f(x(s), x(qs))) \leq -a_1|x(s)|^2 + a_3 e^{-(1-q)\varepsilon s}|x(qs)|^2, \]  
(3.1)
\[ |f(x(s), x(qs))|^2 \vee |g(x(s), x(qs))|^2 \leq b_1|x(s)|^2 + b_3 e^{-(1-q)\varepsilon s}|x(qs)|^2, \]  
(3.2)

Clearly, let \(a = 0, a_2 = 0, a_4 = 0, b_2 = 0\), and (C4) implies (C6), then Theorem 2.1 and Theorem 2.2 hold.

Define the Euler-Maruyama approximate solution for Eq.(2.1), given a step size \(\Delta \in (0, 1)\), and calculate the approximations \(X_k \approx x_k, t_k = k\Delta\) by setting \(X_0 = x_0\) and letting
\[ \overline{X}_{k+1} = \overline{X}_k + f(X_k, X_{[qk]}\Delta + g(X_k, X_{[qk]})\Delta w_k, \]
(3.3)
where \(\overline{X}_k = X_k - u(X_{[qk]}). \Delta w_k = w(t_{k+1}) - w(t_k), k = 1, 2, \ldots\) are independent \(N(0, \Delta)\) -distributed Gaussian random variables, and they are \(\mathcal{F}_{t_k}\) measurable at the mesh-point \(t_k\).

Lemma 3.1[9] Let \(\{A_i\}\) and \(\{U_i\}\) be two sequences of nonnegative random variables such that both \(A_i\) and \(U_i\) are \(\mathcal{F}_{t_i}\) measurable for \(i = 1, 2, \ldots\) with \(A_0 = U_0 = 0\) a.s. Let \(\{M_i\}\) be a real-valued local martingale with \(M_0 = 0\) a.s. Let \(\zeta\) be a nonnegative \(\mathcal{F}_0\)-measurable
random variable. Assume that \( \{X_i\} \) is a nonnegative semi-martingale with the Doob-Mayer decomposition\( X_i = \zeta + A_i - U_i + M_i \).

If \( \lim_{i \to \infty} A_i < \infty \) a.s., then for almost all \( \omega \in \Omega \), \( \lim_{i \to \infty} X_i < \infty \), \( \lim_{i \to \infty} U_i < \infty \).

**Theorem 3.2** Assume that (C1),(C5) and (C6) hold with \( a_1 > 2b_1 + \frac{a_3 + 2b_3}{q} \), then there exists a sufficiently small \( \Delta \in (0,1) \) such that the approximate solution \( X_k \) defined by Eq.(3.3) satisfies

\[
\limsup_{k \to \infty} \frac{1}{k \Delta} \log |X_k| \leq -\frac{\varepsilon}{2} \text{ a.s.}
\]

where \( \varepsilon \leq 1 - \frac{\ln q}{2} \), and \( \varepsilon \) satisfies \( a_2 - 2b_1 - 2\varepsilon - e^\varepsilon \geq 0 \).

**Proof.** By (3.3), we may obtain

\[
|\tilde{X}_{k+1}|^2 = |\tilde{X}_k|^2 + |f(X_k, X_{[qk]})|^2 \Delta^2 + |g(X_k, X_{[qk]})|^2 |\Delta w_k|^2 + 2|\tilde{X}_k, f(X_k, X_{[qk]})| \Delta + 2|\tilde{X}_k, g(X_k, X_{[qk]})| \Delta w_k + 2|f(X_k, X_{[qk]}) \Delta, g(X_k, X_{[qk]})| \Delta w_k.
\]

Applying (C6) and noting \( \Delta \in (0,1) \), we may compute

\[
|\tilde{X}_{k+1}|^2 \leq |\tilde{X}_k|^2 + b_1 \Delta (1 + \Delta) |X_k|^2 + b_3 \Delta (1 + \Delta) e^{-1-q} \Delta |X_k|^2 - a_1 \Delta |X_k|^2 + a_3 \Delta e^{-1-q} \Delta |X_k|^2 + 2|\tilde{X}_k, f(X_k, X_{[qk]})| \Delta + 2|\tilde{X}_k, g(X_k, X_{[qk]})| \Delta w_k + |g(X_k, X_{[qk]})|^2 |\Delta w_k|^2 - \Delta) \]

where

\[
s_k = 2|\tilde{X}_k, f(X_k, X_{[qk]})| \Delta + g(X_k, X_{[qk]})| \Delta w_k + |g(X_k, X_{[qk]})|^2 |\Delta w_k|^2 - \Delta).
\]

According to (3.4), we may obtain immediately

\[
e^{\varepsilon(k+1)\Delta} |\tilde{X}_{k+1}|^2 - e^{\varepsilon k \Delta} |\tilde{X}_k|^2 \leq e^{\varepsilon(k+1)\Delta} (1 - e^{-\varepsilon \Delta}) |\tilde{X}_k|^2 - (a_1 - b_1 \Delta) \Delta e^{\varepsilon(k+1)\Delta} |X_k|^2 + (a_3 + b_3 + b_2 \Delta) \Delta e^{\varepsilon(qk+1)\Delta} |X_{[qk]}|^2 + e^{\varepsilon(k+1)\Delta} s_k.
\]

Recalling that \( \tilde{X}_k = X_k - a(X_{[qk]}) \), we have \( |\tilde{X}_k|^2 \leq 2|X_k|^2 + 2qe^{-1-q} \Delta |X_{[qk]}|^2 \). This, together with (3.5), yields

\[
e^{\varepsilon(k+1)\Delta} |\tilde{X}_{k+1}|^2 - e^{\varepsilon k \Delta} |\tilde{X}_k|^2 \leq -[(a_1 - b_1 - b_1 \Delta) \Delta - 2(1 - e^{-\varepsilon \Delta})] e^{\varepsilon(k+1)\Delta} |X_k|^2 + [(a_3 + b_3 + b_2 \Delta) \Delta + 2q(1 - e^{-\varepsilon \Delta})] e^{\varepsilon(qk+1)\Delta} |X_{[qk]}|^2 + e^{\varepsilon(k+1)\Delta} s_k.
\]

By \( |\tilde{X}_k|^2 \leq 2|X_k|^2 + 2qe^{-1-q} \Delta |X_{[qk]}|^2 \), we get that \( |\tilde{X}_k|^2 \leq 4|x_0|^2 \). Summing up inequality (3.6) from 1 to \( k \), we obtain

\[
e^{\varepsilon k \Delta} |\tilde{X}_k|^2 \leq 4|x_0|^2 - [(a_1 - b_1 - b_1 \Delta) \Delta - 2(1 - e^{-\varepsilon \Delta})] \sum_{i=0}^{k-1} e^{\varepsilon(i+1)\Delta} |X_i|^2 + [(a_3 + b_3 + b_2 \Delta) \Delta + 2q(1 - e^{-\varepsilon \Delta})] \sum_{i=0}^{k-1} e^{\varepsilon(qi+1)\Delta} |X_{[qi]}|^2 + \sum_{i=0}^{k-1} e^{\varepsilon(i+1)\Delta} s_i.
\]
Substituting (3.8) into (3.7) yields
\[ e^{\epsilon \Delta} |\hat{X}_k|^2 \leq |x_0|^2 - \{(a_1 - b_1 - b_1 \Delta)\Delta - 2(1 - e^{-\epsilon \Delta}) - e^{\epsilon \Delta}[(a_3 + b_3 + b_3 \Delta)\Delta + 2q(1 - e^{-\epsilon \Delta})] \sum_{i=0}^{k-1} e^{\epsilon(i+1)\Delta} s_i \}
- \{(a_3 + b_3 + b_3 \Delta)\Delta + 2q(1 - e^{-\epsilon \Delta}) \sum_{i=0}^{k-1} e^{\epsilon(i+2)\Delta} |X_i|^2 \}. \] (3.9)

Denote
\[ f(\Delta, \varepsilon) = a_1 - b_1 - b_1 \Delta - 2\frac{1 - e^{-\epsilon \Delta}}{\Delta} - e^{\epsilon \Delta}[(a_3 + b_3 + b_3 \Delta) + 2q(1 - e^{-\epsilon \Delta})]. \] (3.10)

By the Taylor expansions, we get
\[ e^{-\epsilon \Delta} = 1 - \varepsilon \Delta + \frac{(\varepsilon \Delta)^2}{2} - \frac{(\varepsilon \Delta)^3}{3!} + o(\varepsilon \Delta) > 1 - \varepsilon \Delta, \]
so \( \frac{e^{-\epsilon \Delta} - 1}{\Delta} > -\varepsilon, \Delta \in (0, 1) \) which implies that
\[ f(\Delta, \varepsilon) > a_1 - b_1 - b_1 \Delta - 2\varepsilon - e^{\epsilon \Delta}(a_3 + b_3 + b_3 \Delta + 2q\varepsilon)
> a_1 - 2b_1 - 2\varepsilon - e^{\epsilon}(a_3 + b_3 + b_3 \varepsilon). \]

Since \( a_1 > 2b_1 + \frac{\alpha + 2b_2}{\beta} \), choosing sufficiently small \( \Delta \in (0, 1) \) and \( \varepsilon \in (0, 1) \), such that \( f(\Delta, \varepsilon) > 0 \), by Lemma 3.1, there exists a positive constant \( d_3 \) such that
\[ \limsup_{k \to \infty} e^{\epsilon \Delta} |\hat{X}_k|^2 < d_3 \quad \text{a.s.} \] (3.11)

Since \( X_k = \hat{X}_k + u|X_{[q]}| \), for any \( 0 < \delta_1 < 1 \), we have
\[ \sup_{0 \leq m \leq k} e^{\epsilon \Delta} |X_m|^2 \leq \frac{1}{1 - \delta_1} \sup_{0 \leq m \leq k} e^{\epsilon \Delta} |\hat{X}_m|^2 + \frac{\delta_1}{\delta_1} \sup_{0 \leq m \leq k} e^{\epsilon \Delta} |u(X_{[q]})|^2. \] (3.12)

Since
\[ \sup_{0 \leq m \leq k} e^{\epsilon \Delta} |u(X_{[q]})|^2 = \sup_{0 \leq m \leq k} e^{\epsilon \Delta} qe^{-\epsilon \Delta} |X_{[q]})|^2 = \sup_{0 \leq m \leq k} qe^{\epsilon \Delta} |X_m|^2 \leq qe^{\epsilon \Delta} \sup_{0 \leq m \leq k} e^{\epsilon \Delta} |X_m|^2 \leq qe^{\epsilon \Delta} \sup_{0 \leq m \leq k} e^{\epsilon \Delta} |X_m|^2, \] (3.13)

taking \( \delta_1 = \sqrt{q} \), noting that \( \varepsilon < -\frac{\ln \delta_1}{\Delta} \), so \( 1 - \sqrt{q}e^{\epsilon} > 0 \). Substituting (3.11),(3.13) into (3.12) yields
\[ \sup_{0 \leq m \leq k} e^{\epsilon \Delta} |X_m|^2 \leq \frac{1}{\sqrt{q}} d_4 + \sqrt{q}e^{\epsilon} \sup_{0 \leq m \leq k} e^{\epsilon \Delta} |X_m|^2. \] (3.14)

So
\[ \limsup_{k \to \infty} e^{\epsilon \Delta} |X_k|^2 \leq \frac{d_3}{(1 - \sqrt{q})(1 - \sqrt{q}e^{\epsilon})} \quad \text{a.s.}, \]
which implies that
\[ \limsup_{k \to \infty} \frac{1}{k\Delta} \log |X_k|^2 < -\frac{\varepsilon}{2} \quad \text{a.s.} \]

4. Exponential Stability of the Backward EM Approximation

In the section we will shall show the backward EM method can reserve almost surely exponential stability under the polynomial growth conditions.
(C7) (The polynomial growth conditions) Assume that there exist positive constants \( \alpha, a_1, a_2, a_3, a_4, b_1, b_2, b_3 \) such that
\[
2\langle x(s) - u(x(qs)), f(x(s), x(qs)) \rangle \leq -a_1|x(s)|^2 + a_2e^{-(1-q)\varepsilon s}|x(qs)|^2 \\
+ a_3e^{-(1-q)\varepsilon s}|x(qs)|^{\alpha+2} - a_4|x(s)|^{\alpha+2},
\]
\[
|g(x(s), x(qs))|^2 \leq b_1|x(s)|^2 + b_2|x(s)|^{\alpha+2} + b_3e^{-(1-q)\varepsilon s}|x(qs)|^{\alpha+2}.
\]
By Theorems 2.1 and 2.2, the equation (2.1) has a unique global solution and the solution is almost surely exponentially stable.

Assume the step size \( \Delta \in (0, 1) \), and \( t \in [0, T] \), \( M\Delta = T \). \( M \) is a positive integer, Let \( t_k = k\Delta(k \geq 0) \), \( k \) be the integer part of \( \frac{t}{\Delta} \). Define that
\[
\tilde{X}_{k+1} = \tilde{X}_k + f(X_{k+1}, X_{(qk)}\Delta + g(X_{k+1}, X_{(qk)})\Delta w_k,
\]
where \( \tilde{X}_k = X_k - u(X_{(qk)}) \), \( \Delta w_k = w(t_{k+1}) - w(t_k) \), \( k = 1, 2, \ldots \)

(C8) (One-sided Lipschitz conditions) There exists a constant \( \lambda > 0 \) such that for any \( x_i \in \mathbb{R}^n, (i = 1, 2) \)
\[
\langle x_1 - x_2, f(x_1, y) - f(x_2, y) \rangle \leq \lambda|x_1 - x_2|^2.
\]
By the one-sided Lipschitz conditions above, if \( \lambda \Delta < 1 \), we can get that the backward EM scheme (4.2) is well defined (see[13]). From now on we always assume that \( \Delta < \frac{1}{q} \).

**Theorem 4.1** Assume that (C1), (C5), (C7) and (C8) hold with \( a_1 > \frac{a_2+b_2}{q}, a_4 > \frac{a_3+b_3}{q^2} \), then there exists a sufficiently small \( \Delta \in (0, 1) \) such that the approximate solution \( X_k \) defined by Eq.(2.1) satisfies
\[
\lim_{k \rightarrow \infty} \sup_{|k|} \frac{1}{k\Delta} \log|X_k| \leq -\frac{\varepsilon}{2} \text{ a.s.},
\]
where \( \varepsilon \in 1 \land \frac{q^2}{4}[a_1 - \frac{a_2+b_2}{q}] \).

**Proof** By (4.2), we may obtain
\[
|\tilde{X}_{k+1}|^2 = \langle \tilde{X}_{k+1}, f(X_{k+1}, X_{(qk)})\Delta + g(X_{k+1}, X_{(qk)})\Delta w_k \rangle \\
\leq \langle \tilde{X}_{k+1}, f(X_{k+1}, X_{(qk)})\Delta \rangle + \frac{1}{2}|\tilde{X}_{k+1}|^2 + \frac{1}{2}g(X_{k+1}, X_{(qk)})\Delta w_k|^2 \\
= \langle \tilde{X}_{k+1}, f(X_{k+1}, X_{(qk)})\Delta \rangle + \frac{1}{2}|\tilde{X}_{k+1}|^2 + |\tilde{X}_{k+1}|^2 \\
+ \frac{1}{2}g(X_{k+1}, X_{(qk)})\Delta w_k|^2 + \langle \tilde{X}_{k+1}, g(X_{k+1}, X_{(qk)})\Delta w_k \rangle
\]
Applying (C7), we may compute
\[
|\tilde{X}_{k+1}|^2 \leq |\tilde{X}_k|^2 - a_1\Delta|X_{k+1}|^2 + a_2\Delta e^{-(1-q)\varepsilon s}|X_{(qk)}\Delta|^2 \\
+ a_3\Delta e^{-(1-q)\varepsilon s}|X_{(qk)}\Delta|^{\alpha+2} - a_4\Delta|X_{k+1}|^{\alpha+2} \\
+ b_1\Delta|X_k|^2 + b_2\Delta|X_k|^{\alpha+2} + b_3\Delta e^{-(1-q)\varepsilon s}|X_{(qk)}^{|\alpha+2} \\
+ |g(X_{k+1}, X_{(qk)})\Delta w_k|^2 - 2\langle \tilde{X}_k, g(X_{k+1}, X_{(qk)})\Delta w_k \rangle
\]
Denote
\[
s_k = |g(X_{k+1}, X_{(qk)})\Delta w_k|^2 + 2\langle \tilde{X}_k, g(X_{k+1}, X_{(qk)})\Delta w_k \rangle
\]
So it is not difficult to get that
\[
e^{c(k+1)\Delta}|\tilde{X}_{k+1}|^2 - e^{c(k)\Delta}|\tilde{X}_k|^2 \leq e^{c(k+1)\Delta}|(1 - e^{-\varepsilon\Delta})|\tilde{X}_k|^2 - a_1\Delta e^{c(k+1)\Delta}|X_{k+1}|^2 \\
+ a_2\Delta e^{c(k+1)\Delta}|X_{(qk)}\Delta|^2 + a_3\Delta e^{c(k+1)\Delta}|X_{(qk)}\Delta|^{\alpha+2} \\
- a_4\Delta e^{c(k+1)\Delta}|X_{k+1}|^{\alpha+2} + b_1\Delta e^{c(k+1)\Delta}|X_k|^2
\]
+ b_2 \Delta e^{e(k+1)\Delta} |X_k|^{\alpha+2} + b_3 \Delta e^{e(qk+1)\Delta} |X_{[qk]}|^{\alpha+2} \\
+ e^{e(k+1)\Delta} s_k.  \quad (4.4)

Summing up these inequalities from 1 to \( k - 1 \), we obtain

\begin{align*}
& e^{e\Delta} |\tilde{X}_k|^2 \leq |\tilde{X}_0|^2 + (e^{e\Delta} - 1) \sum_{i=0}^{k-1} e^{e\Delta} |\tilde{X}_i|^2 - a_1 \Delta \sum_{i=0}^{k-1} e^{e(i+1)\Delta} |X_{[i]}|^2 \\
& + a_2 \Delta \sum_{i=0}^{k-1} e^{e(q(i+1))\Delta} |X_{[q(i+1)]}|^2 + a_3 \Delta \sum_{i=0}^{k-1} e^{e(q(i+1))\Delta} |X_{[q(i+1)]}|^{\alpha+2} \\
& - a_4 \Delta \sum_{i=0}^{k-1} e^{e(i+1)\Delta} |X_{i+1}|^{\alpha+2} + b_1 \Delta \sum_{i=0}^{k-1} e^{e(i+1)\Delta} |X_i|^2 \\
& + b_2 \Delta \sum_{i=0}^{k-1} e^{e(i+1)\Delta} |X_i|^{\alpha+2} + b_3 \Delta \sum_{i=0}^{k-1} e^{e(q(i+1))\Delta} |X_{[q]}|^{\alpha+2} + \sum_{i=0}^{k-1} e^{e(i+1)\Delta} s_i. \quad (4.5)
\end{align*}

Recalling that \( \tilde{X}_k = X_k - u(X_{[qk]}) \), we have \( |\tilde{X}_k|^2 \leq 2 |X_k|^2 + 2 q e^{-e(1-q)k\Delta} |X_{[qk]}|^2 \). This, together with (4.5), yields

\begin{align*}
& e^{e\Delta} |\tilde{X}_k|^2 \leq |\tilde{X}_0|^2 + \sum_{i=0}^{k-1} e^{e(i+1)\Delta} s_i + 2 q (e^{e\Delta} - 1) \sum_{i=0}^{k-1} e^{e\Delta} |X_{[q]}|^2 \\
& + 2 (e^{e\Delta} - 1) \sum_{i=0}^{k-1} e^{e\Delta} |X_i|^2 - a_1 \Delta \sum_{i=0}^{k} e^{e\Delta} |X_i|^2 + a_2 \Delta \sum_{i=0}^{k-1} e^{e(q(i+1))\Delta} |X_{[q(i+1)]}|^2 \\
& + a_3 \Delta \sum_{i=0}^{k-1} e^{e(q(i+1))\Delta} |X_{[q(i+1)]}|^{\alpha+2} - a_4 \Delta \sum_{i=0}^{k} e^{e\Delta} |X_{i+1}|^{\alpha+2} + b_1 \Delta \sum_{i=0}^{k-1} e^{e(i+1)\Delta} |X_i|^2 \\
& + b_2 \Delta \sum_{i=0}^{k-1} e^{e(i+1)\Delta} |X_i|^{\alpha+2} + b_3 \Delta \sum_{i=0}^{k-1} e^{e(q(i+1))\Delta} |X_{[q]}|^{\alpha+2}. \quad (4.6)
\end{align*}

Let \( |q| = j \), then \( j \leq q \leq j + 1 \), so \( q_i - 1 < j \leq q_i \). If \( 0 \leq i \leq k - 1 \), then \( -1 < j \leq q(k-1) < |q| + 1 - q \leq |q| + 1 \). So

\begin{align*}
& \sum_{i=0}^{k-1} e^{e(q(i+1))\Delta} |X_{[q]}|^{\alpha+2} - \sum_{i=0}^{k-1} e^{e(i+2)\Delta} |X_i|^{\alpha+2} \leq \sum_{i=0}^{k-1} e^{e(i+2)\Delta} |X_i|^{\alpha+2} - \sum_{i=0}^{k-1} e^{e(i+2)\Delta} |X_i|^2 \\
& \leq - \sum_{|q|+2}^{k-1} e^{e(i+2)\Delta} |X_{i+1}|^{\alpha+2}. \quad (4.7)
\end{align*}

So we get

\begin{align*}
& \sum_{i=0}^{k-1} e^{e(q(i+1))\Delta} |X_{[q]}|^{\alpha+2} \leq \sum_{i=0}^{k-1} e^{e(i+2)\Delta} |X_i|^{\alpha+2} - \sum_{|q|+2}^{k-1} e^{e(i+2)\Delta} |X_{i+1}|^{\alpha+2}. \quad (4.8)
\end{align*}

Similarly, we get

\begin{align*}
& \sum_{i=0}^{k-1} e^{e(q(i+1))\Delta} |X_{[q]}|^2 \leq \sum_{i=0}^{k-1} e^{e(i+2)\Delta} |X_i|^2 - \sum_{|q|+2}^{k-1} e^{e(i+2)\Delta} |X_{i+1}|^2. \quad (4.9)
\end{align*}

Applying the same technique as (4.7), we can get

\begin{align*}
& \sum_{i=0}^{k-1} e^{e(q(i+1))\Delta} |X_{[q(i+1)]}|^{\alpha+2} \leq \sum_{i=0}^{k-1} e^{e(i+1)\Delta} |X_i|^{\alpha+2} - \sum_{|q|+2}^{k-1} e^{e(i+1)\Delta} |X_{i+1}|^{\alpha+2} \quad (4.10)
\end{align*}
and
\[\sum_{i=0}^{k-1} e^{\rho(i+1)\Delta} |X_{q(i+1)}|^2 \leq \sum_{i=0}^{k-1} e^{\rho(i+1)\Delta} |X_i|^2 - \sum_{[qk]+2}^{k-1} e^{\rho(i+1)\Delta} |X_i|^2. \]  
(4.11)

Substituting (4.8)-(4.11) into (4.6) yields
\[e^{\rho\Delta} |\tilde{X}_k|^2 \leq |\tilde{X}_0|^2 + \sum_{i=0}^{k-1} e^{\rho(i+1)\Delta} s_i + 2(e^{\rho\Delta} - 1) \sum e^{\rho|i|} |X_i|^2 \]
\[+ 2q(1 - e^{-\rho\Delta}) \sum e^{\rho(i+2)\Delta} |X_i|^2 - \sum_{[qk]+2}^{k-1} e^{\rho(i+2)\Delta} |X_i|^2 - a_1 \Delta \sum e^{\rho|i|} |X_i|^2 - |X_0|^2 + e^{\rho\Delta} |X_k|^2 \]
\[+ a_2 \Delta \sum e^{\rho(i+1)\Delta} |X_i|^2 - \sum_{[qk]+2}^{k-1} e^{\rho(i+1)\Delta} |X_i|^2 + a_3 \Delta \sum e^{\rho(i+1)\Delta} |X_i|^2 + a_4 e^{\rho\Delta} |X_0|^2 + e^{\rho\Delta} |X_k|^2 \]
\[+ b_2 \Delta \sum e^{\rho(i+1)\Delta} |X_i|^2 + a_3 \Delta \sum e^{\rho(i+2)\Delta} |X_i|^2 - \sum_{[qk]+2}^{k-1} e^{\rho(i+2)\Delta} |X_i|^2 + b_3 \Delta \sum e^{\rho(i+1)\Delta} |X_i|^2. \]  
(4.12)

Organizing the above inequalities, we get
\[e^{\rho\Delta} |\tilde{X}_k|^2 \]
\[\leq |\tilde{X}_0|^2 + \sum_{i=0}^{k-1} e^{\rho(i+1)\Delta} s_i + a_1 \Delta |X_0|^2 + a_4 \Delta |X_0|^2 \]
\[- \left[ a_1 - \frac{2(e^{\rho\Delta} - 1)}{\Delta} - 2q(1 - e^{-\rho\Delta}) e^{\rho\Delta} - a_2 e^{\rho\Delta} - b_1 e^{\rho\Delta} \right] \sum e^{\rho|i|} |X_i|^2 \Delta \]
\[- \left[ a_4 - a_3 e^{\rho\Delta} - b_2 e^{\rho\Delta} - b_3 e^{\rho\Delta} \right] \sum e^{\rho|i|} |X_i|^2 \Delta - [2q(1 - e^{-\rho\Delta}) e^{\rho\Delta} + a_2 e^{\rho\Delta}] \sum e^{\rho|i|} |X_i|^2 \]
\[- \left[ a_3 e^{\rho\Delta} + b_3 e^{\rho\Delta} \right] \sum e^{\rho|i|} |X_i|^2 + 2. \]  
(4.13)

Denote that
\[F(\Delta, \varepsilon) = a_1 - \frac{2(e^{\rho\Delta} - 1)}{\Delta} - 2q(1 - e^{-\rho\Delta}) e^{\rho\Delta} - a_2 e^{\rho\Delta} - b_1 e^{\rho\Delta}, \]
\[G(\Delta, \varepsilon) = a_4 - a_3 e^{\rho\Delta} - b_2 e^{\rho\Delta} - b_3 e^{\rho\Delta}. \]

By the Taylor expansion of $e^{-\rho\Delta}$ at the neighborhoods of the point $\Delta = 0$, we obtain that
\[\frac{e^{-\rho\Delta}}{\Delta} \leq -\varepsilon, F(\Delta, \varepsilon) \geq a_1 - 2\varepsilon e^{\rho\Delta} - 2q\varepsilon e^{\rho\Delta} - a_2 e^{\rho\Delta} - b_1 e^{\rho\Delta}, \] since that $0 < q < 1$, which implies $F(\Delta, \varepsilon) \geq a_1 - 4\varepsilon e^{\rho\Delta} - a_2 e^{\rho\Delta} - b_1 e^{\rho\Delta}$.

Since $a_1 > a_2 + b_2$, letting $e^{\rho\Delta} = \frac{1}{q}$, and choosing that $\varepsilon < \frac{q}{4} (a_1 - a_2 + b_2)$, we get $F(\Delta_1, \varepsilon) > 0$. Obviously, $F(\Delta, \varepsilon)$ is a monotonically decreasing function of $\Delta$, so if $\Delta < \Delta_1$, we have
\[F(\Delta, \varepsilon) > 0. \]  
(4.14)

On the other hand, since $a_4 > a_2 + b_2$, choosing $e^{\rho\Delta} = \frac{1}{q}$, when $\Delta < \Delta_2$, we get
\[G(\Delta, \varepsilon) > 0. \]  
(4.15)
Choose that $\Delta < \Delta_1 \wedge \Delta_2$. Thus, (4.14),(4.15) both hold. By this together with (4.13) and the discrete semimartingale convergence theorem, we can get that there exists a constant $d_4$ such that $\limsup_{k \to \infty} e^{\kappa \Delta_k} |\tilde{X}_k|^2 \leq d_4$ a.s.

Applying the same method as Theorem 3.2, we can get the desired conclusion.

References:

中立型随机比例微分方程的数值解的指数稳定性

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摘要：本文主要利用半鞅收敛定理, 研究中立型随机比例微分方程的数值稳定性. 该文建立了线性的和非线性的中立型随机比例微分方程新的细则. 我们将证明, 在线性增长条件下, 欧拉方法可以保留中立型随机比例微分方程的几乎处处指数稳定性, 并且反向的欧拉方法能保留非线性的中立型随机比例微分方程的几乎处处指数稳定性.

关键词：中立型随机比例微分方程, 数值稳定性; 几乎处处指数稳定性; 反向的欧拉方法