Dimension of the Global Attractor for Damped KdV-Burgers Equations on $\mathbb{R}$

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Abstract: In this paper, we study the dimension of the global attractor for damped KdV-Burgers equation on $\mathbb{R}$. We investigate the quasi-stability of the solutions and prove that the global attractor has finite fractal dimension in $H^1$ by the method that Chueshov and Lasiecka developed (2008).

Key words: Global attractor; Fractal dimension; KdV-Burgers equation

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1. Introduction

In this paper, we estimate the dimension of the global attractor of the following damped KdV-Burgers equation

$$\begin{cases}
    u_t + u_{xxx} - u_{xx} + \lambda u + uu_x = f(x), \ x \in \mathbb{R}, \ t > 0, \\
    u(0,x) = u_0(x), \ x \in \mathbb{R},
\end{cases} \tag{1.1}$$

which represents the union of the Korteweg-de Vries equation and the Burgers equation to describes the propagation of small amplitudes long waves in nonlinear dispersive media when dissipative effects occur\(^{[1-2]}\). Molinet and Riband\(^{[3]}\) proved that the equation (1.1) is globally well-posed in the low regularity space, Dlotko and SUN\(^{[4-5]}\) studied the global solvability of a general KdV-Burgers equation by the parabolic regularization technique, and Cavalcanti et al.\(^{[6]}\) obtained the exponential decay of energy for the KdV-Burgers equation with indefinite damping.

There are some scholars who have paid attention to the longtime behavior of the solutions for the KdV-Burgers equation and KdV equation. GUO and WU\(^{[7]}\) established the existence of a global attractor in $L^2$. Dlotko\(^{[4]}\) obtained the existence of an $(H^2, H^3)$ global attractor. Moreover, Dlotko and SUN\(^{[5]}\) showed that the $(H^1, H^3)$ global attractor exists. GUO et al.\(^{[8]}\) proved the $(H^2, H^5)$ global attractor of the fractional dissipative KdV equation if the external force in space $H^2$. GUO and WANG\(^{[9]}\) obtained the global attractor exists in $L^p$ type Sobolev spaces.

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Finite dimensionality is an important property of global attractors. Teman\cite{teman10} developed the Lyapunov method to prove dynamical systems have finite Hausdorff dimension or finite fractal dimension. Ghidaglia\cite{ghidaglia11,ghidaglia12} proved finite fractal dimension of the global attractors of the Korteweg-de Vries equations with periodic boundary. Abergel\cite{abergel13} proved that the maximal attractor of evolution equations on unbounded domains is actually finite-dimensional in the sense of the fractal dimension and provided an upper bound for its dimension. LIU\cite{liu14} estimated the dimension of unstable manifold around a steady state to give a sharp lower bound for the Hausdorff dimension of the global attractors of the 2D Navier-Stokes equations. YOU and ZHOU\cite{you15,you16} studied finite dimensionality of the global attractor of some chemical models such as Brusselator system, Oregonator system.

Chueshov and Lasiecka\cite{chueshov17} established the criteria for the finite dimensionality of attractors don’t require $C^1$ smoothness of the evolutionary operator. Some dynamical systems don’t display smoothing effects, for example, evolutions of the second order in time\cite{wang18,wang19,wang20}. WANG and TANG\cite{wang21,wang22} applied this method to investigate the fractal dimension of the Quasi-geostrophic equation and Benjamin-Bona-Mahony equation. Goubet and Zahrouni\cite{goubet23} showed that the global attractor of Schrodinger equation has finite fractal dimension in suitable weighted space. In this work we show the global attractor of KdV-Burgers equations (1.1) has finite fractal dimension in $H^1$.

The paper is organized as follows. In Section 2, we recall some function spaces and preliminary results. In section 3, we prove the global attractor has finite fractal dimension.

2. Preliminaries

In this section, we review some notations and preliminary results used in this paper. $A \lesssim B$ means $A \leq cB$ for some absolute constant $c$. Denote by $S$ the Schwartz class, then for any $f(x) \in S$, the Fourier transform of $f(x)$ is defined by

$$\hat{f}(\xi) = \mathcal{F}(f(x)) = \int_{\mathbb{R}} f(x)e^{-2\pi i x \xi}dx,$$

and the inverse Fourier transform $\mathcal{F}^{-1}(\hat{f}(\xi))$ of $\hat{f}(\xi)$ defined by

$$f(x) = \mathcal{F}^{-1}(\hat{f}(\xi)) = \int_{\mathbb{R}} \hat{f}(\xi)e^{2\pi i x \xi}d\xi.$$

As usual, $L^p$ denotes the space of the $p$ th-power integrable functions equipped with the norm

$$\|f\|_{L^p} = \left(\int_{\mathbb{R}} |f(x)|^pdx \right)^{\frac{1}{p}}, \quad \|f\|_{L^\infty} = \text{ess sup}_{x \in \mathbb{R}} |f(x)|.$$

For $s \in \mathbb{R}$, $1 \leq p \leq \infty$, we use $H^{s,p}(\mathbb{R})$ to denote the Bessel potential space defined by

$$\|f\|_{H^{s,p}} = \|\mathcal{F}^{-1}(\langle \xi \rangle^s \hat{f}(\xi))\|_{L^p} < \infty,$$

where $\langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}$.

We recall some results about the regularity and tails estimate of solution of the Cauchy problem (1.1) and the existence of the global attractors proved in [9].

**Lemma 2.1**\cite{9} Assume that $f(x) \in L^p \cap L^q(1 < p \leq 2, 2 < q < +\infty), u_0 \in B_0$, and $B_0$ is an absorbing set in $L^2$. Then there exist $T > 0$, such that for all $t \geq T_0$, we have

$$\|u(t)\|_{H^{s+\frac{1}{2}} \cap H^{s,q}} \leq c(\|f\|_{L^p} \cap L^q, \|u_0\|_{L^2}).$$
Lemma 2.2 Assume that \( f(x) \in L^p \cap L^q(1 < p \leq 2, 2 < q < +\infty), u_0 \in B_0, \) and \( B_0 \) is an absorbing set in \( L^2 \). Then for any \( \varepsilon > 0 \), there exist \( T_1 > 0 \) and \( K = K(\varepsilon) \), such that for all \( t \geq T_1, k > K \), we have

\[
\int_{|x| > k} u^2(t)dx \leq \varepsilon.
\]

Lemma 2.3 Assume \( f(x) \in L^p \cap L^q(1 < p \leq 2, 2 < q < +\infty) \). Then the solution semigroup \( \{S(t)\}_{t \geq 0} \) of (1.1) has an \((L^2, H^{\frac{5}{2}} - \frac{3}{2}) \cap H^3(q)\) global attractor \( \mathcal{A} \).

3. Dimension of the Global Attractor

In this section we estimate the dimension of the global attractor of the Cauchy problem (1.1). Let \( M \) be a compact set in a metric space \( X \). The fractal dimension \( \dim_f M \) of \( M \) is defined by

\[
\dim_f M = \lim_{\varepsilon \to 0} \sup n(M, \varepsilon),
\]

where \( n(M, \varepsilon) \) is the minimal number of closed balls of the radius \( \varepsilon \) which cover the set \( M \).

Recall that a seminorm \( n(\cdot) \) on \( X \) is said to be compact if for any bounded set \( B \subset X \) there exists a sequence \( \{x_n\} \subset B \) such that \( n(x_m - x_n) \to 0 \) as \( m, n \to \infty \). In order to investigate the fractal dimension of the global attractor, we need the following useful result due to [17].

Lemma 3.1 Let \( X \) be a Banach space and \( M \) be a bounded closed set in \( H \). Assume that there exists a mapping \( V : M \mapsto X \) such that \( M \subseteq VM \) and

(i) \( V \) is Lipschitz on \( M \), i.e., there exists \( L > 0 \) such that

\[
\|Vv_1 - Vv_2\|_X \leq L\|v_1 - v_2\|_X, v_1, v_2 \in M;
\]

(ii) There exist compact seminorm \( n_1(\cdot) \) and \( n_2(\cdot) \) on \( X \) such that

\[
\|Vv_1 - Vv_2\|_X \leq \eta\|v_1 - v_2\|_X + K[n_1(v_1 - v_2) + n_2(v_1 - v_2)],
\]

for any \( v_1, v_2 \in M \), where \( 0 < \eta < 1 \) are constants.

Then \( M \) is a compact set in \( X \) of a finite fractal dimension. Moreover, we have the estimate

\[
\dim_f M = [\ln \left( \frac{2}{1 + \eta} \right)]^{-1} \cdot \ln m_0(\frac{4K(1 + L^2)^{\frac{1}{2}}}{1 - \eta}),
\]

where \( m_0(\mathbb{R}) \) is the maximal number of pairs \((x, y)\) in \( X \times X \) possessing the properties

\[
\|x_i\|^2 + \|y_i\|^2 \leq R^2, n_1(x_i - y_j) + n_2(x_i - y_j) > 1, i \neq j.
\]
Now we give the main result of this paper that the fractal dimension of the global attractor of the Cauchy problem (1.1) is finite.

**Theorem 3.1** The global attractor $\mathcal{A}$ of the KdV-Burger equation has a finite fractal dimension in $H^1$.

**Proof** We need show the Lipschitz stability and quasi-stability of the solution semigroup \( \{S(t)\}_{t \geq 0} \) in space $H^1$. Let $u_1, u_2$ be the solutions of the Cauchy problem (1.1) with initial data $u_{10}, u_{20} \in \mathcal{A}$, respectively. Denote $w = u_1 - u_2$. Then

$$
\begin{cases}
  w_t + w_{xxx} - w_{xx} + \lambda w + \left( \frac{u_1 + u_2}{2} \right)_x = 0; \\
  w(0, x) = u_{10} - u_{20}.
\end{cases}
$$

(3.3)

Since the regularity of $\frac{u_1 + u_2}{2}$ is similar to solution trajectory $u$ of the problem (1.1) in global attractor $\mathcal{A}$, for simplicity, we substitute $u$ for $\frac{u_1 + u_2}{2}$, and consider the following equation

$$
\begin{cases}
  w_t + w_{xxx} - w_{xx} + \lambda w + (uw)_x = 0; \\
  w(0, x) = u_{10} - u_{20}.
\end{cases}
$$

(3.4)

Taking the $L^2$ inner product with $w$, and using Lemma 2.1, we get

$$
\frac{d}{dt} \|w\|_{L^2}^2 + 2 \|w\|_{H^1}^2 + 2\lambda \|w\|_{L^2}^2 = (u_x w, w^2) \leq \|u_x\|_{L^\infty} \|w\|_{L^2}^2 \leq c \|w\|_{L^2}^2.
$$

By the Granwall lemma, we have

$$
\|w\|_{L^2}^2 \leq e^{-(2\lambda - c)t} \|w_0\|_{L^2}^2.
$$

(3.5)

(3.6)

Taking the $L^2$ inner product of (3.4) with $-w_{xx}$, we get

$$
\frac{d}{dt} \|w\|_{H^1}^2 + 2 \|w\|_{H^2}^2 + 2\lambda \|w\|_{L^2}^2 = (u_x w, w_{xx}) + (uw_x, w_{xx}).
$$

(3.7)

We estimate every term on the right hand of (3.7). For the fist term, by the Cauchy inequality and Sobolev embedding theorem, we have

$$(u_x w, w_{xx}) \leq \frac{1}{2} \|u_x w\|_{L^2}^2 + \frac{1}{2} \|w\|_{L^2}^2 \leq \frac{1}{2} \|u_x\|_{L^\infty} \|w\|_{L^2}^2 + \frac{1}{2} \|w\|_{H^2}^2 \leq c \|w\|_{H^1}^2 + \frac{1}{2} \|w\|_{H^2}^2.
$$

(3.8)

Repeating the similar argument, one easily deduces that

$$(uw_x, w_{xx}) \leq c \|w\|_{H^1}^2 + \frac{1}{2} \|w\|_{H^2}^2.
$$

(3.9)

Substituting (3.8) and (3.9) into (3.7) and the Granwall lemma, we obtain

$$
\|w\|_{H^1}^2 \leq e^{-(2\lambda - c)t} \|w_0\|_{H^1}^2.
$$

(3.10)

Let $\phi$ be a smooth function defined on $\mathbb{R}$, such that $0 \leq \phi(x) \leq 1$ for all $x \in \mathbb{R}$, and

$$
\phi(x) = \begin{cases}
  0, & |x| \leq 1, \\
  1, & |x| \geq 2.
\end{cases}
$$

Denote $\phi_k = \phi(\frac{x}{k})$, then supp$\phi_k \subset \{x \in \mathbb{R} : |x| \geq k\}$, and

$$
\left| \frac{d^n \phi_k}{dx^n} \right| \leq c k^{-n}, \quad k \leq |x| \leq 2k,
$$

(3.11)

Set $v = \phi_k w$, then

$$
v_t + v_{xxxx} - v_{xx} + \lambda v + (uv)_x = \phi_k'' w + 3\phi_k'' w_w + 3\phi_k'' w_{wx} - \phi_k'' w - 2\phi_k' w_x + \phi_k' w
$$

Taking the $L^2$ inner product of (3.11) with $-v_{xx}$, we get

$$
\frac{d}{dt} \|v\|_{H^1}^2 + 2 \|v\|_{H^2}^2 + 2\lambda \|v\|_{L^2}^2 = 2(\phi_k'' w + 3\phi_k'' w_w + 3\phi_k'' w_{wx}, -v_{xx}) + ((uv)_x, v_{xx})
$$
where \( Q(t) \) is a function of \( t \).

Thus, we can repeatedly finite time to get
\[
|w(t)|_{H^2} \leq Q_1(t) |w_0|_{H^1},
\]
where \( Q_1(t) \) is a function of \( t \).

Substituting inequality (3.14)-(3.20) into (3.12), and choosing \( k \) large enough such that \( \frac{c}{k} + \varepsilon(k) \leq \min \{ \lambda, 1 \} \), we arrive at
\[
\frac{\lambda}{k} \|v\|_{H^1}^2 + \lambda |v|^2 \leq \frac{C}{k} Q_1(t) |w_0|_{H^1}.
\]
By the Gronwall lemma, we obtain
\[
\|v\|_{H^1}^2 \leq e^{-\lambda t} \|v_0\|_{H^1}^2 + \frac{c}{k} \int_0^t e^{-\lambda(t-\tau)} Q_1(\tau) d\tau \|w_0\|_{H^1} \leq e^{-\lambda t} \|v_0\|_{H^1}^2 + \frac{c}{k} Q_2(t) \|w_0\|_{L^2}. 
\] (3.21)

Let \( \varphi_k(x) = 1 - \phi_k(x) \), and set \( \tilde{v} = (1 - \varphi_k(x)) w \), then
\[
\begin{align*}
\tilde{v}_t + \tilde{v}_{xxx} - \tilde{v}_{xx} + \lambda \tilde{v} + (u\tilde{v})_x &= \varphi_k'' w + 3\varphi_k' w_x + 3\varphi_k'' w_{xx} - \varphi_k'' w - 2\varphi_k' w_x + \varphi_k' uw. 
\end{align*}
\] (3.22)

Taking the \( L^2 \) inner product of (3.22) with \(-\tilde{v}_{xx}\), we get
\[
\begin{align*}
\frac{d}{dt} \|\tilde{v}\|_{H^1}^2 + 2\|\tilde{v}\|_{H^2}^2 + 2\lambda \|\tilde{v}\|_{H^1}^2 &= 2(\varphi_k'' w + 3\varphi_k' w_x + 3\varphi_k'' w_{xx}, -\tilde{v}_{xx}) + ((u\tilde{v})_x, \tilde{v}_{xx}) \\
&\quad + 2(-\varphi_k'' w - 2\varphi_k' w_x + \varphi_k' uw, -\tilde{v}_{xx}). 
\end{align*}
\] (3.23)

Proceed as (3.13), (3.14), one can show that
\[
\begin{align*}
2(\varphi_k'' w + 3\varphi_k' w_x + 3\varphi_k'' w_{xx}, -\tilde{v}_{xx}) + 2(-\varphi_k'' w - 2\varphi_k' w_x + \varphi_k' uw, -\tilde{v}_{xx}) \\
&\leq \frac{c}{k} \|w\|_{H^2}^2 + \varepsilon(k) \|\tilde{v}\|_{H^1}^2. 
\end{align*}
\] (3.24)

Since \( u \) is bounded in \( H^2 \) and \( \text{supp} \tilde{v} \subset \{x \|x\| \leq k\} \), we have
\[
((u\tilde{v})_x, \tilde{v}_{xx}) = \int_{|x| \leq k} u_x \tilde{v}_{xx} + u\tilde{v}_{x} \tilde{v}_{xx} dx \\
\leq \|u_x\|_{L^\infty} \int_{|x| \leq k} \tilde{v}_x^2 dx + \varepsilon(k) \|\tilde{v}\|_{H^2}^2 + \int_{|x| \leq k} u_x \tilde{v}_x^2 dx \\
\leq \|u_x\|_{L^\infty} \|w\|_{L^2(|x| \leq k)}^2 + \varepsilon(k) \|\tilde{v}\|_{H^2}^2 + \|u_x\|_{L^\infty} \|\tilde{v}\|_{H^1} \|\tilde{v}\|_{L^2} \\
\leq c \|w\|_{L^2(|x| \leq k)}^2 + \varepsilon(k) \|\tilde{v}\|_{H^1}^2. 
\] (3.25)

Substituting (3.19), (3.24), (3.25) into (3.23), we have
\[
\frac{d}{dt} \|\tilde{v}\|_{H^1}^2 + \|\tilde{v}\|_{H^2}^2 + \lambda \|\tilde{v}\|_{H^1}^2 \leq \frac{Q(t)}{k} \|w_0\|_{H^1}^2 + c \|w\|_{L^2(|x| \leq k)}^2. 
\] (3.26)

Then by the Gronwall lemma and Lipschitz property of \( w \), we have
\[
\begin{align*}
\|\tilde{v}\|_{H^1}^2 &\leq e^{-\lambda t} \|\tilde{v}_0\|_{H^1}^2 + \frac{1}{k} \int_0^t e^{-\lambda(t-\tau)} Q(\tau) d\tau \|w_0\|_{H^1}^2 + c \int_0^t e^{-\lambda(t-\tau)} \|w\|_{L^2(|x| \leq k)}^2 d\tau \\
&\leq e^{-\lambda t} \|\tilde{v}_0\|_{H^1}^2 + \frac{1}{k} \int_0^t e^{-\lambda(t-\tau)} Q(\tau) d\tau \|w_0\|_{H^1}^2 + c \int_0^t e^{-\lambda(t-\tau)} e^{-\varphi_k''(\varphi_k'' - \varphi_k)} d\tau \|w_0\|_{L^2(|x| \leq k)}^2 \\
&\leq e^{-\lambda t} \|\tilde{v}_0\|_{H^1}^2 + \frac{1}{k} Q_3(t) \|w_0\|_{H^1}^2 + Q_4(t) \|w_0\|_{L^2(|x| \leq k)}^2, 
\end{align*}
\] (3.27)

where \( Q_3, Q_4 \) are increasing functions of \( t \).

The combination of (3.21) and (3.27) gives
\[
\|w\|_{H^1}^2 \leq e^{-\lambda t} \|w_0\|_{H^1}^2 + \frac{1}{k} (Q_2(t) + Q_3(t)) \|w_0\|_{H^1}^2 + Q_4(t) \|w_0\|_{L^2(|x| \leq k)}^2. 
\] (3.28)

Let \( t \) large enough such that \( e^{-\lambda t} < \frac{1}{2} \), then choose \( k \) large enough such that \( \frac{1}{k} (Q_2(t) + Q_3(t)) < \frac{1}{2} \), and \( H^1 \to L^2(|x| \leq k) \) is compact. By Lemma 3.1, we obtain the conclusion. The proof is complete.

References:
阻尼KdV-Burgers方程全局吸引子的维数

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摘要：本文研究定义在$\mathbb{R}$上KdV-Burgers方程全局吸引子的分形维数，利用Chueshov和Lasiecka提出的拟稳态估计方法证明方程全局吸引子分形维数有限。

关键词：全局吸引子; 分形维数; KdV-Burgers方程

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