The Center Problems and Time-Reversibility with Respect to a Linear Involution

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Abstract: In this paper, the relationship between time-reversibility and the center of a planar quadratic polynomial system in $\mathbb{R}^2$ is considered. The necessary and sufficient conditions for the system to be time-reversible w.r.t. a linear involution are obtained. These conditions guarantee that the system has a center at the origin which is symmetric w.r.t. a straight line.

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1. Introduction

In the qualitative theory of differential systems, of particular significance are the existence and the number of limit cycles from famous Hilbert 16th problems. In order to know how many limit cycles can bifurcate, it is necessary to have an in-depth understanding of the conditions under which the singular point is a center. If an analytic differential system has a non-degenerate center at the origin, then after making a linear transformation of the variables and rescaling the time variable, it can be transformed into the form as follows:

$$\begin{align*}
\dot{x} &= -y + P(x, y), \\
\dot{y} &= x + Q(x, y),
\end{align*}$$

(1.1)

where $P(x, y)$ and $Q(x, y)$ are analytic functions without constant term. For the purpose of obtaining the necessary and sufficient conditions for the system (1.1) to have a center, the first step is to obtain the necessary conditions by computing the first several focal values and then to prove the sufficiency of these conditions. The construction of integrating factors\textsuperscript{[14]} and the verification of Poincaré symmetric principle\textsuperscript{[10]} which has been presented in 1892 are two major methods of checking the sufficiency. Actually, a system satisfying the Poincaré
principle is a time-reversible one w.r.t. a special linear map such that the system has an axisymmetric center.

The time-reversible theory of differential systems emerged in 1915 with Birkhoff’s work on three bodies\cite{1}. In 1976, Devaney\cite{6} developed the theory of reversibility such that the study of it is expanded to analytic differential systems. For planar analytic differential systems, a reversible center has always received attention\cite{11}. Giné and Maza\cite{7} obtained some results about degenerate center and time-reversibility. In 2018, by using the time-reversibility w.r.t. a special linear map, Boros et al.\cite{2} obtained some new sufficient conditions for Lotka reactions with generalized mass-action kinetics to have a center. Recently, HAN et al.\cite{8} considered a general polynomial quadratic system without constant term in $\mathbb{R}^2$ and obtained some algebraic conditions for the system to be time-reversible one w.r.t. a linear involution. In their computational procedure, the Closure Theorem\cite{5} which just holds over an algebraically closed field has been misused.

The purpose of the present paper is to give the necessary and sufficient conditions for the following quadratic system to be time-reversible w.r.t. a linear involution.

$$\begin{align*}
\dot{x} &= -y + a_1x^2 + a_2xy + a_3y^2, \\
\dot{y} &= x + b_1x^2 + b_2xy + b_3y^2,
\end{align*}$$

(1.2)

where $(x, y) \in \mathbb{R}^2$, $a_i, b_i \in \mathbb{R}$ ($i = 1, 2, 3$). This kind of problems can be converted into ones of obtaining the set of zero roots of some polynomials with parameters. The concept of triangular decomposition emerges in a lot of computer algebra methods for computing the roots of polynomial systems. For example, the characteristic set of Ritt-Wu’s method\cite{15,16}, Groebner bases\cite{3}, resultant\cite{5} and so on. The regular chain which was introduced by Kalkbrener\cite{9} and YANG and ZHANG\cite{13} independently is also a kind of triangular decompositions and well-developed in complex field. In 2013, CHEN et al.\cite{4} considered the real solutions for semi-algebraic systems and proposed the following Lemma.

**Lemma 1.1**\cite{4} Let $\mathcal{S}$ be a semi-algebraic system of $\mathbb{Q}[x]$. Then one can compute a triangular decomposition of $\mathcal{S}$, that is finitely many regular semi-algebraic systems $\mathcal{B}_i$ such that $Z_\mathcal{S}(\mathcal{S}) = \bigcup_i Z_\mathcal{B}_i(\mathcal{B}_i)$, where $Z_\mathcal{S}$ denotes the zero set in $\mathbb{R}^n$ of a semi-algebraic system.

In this paper, by using the regular chain theory\cite{4}, the necessary and sufficient conditions for the system (1.2) to be time-reversible w.r.t. a linear involution are obtained in Section 2. In Section 3, we show that these conditions can ensure the system to have a center by using the result of Teixeire and YANG\cite{12} which indicates that the system (1.1) is time-reversible if and only if it is a center.

2. Time-Reversible with Respect to a Linear Map

Firstly, we introduce the definition of a time-reversible system.

**Definition 2.1**\cite{12} A $C^r$ planar differential system ($r \in \mathbb{N} \cup \{\infty, \omega\}$),

$$\begin{align*}
\dot{x} &= P(x, y), \quad \dot{y} = Q(x, y), \quad (x, y) \in \mathbb{R}^2,
\end{align*}$$

(2.1)

having a singularity at the origin is time-reversible if there exists a diffeomorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying $\phi \circ \phi = \text{Id}$ such that

$$\phi \ast \mathcal{X} = -\mathcal{X} \circ \phi,$$

(2.2)

where $\mathcal{X}$ is the vector field associated to the system (2.1), and $\phi_*$ represents the tangent map of $\phi$, i.e., $\phi_*\mathcal{X} = D(x,y)\phi(x,y)\mathcal{X}$ in the local coordinates. The map $\phi$ is called an involution. Fix($\phi$) = $\{(x, y) \in \mathbb{R}^2 \mid \phi(x, y) = (x, y)\}$ is called fixed point set of $\phi$. 
Let
\[ \phi(x, y) = (\lambda_1 x + \lambda_2 y, \delta_1 x + \delta_2 y) = B \cdot (x, y)^T, \]
where \( \lambda_i, \delta_i \in \mathbb{R} \) \((i = 1, 2)\). According to Definition 2.1, when \( \phi \circ \phi = \text{Id} \), \( \phi \) is an involution. Hence, we have
\[ B^2 = \begin{pmatrix} \lambda_1^2 + \lambda_2 \delta_1 & \lambda_1 \lambda_2 + \lambda_2 \delta_2 \\ \lambda_1 \delta_1 + \delta_1 \lambda_2 & \lambda_2 \delta_1 + \delta_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \]
(2.3)
Clearly, (2.3) is equivalent to
\[ \lambda_2 = \delta_1 = 0, \quad \lambda_1 = \delta_2 = \pm 1, \]
(2.4)
or
\[ \lambda_1 = -\delta_2, \quad \lambda_1^2 + \lambda_2 \delta_1 = 1. \]
(2.5)
Note that condition (2.4) corresponds to two trivial situations: \( \phi(x, y) = \pm (x, y) \). From the definition, the following is a direct result.

**Lemma 2.1** A linear map \( \phi(x, y) = (\lambda_1 x + \lambda_2 y, \delta_1 x + \delta_2 y) \) is an involution if and only if one of the following conditions is satisfied:
1) \( \lambda_2 = \delta_1 = 0, \quad \lambda_1 = \delta_2 = \pm 1, \)
2) \( \lambda_1 = -\delta_2, \quad \lambda_1^2 + \lambda_2 \delta_1 = 1, \)
where \( \lambda_i, \delta_i \in \mathbb{R} \) \((i = 1, 2)\).

It is clear that the fixed point set of a linear map satisfying the condition 1) of Lemma 2.1 is either the origin or the whole space. If a linear map satisfies the condition 2) of Lemma 2.1, the fixed point set is
\[ \{(x, y) | (\lambda_1 - 1)x + \lambda_2 y = 0, \quad \delta_1 x - (\lambda_1 + 1)y = 0, \quad \lambda_1^2 + \lambda_2 \delta_1 = 1\}. \]
This set is a straight line passing through the origin.

Let
\[ P(x, y) = -y + a_1 x^2 + a_2 xy + a_3 y^2, \quad Q(x, y) = x + b_1 x^2 + b_2 xy + b_3 y^2. \]
From Definition 2.1 and Lemma 2.1, we have

**Lemma 2.2** The system (1.2) is time-reversible w.r.t. a linear map \( \phi(x, y) = (\lambda_1 x + \lambda_2 y, \delta_1 x + \delta_2 y) \) if and only if the map \( \phi \) is an involution such that
\[ \phi(P(x, y), Q(x, y)) = -(P(\phi(x, y)), Q(\phi(x, y))). \]
(2.6)
In the case where the linear map \( \phi \) satisfies the condition 1) of Lemma 2.1, the map \( \phi \) is an involution satisfying \( \phi(x, y) = (x, y) \) or \( \phi(x, y) = -(x, y) \). Furthermore, the equality (2.6) in Lemma 2.2 is equivalent to \( P(x, y) = Q(x, y) = 0 \). Hence, the system (1.2) is not time-reversible w.r.t. these involutions.

Now we consider the case where the linear map \( \phi \) satisfies the condition 2) of Lemma 2.1. Since the fixed point set of the linear map, in this case, is a straight line, then the phase diagram of the system (1.2) is symmetric w.r.t. a straight line if the system is time-reversible w.r.t. a linear map. The linear part of (2.6) of Lemma 2.2 is as follows,
\[ \lambda_2 x = \delta_1 x, \quad \delta_1 y = \lambda_2 y, \]
therefore, we have the following Lemma.

**Lemma 2.3** The system (1.2) is time-reversible w.r.t. a linear map \( \phi(x, y) = (\lambda_1 x + \lambda_2 y, \delta_1 x + \delta_2 y) \) if and only if the following conditions are satisfied:
\[ \delta_1 = \lambda_2, \quad \delta_2 = -\lambda_1, \quad \lambda_1^2 + \lambda_2^2 = 1, \]
(2.7)
\[ \phi(P(x,y),Q(x,y)) = -(P(\phi(x,y)),Q(\phi(x,y))), \] (2.8)

where \((x,y) \in (0, \mathbb{R}^2), \lambda_i \text{ and } \delta_i \in \mathbb{R}, i = 1, 2.\)

According to Lemma 2.3, we just need to consider the involution in the form of \(\phi(x,y) = (\lambda_1 x + \lambda_2 y, \lambda_2 x - \lambda_1 y).\) Let
\[ H(x,y) = \phi(P(x,y),Q(x,y)) + (P(\phi(x,y)),Q(\phi(x,y))) \]
\[ = \left( \sum_{i+j=1}^{2} h_{ij}^{(1)} x^i y^j, \sum_{i+j=1}^{2} h_{ij}^{(2)} x^i y^j \right), \]
where \(h_{ij}^{(k)}\) are polynomials in two variables \(\lambda_1\) and \(\lambda_2\) with parameters of the system (1.2) for \(k = 1, 2.\) Clearly, the equality (2.8) is equivalent to \(H(x,y) = 0.\) If
\[ h_{ij}^{(k)} = 0, \quad i + j = 1, 2 \quad \text{for } k = 1, 2, \]
then \(H(x,y) = 0\) holds for all value of \((x,y)\) in \((0, \mathbb{R}^2).\) Let
\[ Cof = \left\{ h_{ij}^{(k)}, \lambda_1^2 + \lambda_2^2 - 1 \mid i + j = 1, 2, \quad k = 1, 2 \right\}. \]

By Lemma 2.3, the necessary and sufficient conditions for the system (1.2) to be time-reversible w.r.t. a linear map is equivalent to the necessary and sufficient conditions of the polynomial set \(Cof\) to have a real root in variables \(\lambda_1\) and \(\lambda_2\).

The following is the main result of the present paper which gives a complete description of a time-reversibility w.r.t. a linear involution.

**Theorem 2.1** The system (1.2) is time-reversible w.r.t. a linear map if and only if one of the following conditions is satisfied:

\begin{align*}
(A_1) & \quad b_2 = a_3 = a_1 = 0; \\
(A_2) & \quad b_2 = b_3 = b_1 = 0; \\
(A_3) & \quad b_2 = \pm a_2, b_3 = \pm a_1, b_1 = \pm a_3; \\
(A_4) & \quad a_1 = a_2 = a_3 = b_1 = b_2 = b_3 = 0; \\
(A_5) & \quad a_1 = a_3 = 0, b_1 = -b_3, a_2 = -2b_3, b_2 = 0, a_2 \neq 0; \\
(A_6) & \quad a_1 = -a_3, b_1 = -b_3, a_2 = -2b_3, b_2 = 2a_3, b_3 \neq 0, a_3 \neq 0, a_3 \neq \pm b_3; \\
(A_7) & \quad a_1 = a_3 = 0, b_1 = -b_3, \omega_5 = 0, a_2 \neq 0, b_2 = 0, a_2 \neq \pm 2b_3, -1 < b_3/a_2 < 1; \\
(A_8) & \quad a_1 = -a_3, b_1 = -b_3, a_2 = 2b_3, b_2 = -2a_3, a_3 \neq 0, b_3 \neq 0, a_3 \neq b_3^2, 3a_3^2 \neq b_3^2; \\
(A_9) & \quad \omega_1 = 0, \omega_2 = 0, a_1 \neq -a_3, b_1 \neq -b_3, (a_1 + a_3)^2 \neq (b_1 + b_3)^2, \quad 3(a_1 + a_3)^2 \neq (b_1 + b_3)^2; \\
(A_{10}) & \quad \pm \sqrt{3}(2a_1 + b_2) = b_1 - 3b_3, \quad \pm \sqrt{3}(a_2 + 2b_1) = -3(a_1 + a_3), \quad \pm \sqrt{3}(b_1 + b_3) = -3(a_1 + a_3); \\
(A_{11}) & \quad a_1 = -a_3, b_1 = -b_3, \omega_3 = 0, a_3 \neq 0, a_2 \neq \pm 2b_3, a_2b_2 + 4a_3b_3 \neq 0, 2a_3 - b_2 \neq 0, 0 < |\omega_4| < 1. \\
\end{align*}

Here
\begin{align*}
\omega_4 & = (a_2 a_3 + b_3 b_2)/(a_2 b_2 + 4a_3 b_3), \\
\omega_5 & = a_2^3 + 3a_2^2 b_3 - a_2 b_3^2 - b_3^2 - 4b_3^3, \\
\omega_3 & = (a_3 - b_3)a_2^3 - 3a_2^2 b_3 a_3 + (4a_3^3 + (-3b_2^2 - 12b_3^2)a_3 + b_2^3) a_2 + 16b_3(a_2^3 - 3/4a_2^3 b_2 - a_2 b_3^2 + 1/16b_2^3 + 1/4b_2 b_3^2), \\
\omega_2 & = (a_2 - 2b_1 - b_3) a_3 + (3a_2 - b_1) a_3^2 + ((3a_2 + 3b_3) a_3^2 - (b_1 + b_3)^2(a_2 - b_3)) a_1 + a_3(-a_2 + b_1 + 2b_3) a_2 - (b_1 + b_3) a_3^2 (a_2 - (b_1 + b_3))^2), \\
\omega_1 & = (a_2 - 2b_1 + 2b_3) a_3^3 + ((3a_2 - 6b_1 + 6b_3) a_3 - 3b_2(b_1 + b_3)) a_2^2 + ((3a_2 - 6b_1 + 6b_3) a_3^2 - 6b_2(b_1 + b_3) a_3 + (b_1 + b_3)^2(a_2 - 2b_1 - 2b_3)) a_1 + (a_2 - 2b_1 + 2b_3) a_3^2 - 3b_2(b_1 + b_3) a_3^2 + (b_1 + b_3)^2(a_2 + 2b_1 - 2b_3) a_3 + b_2(b_1 + b_3)^3. 
\end{align*}
The Proof of Theorem 2.1  The crucial step of proving Theorem 2.1 is to obtain the necessary and sufficient conditions for the existence of the real roots of polynomial set $Cof$. The set $Cof$ corresponding to the system (1.2) consists of seven polynomials:

$$h_{20}^{(1)} = a_1 \lambda_1^2 + \lambda_2 \lambda_1 a_2 + a_3 \lambda_2^2 + a_1 \lambda_1 + b_1 \lambda_2,$$

$$h_{11}^{(1)} = 2 a_1 \lambda_1 \lambda_2 - a_2 \lambda_1^2 + a_2 \lambda_2^2 - 2 a_3 \lambda_1 \lambda_2 + a_2 \lambda_1 + \lambda_2 b_2,$$

$$h_{02}^{(1)} = a_1 \lambda_2^2 - \lambda_2 \lambda_1 a_2 + a_3 \lambda_1^2 + a_3 \lambda_1 + \lambda_2 b_3,$$

$$h_{20}^{(2)} = b_1 \lambda_1^2 + b_2 \lambda_1 \lambda_2 + b_3 \lambda_2^2 + a_1 \lambda_2 - b_1 \lambda_1,$$

$$h_{11}^{(2)} = 2 b_1 \lambda_2 \lambda_1 - b_2 \lambda_1^2 + b_2 \lambda_2^2 - 2 b_3 \lambda_2 \lambda_1 + \lambda_2 a_2 - b_2 \lambda_1,$$

$$h_{02}^{(2)} = b_1 \lambda_2^2 - b_2 \lambda_1 \lambda_2 + b_3 \lambda_1^2 + a_3 \lambda_2 - b_3 \lambda_1,$$

$$\lambda_1^2 + \lambda_2^2 = 1.$$

By using the command in the computer algebra system Maple, i.e.,

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RegularChains:-RealTriangularize,
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we obtain a triangular decomposition of these polynomials. In the order of $b_2 > a_2 > b_3 > b_1 > a_3 > a_1 > \lambda_2 > \lambda_1$, the triangular decomposition has six regular semi-algebraic systems such that the union of their real root sets is exactly the real root set of $Cof$.

Conditions $(A_1), (A_2), (A_3), (A_{10})$ in Theorem 2.1 can be obtained directly from the regular semi-algebraic systems $T_i, i = 2, \ldots, 6$ as follows

$$T_2 : \left\{ \begin{array}{l}
2 \lambda_1 - 1 = 0, 
4 \lambda_2^2 - 3 = 0, 
2 \lambda_2 b_2 + 4 a_1 \lambda_2 + 3 b_3 - b_1 = 0,
\end{array} \right\},$$

$$T_3 : \left\{ \begin{array}{l}
\lambda_1 = 0, 
\lambda_2 + 1 = 0, 
b_2 - a_2 = 0, 
b_3 - a_1 = 0, 
b_1 - a_3 = 0,
\end{array} \right\},$$

$$T_4 : \left\{ \begin{array}{l}
\lambda_1 = 0, 
\lambda_2 - 1 = 0, 
b_2 + a_2 = 0, 
b_3 + a_1 = 0, 
b_1 + a_3 = 0,
\end{array} \right\},$$

$$T_5 : \left\{ \begin{array}{l}
a_2 = 0, 
b_3 = 0, 
b_1 = 0, 
\lambda_2 = 0, 
\lambda_1 + 1 = 0,
\end{array} \right\},$$

$$T_6 : \left\{ \begin{array}{l}
b_2 = 0, 
a_3 = 0, 
a_1 = 0, 
\lambda_2 = 0, 
\lambda_1 - 1 = 0,
\end{array} \right\}.$$

The other conditions can be obtained by analysing the necessary and sufficient conditions of the regular semi-algebraic system $T_1$ to have real roots in variables $\lambda_1, \lambda_2$. The system $T_1$ is as follows:

$$\lambda_2^2 + \lambda_1^2 - 1 = 0, \quad (2.10)$$

$$\lambda_2 b_3 + b_1 \lambda_2 + (\lambda_1 + 1) a_3 + (\lambda_1 + 1) a_1 = 0, \quad (2.11)$$

$$\lambda_2 \lambda_1 a_2 + b_1 \lambda_2 + (- \lambda_1^2 + 1) a_2 + (\lambda_1^2 + \lambda_1) a_1 = 0, \quad (2.12)$$

$$2 \lambda_1^2 + \lambda_2 + 1 b_2 - 2 b_1 \lambda_2 \lambda_1 + 2 b_3 \lambda_2 \lambda_1 - 2 \lambda_2 a_2 = 0, \quad (2.13)$$

$$-1 < \lambda_1 < 1, \lambda_1 \neq 0, 2 \lambda_1 - 1 \neq 0. \quad (2.14)$$

1) When $a_1 \neq -a_3$, by the equalities (2.10) and (2.11), we have

$$\lambda_1 = \frac{(b_1 + b_3)^2 - (a_1 + a_3)^2}{(a_1 + a_3)^2 + (b_1 + b_3)^2}, \quad (2.15)$$

$$\lambda_2 = -\frac{2(a_1 + a_3)(b_1 + b_3)}{(a_1 + a_3)^2 + (b_1 + b_3)^2}. \quad (2.16)$$
By substituting $\lambda_1$, $\lambda_2$ into (2.12) and (2.13), we have

\[
\frac{2(b_1 + b_3)\omega_1}{(a_1 + a_3)^2 + (b_1 + b_3)^2} = 0, \quad (2.17)
\]

\[
\frac{2(b_1 + b_3)\omega_2}{(a_1 + a_3)^2 + (b_1 + b_3)^2} = 0. \quad (2.18)
\]

Inequalities in (2.14) lead to $b_1 + b_3 \neq 0$, $(a_1 + a_3)^2 \neq (b_1 + b_3)^2$ and $3(a_1 + a_3)^2 \neq (b_1 + b_3)^2$. Hence, $\omega_1 = \omega_2 = 0$. Condition (A9) in Theorem 2.1 is proved.

2) When $a_1 = -a_3$, equality (2.11) leads to

\[
\lambda_2(b_1 + b_2) = 0. \quad (2.19)
\]

Inequalities $-1 < \lambda_1 < 1$ together with equality $\lambda_1^2 + \lambda_2^2 = 1$ lead to $\lambda_2 \neq 0$. Hence, $b_1 + b_2 = 0$.

By substituting $a_1 = -a_3$ and $b_1 = -b_3$ into $T_1$, we have

\[
\begin{cases}
\lambda_2^2 + \lambda_1^2 - 1 = 0, \\
(a_2 \lambda_1 - b_3) \lambda_2 - 2a_3 \lambda_1^2 - a_3 \lambda_1 + a_3 = 0, \\
(4b_3 \lambda_1 - a_2) \lambda_2 + 2b_2 \lambda_1^2 + 2b_1 \lambda_1 - b_2 = 0, \\
-1 < \lambda_1 < 1, \quad \lambda_1 \neq 0, \quad 2\lambda_1 - 1 \neq 0.
\end{cases} \quad (2.20)
\]

(i) When $a_1 = -a_3$, $b_1 = -b_3$ and $a_2 \lambda_1 - b_3 \neq 0$, we can rewrite the equality (2.21) as

\[
\lambda_2 = \frac{(2 \lambda_1^2 + \lambda_1 - 1) a_3}{a_2 \lambda_1 - b_3}. \quad (2.24)
\]

Substituting $\lambda_2$ into equality (2.22), we obtain

\[
\frac{(\lambda_1 + 1)(2 \lambda_1 - 1)((a_2 b_2 + 4a_3 b_3)\lambda_1 + a_3 a_2 + b_2 b_3)}{a_2 \lambda_1 - b_3} = 0. \quad (2.25)
\]

Since the inequalities in (2.14) lead to $(\lambda_1 + 1)(2 \lambda_1 - 1) \neq 0$, the equality (2.25) is equivalent to

\[
((a_2 b_2 + 4a_3 b_3)\lambda_1 + a_3 a_2 + b_2 b_3) = 0. \quad (2.26)
\]

(a) When $a_2 b_2 + 4a_3 b_3 \neq 0$, by the equality (2.26), we have

\[
\lambda_1 = \frac{a_3 a_2 + b_2 b_3}{a_2 b_2 + 4a_3 b_3}. \quad (2.27)
\]

The inequalities in (2.14) lead to $0 < |\omega_3| < 1$, $a_2 a_3 + b_3 b_2 \neq 0$, $(2a_3 - b_2)(a_2 - 2b_3) \neq 0$. By the equality (2.27), we have

\[
a_2 \lambda_1 - b_3 = \frac{a_3 (a_2 - 2b_3)(a_2 + 2b_3)}{a_2 b_2 + 4a_3 b_3}.
\]

Therefore, $a_2 \lambda_1 - b_3 \neq 0$ is equivalent to $a_3 \neq 0$ and $a_2 \neq \pm 2b_3$. Furthermore, $(2a_3 - b_2)(a_2 - 2b_3) \neq 0$ is equivalent to $2a_3 - b_2 \neq 0$. Substituting $\lambda_1$ and $\lambda_2$ into the equality (2.20) leads to $\omega_3 = 0$. Therefore, the condition (A11) in Theorem 2.1 is proved.

(b) When $a_2 b_2 + 4a_3 b_3 = 0$, the equality (2.26) is simplified to

\[
a_3 a_2 + b_2 b_3 = 0. \quad (2.28)
\]

Using the command

\texttt{RegularChains:-RealTriangularize},

we can solve the polynomial equations

\[
a_2 b_2 + 4a_3 b_3 = 0, \\
a_3 a_2 + b_2 b_3 = 0,
\]

thereby proving Theorem 2.1.
and obtain
\[
\begin{cases}
    a_2 + 2b_3 = 0, \\
    a_2 - 2b_3 = 0, \\
    2a_3 - b_2 = 0, \\
    2a_3 + b_2 = 0, \\
    a_3 = 0, \\
    b_2 = 0, \\
    b_3 = 0.
\end{cases}
\]

When \(a_1 = -a_3, b_1 = -b_3, a_2\lambda_1 - b_3 \neq 0, a_2 = -2b_3\) and \(b_2 = 2a_3, T_1\) is simplified to the following form.
\[
\begin{cases}
    \lambda_2^2 + \lambda_1^2 - 1 = 0, \\
    -2a_3\lambda_1^2 - 2b_3\lambda_2 - a_3\lambda_1 - 2\lambda_2 + a_3 = 0, \\
    -1 < \lambda_1 < 1, \lambda_1 \neq 0, 2\lambda_1 - 1 \neq 0.
\end{cases}
\]

Since \(a_2 = -2b_3, a_2\lambda_1 - b_3 \neq 0\) is equivalent to \(b_3 \neq 0\) as well as \(2\lambda_1 + 1 \neq 0\), the equality (2.30) leads to
\[
\lambda_2 = -\frac{(2\lambda_1^2 + \lambda_1 - 1)a_3}{b_3(2\lambda_1 + 1)}.
\]
By substituting \(\lambda_2\) into the equality (2.29), we have
\[
\frac{(\lambda_1 + 1)\Delta(\lambda_1)}{b_3^2(2\lambda_1 + 1)^2} = 0,
\]
where \(\Delta(\lambda_1) = 4(a_3^2 + b_3^2)\lambda_1^3 - 3(a_3^2 + b_3^2)\lambda_1 + a_3^2 - b_3^2\). Since \(\lambda_1 \neq -1\), the equality (2.33) is equivalent to \(\Delta(\lambda_1) = 0\). According to the inequalities in (2.14), \(\lambda_1\) does not belong to \(\{1, -1, 0, \frac{1}{2}\}\). Thus, \(\Delta(-1) = -2b_3^2 \neq 0, \Delta(1) = 2a_3^2 \neq 0, \Delta(0) = (a_3 + b_3)(a_3 - b_3) \neq 0\) and \(\Delta(\frac{1}{2}) = -2b_3^2 \neq 0\) lead to \(a_3 \neq 0, b_3 \neq 0\) and \(a_3 \neq \pm b_3\). Furthermore, since \(\Delta(-1) = -2b_3^2 < 0\) and \(\Delta(1) = 2a_3^2 > 0\), then \(\Delta(\lambda_1) = 0\) has real roots in the interval \((-1, 1)\). If \(\lambda_1 = -\frac{1}{2}\), then \(\Delta(\frac{1}{2}) = 2a_3 \neq 0\) contradicts \(\Delta(\lambda_1) = 0\). Hence \(a_3 \neq 0\) implies \(2\lambda_1 + 1 \neq 0\). Condition (A6) of Theorem 2.1 is proved.

When \(a_1 = -a_3, b_1 = -b_3, a_2\lambda_1 - b_3 \neq 0, a_2 = 2b_3\) and \(b_2 = -2a_3, T_1\) takes the form
\[
\begin{cases}
    \lambda_2^2 + \lambda_1^2 - 1 = 0, \\
    (2\lambda_1 - 1)(a_3\lambda_1 - \lambda_2b_3 + a_3) = 0, \\
    -1 < \lambda_1 < 1, \lambda_1 \neq 0, 2\lambda_1 - 1 \neq 0.
\end{cases}
\]

Because \(a_2 = 2b_3\) and \(\lambda_1 \neq \frac{1}{2}, a_2\lambda_1 - b_3 \neq 0\) is equivalent to \(b_3 \neq 0\). And since \(\lambda_1 \neq \frac{1}{2}, \) by the equalities (2.34) and (2.35), we have
\[
\lambda_1 = \frac{-a_3^2 + b_3^2}{a_3^2 + b_3^2},
\]
\[
\lambda_2 = 2\frac{ba_3}{a_3^2 + b_3^2}.
\]

Inequalities in (2.14) lead to \(a_3 \neq 0, a_3^2 \neq b_3^2\) and \(3a_3^2 \neq b_3^2\). Hence, the condition (A8) of Theorem 2.1 is proved.

When \(a_1 = -a_3, b_1 = -b_3, a_2\lambda_1 - b_3 \neq 0, a_3 = 0\) and \(b_2 = 0, T_1\) becomes
\[
\begin{cases}
    \lambda_2^2 + \lambda_1^2 - 1 = 0, \\
    (a_2\lambda_1 - b_3)\lambda_2 = 0, \\
    -\lambda_2 (-4b_3\lambda_1 + a_2) = 0, \\
    -1 < \lambda_1 < 1, \lambda_1 \neq 0, 2\lambda_1 - 1 \neq 0.
\end{cases}
\]

\(\lambda_2 \neq 0\) and \(a_2\lambda_1 - b_3 \neq 0\) lead to \((a_2\lambda_1 - b_3)\lambda_2 \neq 0,\) which contradicts the equality (2.40).

That \(a_1 = -a_3, b_1 = -b_3, a_2\lambda_1 - b_3 \neq 0, a_2 = 0\) and \(b_3 = 0\) leads to a contradiction.
(ii) When \( a_1 = -a_3, b_1 = -b_3, a_2 \lambda_1 - b_3 = 0 \) and \( a_2 \neq 0 \), we have

\[
\lambda_1 = \frac{b_3}{a_2}. \tag{2.43}
\]

In this case, \( T_1 \) is simplified to

\[
\begin{aligned}
& a_2^2 \lambda_2^2 - a_2^2 + b_3^2 = 0, \\
& a_3 (a_2 + b_3) (a_2 - 2 b_3) = 0, \\
& (a_2 - 2 b_3) (a_2 (a_2 + 2 b_3) \lambda_2 + b_2 (a_2 + b_3)) = 0,
\end{aligned} \tag{2.44}
\]

\[
-1 < \lambda_1 < 1, \quad \lambda_1 \neq 0, \quad 2 \lambda_1 - 1 \neq 0. \tag{2.45}
\]

Since \( \lambda_1 \neq \{-1, \frac{1}{2}\} \), then \( a_2 + b_3 \neq 0 \) and \( a_2 - 2 b_3 \neq 0 \). In this case, the equalities (2.45) and (2.46) are equivalent to

\[
\begin{cases}
  a_3 = 0, \\
  a_2 (a_2 + 2 b_3) \lambda_2 + b_2 (a_2 + b_3) = 0.
\end{cases} \tag{2.48}
\]

(a) When \( a_1 = -a_3 = 0, b_1 = -b_3, a_2 \lambda_1 - b_3 = 0, a_2 \neq 0 \) and \( a_2 + 2 b_3 \neq 0 \), we have

\[
\lambda_2 = -\frac{b_2 (a_2 + b_3)}{a_2 (a_2 + 2 b_3)}. \tag{2.49}
\]

In this case, the equality (2.44) becomes

\[
-\frac{(a_2 + b_3) \omega_5}{(a_2 + 2 b_3)} = 0. \tag{2.50}
\]

Since \( a_2 + b_3 \neq 0 \), we have \( \omega_5 = 0 \). Inequalities in (2.14) lead to \(-1 < \frac{b_3}{a_2} < 1, b_3 \neq 0 \) and \( a_2 \neq 2 b_3 \). Condition (A7) of Theorem 2.1 is proved.

(b) When \( a_1 = -a_3 = 0, b_1 = -b_3, a_2 \lambda_1 - b_3 = 0, a_2 \neq 0 \) and \( a_2 + 2 b_3 = 0 \), the equality (2.49) is equivalent to \( b_2 = 0 \) because of \( a_2 + b_3 \neq 0 \). Hence, by the equalities (2.43) and (2.44), we have

\[
\lambda_1 = -\frac{1}{2}, \quad \lambda_2 = \pm \frac{\sqrt{3}}{2}.
\]

Condition (A5) of Theorem 2.1 is proved.

(iii) When \( a_1 = -a_3, b_1 = -b_3, a_2 \lambda_1 - b_3 = 0 \) and \( a_2 = 0 \), \( T_1 \) is

\[
\begin{cases}
  (\lambda_1 + 1) (2 \lambda_1 - 1) b_2 = 0, \\
  - (\lambda_1 + 1) (2 \lambda_1 - 1) a_3 = 0, \\
  \lambda_2^2 + \lambda_1^2 - 1 = 0, \\
  -1 < \lambda_1 < 1, \quad \lambda_1 \neq 0, \quad 2 \lambda_1 - 1 \neq 0.
\end{cases} \tag{2.52}
\]

Hence \( b_2 = a_3 = 0 \). Condition (A4) of Theorem 2.1 is proved.

The proof of Theorem 2.1 is complete.

3. Center and Phase Diagram

**Lemma 3.1**[12] Consider the \( C^r \) differential system \( (r \in \mathbb{N} \cup \{\omega\}) \),

\[
\dot{x} = -y + f(x, y), \quad \dot{y} = x + g(x, y), \quad (x, y) \in (\mathbb{R}^2, 0), \tag{3.1}
\]

where \( f(x, y) = o(|x^2 + y^2|), \ g(x, y) = o(|x^2 + y^2|), (\mathbb{R}^2, 0) \) is a neighborhood of the origin. The system (3.1) is time-reversible if and only if the origin is a center.

By Lemma 3.1 and according to the relationship between a linear involution and its fixed point sets in Section 2, we have
Theorem 3.1 If the system (1.2) satisfies one of the conditions of Theorem 2.1, the origin is a center. Furthermore, the phase diagram near the origin is symmetric w.r.t. a straight line.

Figure 3.1 is the phase diagram of the system (1.2), if its parameters satisfy $a_1 = b_1 = b_2 = -a_3 = -b_3 = 1/2$, $a_2 = 7/4$ for the condition $(A_{11})$ in Theorem 2.1. In this case, the system takes the form
\[
\begin{align*}
\dot{x} &= -y + 1/2 x^2 + 7/4 xy - 1/2 y^2, \\
\dot{y} &= x + 1/2 x^2 + 1/2 xy - 1/2 y^2,
\end{align*}
\]
which is time-reversible w.r.t. the involution
\[
\phi(x, y) = (-3/5 x - 4/5 y, -4/5 x + 3/5 y).
\]
The fixed point set is $\{(x, y) \mid 2x + y = 0\}$ represented by dotted line. The solid curves represent orbits nearing the origin.

Reference:

线性对合下时间可逆与中心问题

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摘要：本文研究一类实平面二次多项式微分系统时间可逆性与中心的问题，得到此系统关于线性对合时间可逆的充要条件。此条件保证系统在原点处是一个关于直线对称的中心。

关键词：多项式微分系统; 时间可逆性; 线性对合; 中心