Robust Optimal Reinsurance and Investment Strategies with Delay and Default Risk in a Jump-Diffusion Financial Market with Common Shock Dependence

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Abstract: In this paper, we consider a robust optimal reinsurance and investment problem with delay and default risk under jump-diffusion model. The insurer can trade in a risk-free asset, a defaultable bond and two risky assets whose price processes are described by jump-diffusion models and correlated through a common shock. Assume that the insurer is allowed to purchase proportional reinsurance, in particular, the reinsurance premium is assumed to be calculated via the mean-variance premium principle. Under the consideration of the performance-related capital inflow/outflow, the wealth process of the insurer is modeled by a stochastic differential delay equation. The insurer’s aim is to maximize the expected exponential utility of the combination of terminal wealth and average performance wealth to study the pre-default and post-default case respectively. Furthermore, closed-form expressions for the optimal strategies and the corresponding value function are derived. Finally, numerical examples and sensitivity analyses are provided to show the impact of various parameters on the optimal strategies. In addition, ignoring model uncertainty risk will lead to significant utility loss for the Ambiguity-Averse insurers.

Key words: Robust optimal control; Jump-diffusion model; Common shock dependence; Stochastic differential delay equation; Default risk

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1. Introduction

The insurer transfers the risk of large claims to other insurers by purchasing reinsurance, and at the same time invests the premium in different assets to achieve the maximum profit of the company. In recent years, the optimal reinsurance and investment problem in actuarial insurance has been widely investigated.[1-6]

In actuarial practice, most insurance companies base their premium on the expected value principle. The reason is, of course, mathematical convenience but this is also often due
to a shortage of available statistical data. Against this principle, one can convincingly argue that two risks with the same mean may appear very different whereas the price list will give the same amount of premium for both of them. Relative to the expected value criterion, the variance premium criterion fully considers the volatility of losses. As an example, think of the family of the normal distributions with fixed mean $\mu_0$ and parameterized by the variance $\sigma_0^2$. The premium will be constant (and equal to $(1 + \theta)\mu_0$) but everybody agrees that the underlying danger will strongly vary. SUN et al.\cite{7} and ZHANG et al.\cite{8} considered reinsurance control under the variance premium criterion.

In most studies of optimal investment or investment-reinsurance strategies, it is assumed that insurance companies have only one shock investment. In fact, many insurance companies have two or more lines of investment, and most of them are not independent of each other due to the risk of suffering from a common information shock. For example LIANG et al.\cite{9} studied the optimal reinsurance-investment problems in a financial market with jump-diffusion risky asset, where the two jump number processes are correlated by a common shock. In addition, with the rapid development of financial markets, high-yield corporate bonds have become increasingly attractive to investors. Although there is indeed a default risk, it is often sought after due to its relatively high yield. In fact, the problem of portfolio optimization with defaultable securities has become an important research content in the field of insurance actuarial.\cite{7,10–11}.

Combined with practice, most of the literature on the robust optimal reinsurance and investment problem assumes that the ambiguity-averse insurers’ risky asset’s price process follows the diffusion model, which ignores the significant effect that jumps and default have on the optimal strategy. There are pronounced differences between ambiguity aversion with respect to (w.r.t.) diffusion, jumps, default risks. Therefore, in the portfolio selection problem, ignoring ambiguity w.r.t. the jumps and default risk may result in large losses in the financial market.\cite{12–13} In addition, there is also a practical problem to consider that the development of real-world systems depends not only on their current state but also on their previous history. If we believe that financial market exists bounded memory or the performance-related capital inflow/outflow, then the wealth process with delay must be considered\cite{11,14}.

In this paper, under model uncertainty, we adopt a relatively classical diffusion approximation model for insurance surplus process; financial market is diffusive only, and consists of the defaulted bond, one risk-free asset and a pair of stocks where the price processes of the stocks is given by jump processes. We assume that there exists capital inflow into or outflow from the insurer’s current wealth, and the amount of the capital inflow/outflow is proportional to the past performance of the insurer’s wealth. In that case, the wealth process of the insurer is governed by a stochastic delay differential equation (SDDE). Our goal is to maximize the expected exponential utility of the terminal wealth to find an optimal strategy and corresponding optimal value function, in as explicit a closed form as possible.

The remainder of the paper is organized as follows. Section 2, we describe the model and necessary assumptions. In Section 3, we state the robust optimal investment-reinsurance problem with exponential utility under default risk. In Section 4, we study a post-default case and a pre-default case, respectively, and derive explicit solution for the two problems. In Section 5, we present the numerical results and graphs for illustrative our results.
2. Model Assumption

Suppose \((\Omega, \mathcal{F}, P)\) be a given complete probability space with a filtration \(\mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0}\) is the information of the market available up to time \(t\). \([0, T]\) is a fixed time horizon. All the processes introduced below are assumed to be adapted to \(\{\mathcal{F}_t, t \in [0, T]\}\). In the classical risk model, without investment and reinsurance, the insurer’s surplus process is modeled by

\[
dR(t) = cd_t - d \sum_{i=1}^{K(t)} Y_i, \tag{2.1}
\]

where \(c > 0\) is the premium rate, \(\{Y_i\}\) is the \(i\)th claim and \(K(t)\) is a Poisson process with intensity \(\lambda_i > 0\). \(\{Y_i\}\) is assumed to be independent and identically distributed positive random variables with finite mean \(E[Y_i] = \mu_y\) and second moment \(E[(Y_i)^2] = \sigma_y\). Furthermore, we assume that claims \(\{Y_i\}\) are independent of \(K(t)\) and the premium rate \(c\) is calculated by the expected value premium, i.e., \(c = (1 + \theta)\lambda_i \mu_y\), where \(\theta > 0\) is the insurer’s safety loading. To be protected from potential large claims, the insurer is allowed to purchase proportional reinsurance to disperse risk. Let \(\alpha(t) \in [0, 1]\) be the reinsurance retention levels at time \(t\) and the aggregate reinsurance premium under the mean-variance principle takes the form

\[
\rho(\alpha(t)) = (1 + \eta)(1 - \alpha(t))\lambda_i \mu_y + (1 + \eta)\chi(1 - \alpha(t))^2\lambda_i \sigma_y,
\]

where \(\eta, \chi > 0\). From [8], joining the reinsurance and the surplus process for insurance is approximated as

\[
dR_0(t) = [c - \rho(\alpha(t)) - \lambda_i \mu_y \alpha(t)]dt + \sqrt{\lambda_i \sigma_y} \alpha(t)dW_0(t). \tag{2.2}
\]

Remark 2.1 When \(\eta > 0, \chi = 0\), the reinsurance premium rate degenerates to the expectation premium rate.

Remark 2.2 When \(\eta = 0, \chi > 0\), the reinsurance premium rate is calculated under the variance premium principle.

In addition to reinsurance, the insurer also can invest in a risk-free asset, a pair of shocks and a defaultable bond. The price process of the risk-free asset is described by

\[
dS_0(t) = r_0 S_0(t) dt, \tag{2.3}
\]

where \(r_0 > 0\) is the constant risk-free interest rate. The price processes of the other two risky assets are respectively modeled by the following jump-diffusion processes

\[
dS_1(t) = S_1(t-)[r_1 dt + \sigma_1 dW_1(t) + d \sum_{i=1}^{K(t)} Z_{1i}],
\]

\[
dS_2(t) = S_2(t-)[r_2 dt + \sigma_2 dW_2(t) + d \sum_{i=1}^{K(t)} Z_{2i}], \tag{2.4}
\]

where \(r_1, r_2\) are the appreciation rates, \(\sigma_1, \sigma_2 > 0\) are the volatility coefficients. \(W_1(t)\) and \(W_2(t)\) are independent standard Brownian motions. In addition, \(\{K_1(t)\}_{t \geq 0}, \{K_2(t)\}_{t \geq 0}\) are two counting processes. It is assumed that \(Z_{1i}\) and \(Z_{2i}\) are independent jump amplitude of the risky assets’ price \(S_1(t)\) and \(S_2(t)\), respectively, they are independent of \(K_1(t)\) and \(K_2(t)\) and their moments are denoted by \(E[Z_{1i}] = \mu_{1z}, E[(Z_{1i})^2] = \sigma_{1z}\), \(E[Z_{2i}] = \mu_{2z}, E[(Z_{2i})^2] = \sigma_{2z}\).

The two counting processes are correlated in the way that

\[
K_1(t) = N_1(t) + N(t), \quad K_2(t) = N_2(t) + N(t), \tag{2.5}
\]

with \(N_1(t), N_2(t), N(t)\) being three independent Poisson processes with parameters \(\lambda_1, \lambda_2, \lambda\), respectively. It is obvious that the dependence of the two classes of risky asset is due to a common shock governed by the counting process \(N(t)\).
Remark 2.3 Here \( N(t) \) represents the common information in the financial market, and the same common information has a common influence on the financial market. More vividly, in the same financial market, the common information may be caused by news that affects the entire market such as interest rates, credit spreads or oil prices.

In the development of this study, we use Poisson random measure \( K_j(\cdot, \cdot) \) \( j = 1, 2 \) on \( \Omega \times [0, T] \times [-1, +\infty] \) to denote \( \sum_{i=1}^{K_1(t)} Z_{1i} \) and \( \sum_{i=1}^{K_2(t)} Z_{2i} \) as

\[
\sum_{i=1}^{K_1(t)} Z_{1i} = \int_0^t \int_{-1}^{+\infty} z_1 K_1(ds, dz_1), \quad \sum_{i=1}^{K_2(t)} Z_{2i} = \int_0^t \int_{-1}^{+\infty} z_2 K_2(ds, dz_2),
\]

Let \( \nu_1(dt, dz_1) = (\lambda + \lambda_1) dt dF(Z_1), \nu_2(dt, dz_2) = (\lambda + \lambda_2) dt dF(Z_2) \), then

\[
E[\sum_{i=1}^{K_1(t)} Z_{1i}] = \int_0^t \int_{-1}^{+\infty} z_1 \nu_1(ds, dz_1), \quad E[\sum_{i=1}^{K_2(t)} Z_{2i}] = \int_0^t \int_{-1}^{+\infty} z_2 \nu_2(ds, dz_2), \quad \forall t \in [0, T],
\]

and \( \nu_j(\cdot, \cdot) \) is the compensator of the random measure \( K_j(\cdot, \cdot) \). Therefore, the compensated measure \( \tilde{K}_j(\cdot, \cdot) = K_j(\cdot, \cdot) - \nu_j(\cdot, \cdot) \) is related to \( \sum_{i=1}^{K_1(t)} Z_{1i}, \sum_{i=1}^{K_2(t)} Z_{2i} \) as follows

\[
\int_0^t \int_{-1}^{+\infty} z_1 \tilde{K}_1(ds, dz_1) = \sum_{i=1}^{K_1(t)} Z_{1i} - E(\sum_{i=1}^{K_1(t)} Z_{1i}), \quad \int_0^t \int_{-1}^{+\infty} z_2 \tilde{K}_2(ds, dz_2) = \sum_{i=1}^{K_2(t)} Z_{2i} - E(\sum_{i=1}^{K_2(t)} Z_{2i}).
\]

Next, we introduce the defaultable bond. Denote \( \tau \) by the first jump time of a Poisson process with constant intensity \( h^P > 0 \), which represents the default time of the company issuing the bond. For \( t \geq 0 \), define a default process by \( H(t) = 1_{[\tau \leq t]} \), which is a nondecreasing right continuous process and makes a discrete jump at the random time \( \tau \), where \( 1 \) is an indicator function which has value one if there is a jump and zero otherwise. We use \( H_t = \sigma(H(\nu), 0 \leq \nu \leq t) \) and \( G_t = F_t \vee H_t, t \geq 0 \), then \( G = (G_t)_{t \geq 0} \), is the smallest filtration such that \( \tau \) is a stopping time. Furthermore, the associated martingale default process can be written as

\[
M^P(t) = H(t) - \int_0^t (1 - H(s)) h^P ds, \quad (2.6)
\]

Similarly, we assume that there exists a defaultable bond with a maturity date \( T_1 \) and that the default time is \( \tau \). Let \( 1/\Delta \geq 1 \) denote the default risk premium and \( \varsigma \in [0, 1] \) be the loss rate of the bond when a default occurs. By Girsanov’s theorem, we find that there exists a risk neutral measure \( Q \) (which is equivalent to \( P \) ) and under \( Q \), the arrival intensity of default is given by \( h^Q = h^P / \Delta \). Denote \( p(t, T_1) \) to be the price process of the defaultable bond, then based on the definitions of \( H(t) \) and \( M^P(t) \), the dynamics of the defaultable bond price under \( P \) is given by

\[
dp(t, T_1) = p(t, T_1)[r_0 dt + (1 - H(t))\delta(1 - \Delta) dt - (1 - H(t-))\varsigma dM^P(t)], \quad (2.7)
\]

where \( \delta = h^Q \varsigma \) is the risk neutral credit spread.

In what follows, we formulate a wealth process with delay, which is caused by the instantaneous capital inflow or outflow from the insurer’s current wealth. We describe the instantaneous amount of the capital inflow/outflow as proportional to the past performance of the insurer’s current wealth by using a linear delay equation \( f \) defined as

\[
f(t, X(t) - L(t), X(t) - O(t)) = k_1(X(t) - L(t)) + k_2(X(t) - O(t)), \quad (2.8)
\]

where \( k_1 \) and \( k_2 \) are nonnegative constants, and \( L(t) = \int_0^t e^{\rho s} X(t + s) ds, O(t) = X(t - \nu) \) represent given functionals of the path segment \( X_t = \{ X(t + s), s \in [-\nu, 0] \} \). Here \( \rho \) is the
given nonnegative average parameter, \( \nu \) stands for the time of delay and is the duration of the past that the investor usually cares about. In the delay function \( (2.8) \), \( X(t) - L(t) \) represents the average performance of the wealth between \( t - \nu \) and \( t \), while \( X(t) - O(t) \) accounts for the absolute gain/loss of the insurer’s wealth in the period \([t - \nu, t]\).

Denote \( \beta_1(t), \beta_2(t) \) and \( \beta_3(t) \) the money amount invested in the stocks and the defaultable bond, respectively, and \( \alpha(t) \) represents the risk exposure of the insurer. The trading strategy is represented by \( u = \{ \alpha(t), \beta_1(t), \beta_2(t), \beta_3(t) \} \). Thus, the wealth process \( X^u(t) \) can be presented by the following SDDE

\[
dX^u(t) = \{ X^u(t)(r_0 - k_1 - k_2) + k_1L(t) + k_2O(t) + [(\theta - \eta)\lambda_k\mu_y + \eta\alpha(t)\lambda_k\mu_y \\
- (1 + \eta)\chi(1 - \alpha(t))^2\lambda_k\sigma_y] + (r_1 - r_0)\beta_1(t) + (r_2 - r_0)\beta_2(t) \\
+ (1 - H(t))\delta(1 - \Delta)\beta_3(t)\}dt + \alpha(t)\sqrt{\lambda_k\sigma_y}dW_0(t) \\
+ \beta_1(t)\sigma_1dW_1(t) + \beta_2(t)\sigma_2dW_2(t) - \beta_3(t)(1 - H(t))\chi dM^P(t) \\
+ \int_{-1}^{+\infty} \beta_1(t)\z_1K_1(dt, dz_1) + \int_{-1}^{+\infty} \beta_2(t)\z_2K_2(dt, dz_2).
\]

\[ (2.9) \]

3. Robust Optimal Control Under Exponential Utility

Following the existing literature on stochastic control with delay,[4,11] we suppose that the insurer is concerned with not only the terminal wealth \( X^u(T) \) but also the average wealth over the period \([T - \nu, T]\), i.e., \( L(T) \), meanwhile, in the paper we suppose that the aim of the insurer is to maximize the expected exponential utility of the combination of terminal wealth and average performance wealth. So the ambiguity-neutral insurer (ANI) maximizes the expected terminal wealth as

\[
\sup_{u \in \mathcal{H}} E(U(X^u(T) + \nu L(T))) = \sup_{u \in \mathcal{H}} E[-\frac{1}{\gamma}e^{-\gamma(X^u(T) + \nu L(T))}],
\]

where \( \gamma \) is the pre-specified risk aversion coefficient, \( \mathcal{H} \) is the set of all admissible strategies \( u \) in a given market and is given in the following definition 3.1, \( E[\cdot] = E[\cdot|X^u(t) = x, L(t) = l, H(t) = z] \).

Recent studies have shown that investors in the market are most often ambiguity-averse in reality. They propose the optimal strategies under worst case scenario. In this paper, we will incorporate ambiguity into the optimal problem for an ambiguity-averse insurer (AAI). We assume the alternative models are characterized by a class of probability measures which are absolutely continuous with respect to probability measure \( P \).

To introduce the ambiguity on the diffusion, jumps, default risks, we define a set of prior probability measures as follows. First of all, we call the probability distortion function

\[
\phi(t) := (\phi_1(t), \phi_2(t), \phi_3(t), \phi_4(t), \phi_5(t), \phi_6(t)) \)

and satisfy

1) \( \phi(t) \) are \( G \)-progressively measurable, for each \( t \in [0, T] \);
2) \( E[\exp\{\frac{1}{2} \int_0^T \phi(t)\sigma(t)dt\}] < +\infty \).

We denote \( \Sigma \) for the space of all such functions \( \phi \). Each probability distortion function \( \phi \in \Sigma \) associates to a probability measure \( P^* \sim P \) such that the Radon-Nikodym derivative process \( \frac{dP^*}{dP}|_{\mathcal{G}_t} = A^\phi(t) \) is

\[
A^\phi(t) = \exp\left\{ \int_0^t \sum_{i=1}^3 \phi_i(s)dW_{i-1}(s) - \frac{1}{2} \int_0^t \sum_{i=1}^3 \phi_i^2(s)ds + \int_0^t \int_{-1}^{+\infty} 2\ln \phi_{j+3}(s)K_j(ds, dz_j) \right\}
\]
The purpose of this section is to find the portfolio optimization and the optimal reinsurance strategy $\alpha(t)$ under the worst-case scenario. According to the dynamic programming principle, the extended HJB equation can be derived as follows:

$$\sup_{u \in \mathcal{U}} \inf_{\phi \in \mathcal{P}} \{ A^u, \phi V + \sum_{i=1}^{3} \phi_i^2 \frac{\phi_{i+3}^2}{\psi_i} + \sum_{j=1}^{2} (\lambda + \lambda_j) \phi_{j+3} \ln \phi_{j+3} - \phi_{j+3} + 1 \frac{\phi_{j+3}^2}{\psi_{j+3}} + h^p (1 - z) \frac{\phi_6 \ln \phi_6 - \phi_6 + 1}{\psi_6} \} = 0,$$

with the boundary condition $V(T, x, l, z) = -\frac{1}{\gamma} e^{-\gamma x + \gamma l}$, where $A^u, \phi V$ is the variational operator and is defined by $A^u, \phi V(t, x, l, z)$.
\[ V_{t}(x, l, z) = (x - \rho l - e^{-\rho t} \alpha l)V_{t}(x, l, z) + \{(r_0 - k_1 - k_2)x + k_1 l + k_2 \phi \} + \frac{(\theta - \eta)\lambda k_{l} \mu_{y} - (1 + \eta)\chi(1 - \alpha(t))\lambda_{k} \sigma_{y}}{\lambda_{k} \sigma_{y}} + (r_1 - r_0)\beta_1 + (r_2 - r_0)\beta_2 + \delta_{\beta_1}(z - 1) + \alpha \sqrt{\lambda_{k} \sigma_{y} \phi_1 + \beta_1 \sigma_{\phi_2} + \phi_2 \sigma_{\phi_3}}V_{x}(t, x, l, z) + \frac{1}{2}(\sigma^2 \lambda_{k} \sigma_{y} + \beta^2_{\sigma_2} \sigma^2_{l})^{2} \] 
\[ + \frac{\beta^2_{\sigma_2} \sigma^2_{l}}{2} V_{xx}(t, x, l, z) + (\lambda + \lambda_1)\phi_4 \mathbb{E}[V(t, x + \beta_3 Z_1, l, z) - V(t, x, l, z)] + (\lambda + \lambda_2)\phi_5 \cdot \mathbb{E}[V(t, x + \beta_2 Z_2, l, z) - V(t, x, l, z)] + h^p \phi_6 \mathbb{E}[V(t, x - \gamma \phi_3, l, 1) - V(t, x, l, 0)](z - 1). \]

(4.2)

For convenience, we choose a statedependent expression for the preference parameters, and let \( \Psi_i(t, x, l, z) = -\frac{\psi_i}{\mathbb{E}[V(t, x, l, z)]}, i = 1, 2, 3, 4, 5, 6. \)

In the following, we will derive the explicit solutions to the HJB equation (4.1) with preference parameter for the post-default case and the pre-default case, respectively.

As we can see in the equation (3.3), the change of the wealth process \( X^u(t) \) depends on the delay variables \( L(t) \) and \( O(t) \). For the sake of tractability, we impose additional conditions on the controlled delay system,

\[ k_2 = e^{-\rho t} \omega, \]

\[ k_1 e^{-\rho t} = k_2 (r_0 - k_1 - k_2 + \omega + \rho). \]

We will concentrate on the post-default case. Let \( z = 1 \) in (4.1). By the expression of \( A^{u, \phi} \) in (4.2), we rewrite (4.1) as

\[ \sup_{u \in H} \mathbb{E}^{P^*}[V_t(t, x, l, 1) + (x - \rho l - e^{-\rho t} \alpha l)V_t(t, x, l, 1) + \{(r_0 - k_1 - k_2)x + k_1 l + k_2 \phi \} + \frac{(\theta - \eta)\lambda k_{l} \mu_{y} - (1 + \eta)\chi(1 - \alpha(t))\lambda_{k} \sigma_{y}}{\lambda_{k} \sigma_{y}} + (r_1 - r_0)\beta_1 + (r_2 - r_0)\beta_2 + \delta_{\beta_1}(z - 1) + \alpha \sqrt{\lambda_{k} \sigma_{y} \phi_1 + \beta_1 \sigma_{\phi_2} + \phi_2 \sigma_{\phi_3}}V_{x}(t, x, l, 1) + \frac{1}{2}(\sigma^2 \lambda_{k} \sigma_{y} + \beta^2_{\sigma_2} \sigma^2_{l})^{2} \] 
\[ + \frac{\beta^2_{\sigma_2} \sigma^2_{l}}{2} V_{xx}(t, x, l, 1) + (\lambda + \lambda_1)\phi_4 \mathbb{E}[V(t, x + \beta_3 Z_1, l, 1) - V(t, x, l, 1)] + (\lambda + \lambda_2)\phi_5 \mathbb{E}[V(t, x + \beta_2 Z_2, l, 1) - V(t, x, l, 1)] \]
\[ - V(t, x, l, 1)] - \sum_{i=1}^{3} \frac{\phi_i^2}{2\beta_1} + \sum_{j=1}^{2}(\lambda + \lambda_3)\frac{\phi_{j+3} \ln \phi_{j+3} - \phi_{j+3} + 1}{2\beta_{j+3}} \mathbb{E}[g(t, x + \omega l)] = 0, \]

(4.5)

with the boundary condition \( V(T, x, l, 1) = -\frac{1}{\gamma} e^{-\gamma(x + \omega l)}. \) According to the structure of (4.5) and the boundary condition of \( V(t, x, l, 1) \), and for simplicity, we define \( r = r_0 - k_1 - k_2 + \omega \), and conjecture that the value function has the following form

\[ V(t, x, l, 1) = -\frac{1}{\gamma} e^{-\gamma e^{r(t-t)}(x + \omega l) - g(t)}, \]

(4.6)

where \( g(t) \) is a deterministic function with \( g(T) = 0 \). A direct calculation yields

\[ V_t(t, x, l, 1) = \gamma(r(x + \omega l)e^{r(t-t)} + g'(t))V(t, x, l, 1), \]
\[ V_x(t, x, l, 1) = -\gamma e^{r(t-t)}V(t, x, l, 1), \]
\[ V_{xx}(t, x, l, 1) = \gamma^2 e^{2r(t-t)}V(t, x, l, 1), \]
\[ V(t, x, l, 1) = -\gamma \omega e^{r(t-t)}V(t, x, l, 1), \]
\[ E^{P^*}[V(t, x + \beta_3 Z_1, l, 1) - V(t, x, l, 1)] = (M_{Z_1}(-\gamma e^{r(t-t)} \beta_3))V(t, x, l, 1), \]
\[ E^{P^*}[V(t, x + \beta_2 Z_2, l, 1) - V(t, x, l, 1)] = (M_{Z_2}(-\gamma e^{r(t-t)} \beta_2))V(t, x, l, 1), \]

where \( M_{Z_1}(\epsilon) := E_{P^*}[e^{\epsilon Z_1}] = E_{P}[e^{\epsilon Z_1}]; M_{Z_2}(\epsilon) := E_{P^*}[e^{\epsilon Z_2}] = E_{P}[e^{\epsilon Z_2}]. \) Note that the second and the fourth equality are due to the assumption that the distribution of the claim \( \{Z_1\} \), \( \{Z_2\} \) are restricted to be identical under \( P \) and \( P^* \).
Substituting (4.7) into (4.5), we obtain
\[
\sup_{u \in U} \inf_{\Theta \in \Sigma} \{ -\gamma (-(x + \omega l)e^{r(T-t)} - g'(t)) + (x - \rho l - e^{-\rho t} o)(-\gamma \omega e^{r(T-t)}) + ((r_0 - k_1 - k_2)x + k_1 l + k_2 o + [(\theta - \eta) \lambda \kappa_y + \eta \alpha \lambda \kappa_y - (1 + \eta) \chi (1 - \alpha)^2 \lambda \kappa_y] + (r_1 - r_0) \beta_1 + (r_2 - r_0) \beta_2 + \alpha \sqrt{\lambda \kappa \phi_1 + \beta_1 \sigma_2 \phi_1 + \beta_2 \sigma_2 \phi_1}(-\gamma e^{r(T-t)}) + \frac{1}{2} (\alpha^2 \lambda \kappa \phi_2 + \beta_1^2 \sigma_1^2 + \beta_2^2 \sigma_2^2)(\gamma^2 e^{2r(T-t)}) + (\lambda + \lambda_1) \phi_2 (M_{Z_t}(-\gamma e^{r(T-t)} \beta_1 - 1)) + (\lambda + \lambda_2) \phi_2 (M_{Z_t}(-\gamma e^{r(T-t)} \beta_2 - 1)) - \left[ \frac{\gamma^2 \phi_1}{2 \sigma_1^2} + \frac{\phi_1^2}{2 \sigma_2^2} + \frac{\gamma (\lambda + \lambda_1) \phi_2}{\sigma_2} + 1 \right] + (\lambda + \lambda_2) \gamma \frac{\phi_2}{\sigma_2} \phi_2 + \phi_2 + 1 \} \} = 0.
\]

Differentiating (4.8) w.r.t. \(\phi_1(t), \phi_2(t), \phi_3(t), \phi_4(t), \phi_5(t)\) gives
\[
\begin{align*}
\phi_1^*(t) &= -\alpha \sqrt{\lambda \kappa \phi_1 + \sigma_2 \phi_1} e^{r(T-t)}, \\
\phi_2^*(t) &= \exp \left\{ \frac{\phi_1}{\sigma_2} (M_{Z_t}(-\gamma e^{r(T-t)} \beta_1 - 1)) \right\}, \\
\phi_3^*(t) &= \exp \left\{ \frac{\phi_1}{\sigma_2} (M_{Z_t}(-\gamma e^{r(T-t)} \beta_2 - 1)) \right\}.
\end{align*}
\]

Substituting (4.9) into (4.8), we can obtain
\[
\sup_{u \in U} \inf_{\Theta \in \Sigma} \{ -\gamma (-(x + \omega l)e^{r(T-t)} - g'(t)) + (x - \rho l - e^{-\rho t} o)(-\gamma \omega e^{r(T-t)}) + ((r_0 - k_1 - k_2)x + k_1 l + k_2 o + [(\theta - \eta) \lambda \kappa_y + \eta \alpha \lambda \kappa_y - (1 + \eta) \chi (1 - \alpha)^2 \lambda \kappa_y] + (r_1 - r_0) \beta_1 + (r_2 - r_0) \beta_2 - \gamma e^{r(T-t)} + \frac{1}{2} (\alpha^2 \lambda \kappa \phi_2 + \beta_1^2 \sigma_1^2 + \beta_2^2 \sigma_2^2)(\gamma^2 e^{2r(T-t)}) + \alpha \sqrt{\lambda \kappa \phi_1 + \sigma_2 \phi_1} e^{r(T-t)} + \frac{\gamma (\lambda + \lambda_1) \phi_2}{\sigma_2} (M_{Z_t}(-\gamma e^{r(T-t)} \beta_1 - 1)) - 1) \}.
\]

and differentiating (4.10) w.r.t. \(\alpha(t), \beta_1(t), \beta_2(t)\) gives
\[
\alpha^*(t) = \frac{\eta \mu_y + 2(1 + \eta) \chi \sigma_y}{2(1 + \eta) \chi \sigma_y + (\gamma + \phi_1) \gamma e^{r(T-t)}},
\]
\[
\begin{align*}
\beta_1^*(t) &= \frac{e^{-r(T-t)}}{(\gamma + \phi_1) \gamma} \left[ r_1 - r_0 + (\lambda + \lambda_1) \exp \left\{ \frac{\phi_1}{\gamma} (M_{Z_t}(-\gamma e^{r(T-t)} \beta_1^*(t) - 1)) \right\} E^P(Z_t e^{-\beta_1^*(t) e^{r(T-t)} Z_t}), \\
\beta_2^*(t) &= \frac{e^{-r(T-t)}}{(\gamma + \phi_1) \gamma} \left[ r_2 - r_0 + (\lambda + \lambda_2) \exp \left\{ \frac{\phi_1}{\gamma} (M_{Z_t}(-\gamma e^{r(T-t)} \beta_2^*(t) - 1)) \right\} E^P(Z_t e^{-\beta_2^*(t) e^{r(T-t)} Z_t}).
\end{align*}
\]

**Theorem 4.1** Equations (4.9), (4.11) indeed admit a unique solution \(\phi^*(t) = (\phi_1^*(t), \phi_2^*(t), \phi_3^*(t), \phi_4^*(t), \phi_5^*(t))\), \(u^* = (\alpha^*(t), \beta_1^*(t), \beta_2^*(t))\). More specifically, \(\beta_1^*(t), \beta_2^*(t)\) is the unique root of the function respectively defined by
\[
W_1(\beta_1) = \beta_1 (\gamma + \phi_1) \gamma e^{r(T-t)} - (r_1 - r_0 + (\lambda + \lambda_1) \phi_1^*(t) e^{r(T-t)} Z_t),
W_2(\beta_2) = \beta_2 (\gamma + \phi_1) \gamma e^{r(T-t)} - (r_2 - r_0 + (\lambda + \lambda_2) \phi_2^*(t) e^{r(T-t)} Z_t).
\]

**Proof** It is easy to verify that \(W_1(\beta_1), W_2(\beta_2)\) defined by (4.12) is an increasing function of the variable \(\beta_1, \beta_2\), respectively. Moreover, \(W_1(0) = -(r_1 - r_0 + (\lambda + \lambda_1) \mu_1) < 0, W_1(1) = (\gamma + \phi_1) \gamma e^{r(T-t)} - (r_1 - r_0 + (\lambda + \lambda_1) \phi_1^*(t) e^{r(T-t)} Z_t) > 0\) by the fact that \(M_{Z_t}(-\gamma e^{r(T-t)}) > 1\) and \(e^{r(T-t)} Z_t > 1\).

Thus, the function \(W_1(\beta_1) = 0\) admits a unique root. The same reason to analyze \(W_2(\beta_2)\) has a unique solution.
Replacing (4.11) back into (4.10), and using (4.3)-(4.4), we obtain that satisfies the following ordinary differential equation (ODE):

\[ g'(t) + (\theta - \eta)\lambda_k \mu_y - (1 + \eta)\chi \lambda_k \sigma_y e^{r(T-t)} = -\frac{(\eta \mu_y + 2(1 + \eta)\chi \sigma_y)^2 \lambda_k e^{r(T-t)}}{2(1 + \eta)\chi \sigma_y + (\gamma + \vartheta_1)\sigma_y e^{r(T-t)}} - e^{r(T-t)}(r_1 - r_0)\beta_1^*(t) + \frac{1}{2}(\gamma + \vartheta_2)\sigma_\gamma^2(\beta_1^*(t))^2 e^{2r(T-t)} - e^{r(T-t)}(r_2 - r_0)\beta_2^*(t) + \frac{1}{2}(\gamma + \vartheta_3)\sigma_\gamma^2(\beta_2^*(t))^2 e^{2r(T-t)} + \frac{\lambda + \lambda_1}{\vartheta_4} \exp\left(\frac{\vartheta_5}{\gamma} (M_{Z_1}(-\gamma e^{r(T-t)} \beta_1^*(t)) - 1)\right) \]

\[ + \frac{\lambda + \lambda_2}{\vartheta_5} \exp\left(\frac{\vartheta_5}{\gamma} (M_{Z_2}(-\gamma e^{r(T-t)} \beta_2^*(t)) - 1)\right) - \frac{\lambda + \lambda_1}{\vartheta_4} - \frac{\lambda + \lambda_2}{\vartheta_5} = 0, \]

thus

\[ g(t) = \int_t^T \left[ (\theta - \eta)\lambda_k \mu_y - (1 + \eta)\chi \lambda_k \sigma_y e^{r(T-s)} - \frac{(\eta \mu_y + 2(1 + \eta)\chi \sigma_y)^2 \lambda_k e^{r(T-s)}}{2(1 + \eta)\chi \sigma_y + (\gamma + \vartheta_1)\sigma_y e^{r(T-s)}} - e^{r(T-s)}(r_1 - r_0)\beta_1^*(s) + \frac{1}{2}(\gamma + \vartheta_2)\sigma_\gamma^2(\beta_1^*(s))^2 e^{2r(T-s)} - e^{r(T-s)}(r_2 - r_0)\beta_2^*(s) + \frac{1}{2}(\gamma + \vartheta_3)\sigma_\gamma^2(\beta_2^*(s))^2 e^{2r(T-s)} + \frac{\lambda + \lambda_1}{\vartheta_4} \exp\left(\frac{\vartheta_5}{\gamma} (M_{Z_1}(-\gamma e^{r(T-s)} \beta_1^*(s)) - 1)\right) \right] \]

\[ + \frac{\lambda + \lambda_2}{\vartheta_5} \exp\left(\frac{\vartheta_5}{\gamma} (M_{Z_2}(-\gamma e^{r(T-s)} \beta_2^*(s)) - 1)\right) - \frac{\lambda + \lambda_1}{\vartheta_4} - \frac{\lambda + \lambda_2}{\vartheta_5} ds. \quad (4.13) \]

**Remark 4.1** The robust optimal portfolio optimization strategy \( \beta_1^*(t) \), \( \beta_2^*(t) \) depend on the absolute risk aversion coefficient \( \gamma \), the ambiguity aversion coefficient \( \vartheta_2, \vartheta_3 \) respectively and the parameters of insurance market. Moreover, when \( \eta > 0 \), \( \chi > 0 \), the reinsurance strategy is similar to that in [15], which considered the robust optimal portfolio selection under the mean-variance criterion; when \( \eta = 0 \), \( \chi > 0 \), the reinsurance strategy is similar to that in [7].

Next we will deal with the pre-default case. Let \( z = 0 \) in (4.1), the HJB equation (4.1) turns into

\[ \sup \inf \ E^w \{ V(t, x, l, 0) + (x - \rho l) e^{-rT} \} \]

\[ + k_2 \alpha + [(\theta - \eta)\lambda_k \mu_y + \eta \alpha \lambda_k \mu_y - (1 + \eta)\chi (1 - \alpha) \lambda_k \sigma_y + (r_1 - r_0)\beta_1 + (r_2 - r_0)\beta_2 \]

\[ + \alpha \sqrt{\lambda_k \sigma_y} \phi_1 + \beta_1 \phi_2 + \beta_2 \phi_3 + \delta \beta_3 \} V(t, x, l, 0) + \frac{1}{2}(\sigma_2^2 \lambda_k \sigma_y + \beta_2^2 \sigma^2_\gamma) \]

\[ + \beta_2^2 \sigma_\gamma^2 V_{xx}(t, x, l, 0) + (\lambda + \lambda_1) \phi_4 E[V(t, x + \beta_1 Z_1, l, 0) - V(t, x, l, 0)] + (\lambda + \lambda_2) \phi_5 \]

\[ E[V(t, x + \beta_2 Z_2, l - V(t, x, l, 0))] - \sum_{i=1}^{3} \phi_i^2 + \sum_{j=1}^{2} (\lambda + \lambda_j) \frac{\phi_{j+3} \ln \phi_{j+3} - \phi_{j+3} + 1}{2\vartheta_{j+3} + 1} \]

\[ + h^T \phi_6 \ln \phi_6 - \frac{\phi_6}{2} \] \( \gamma V(t, x, l, 0) \} = 0, \quad (4.14) \]

with the boundary condition \( V(T, x, l, 0) = -\frac{1}{\gamma} e^{-\gamma(x+\pi l)}. \) We try the following form of the value function

\[ V(t, x, l, 0) = -\frac{1}{\gamma} e^{-\gamma e^{r(T-t)}(x+\pi l) - \bar{y}(t)}, \quad (4.15) \]
where $\tilde{g}(t)$ is a deterministic function with $\tilde{g}(T) = 0$. A direct calculation yields

$$
\begin{align*}
V_1(t, x, l, 0) &= \gamma(r(x + w l))e^{r(T-t)} + \tilde{g}(t)V(t, x, l, 0), \\
V_2(t, x, l, 0) &= -\gamma e^{r(T-t)}V(t, x, l, 0), \\
V_{x}(t, x, l, 0) &= \gamma e^{2r(T-t)}V(t, x, l, 0), \\
V_{ll}(t, x, l, 0) &= 0.
\end{align*}
$$

(4.16)

$$
E_{P}[V(t, x + \beta_1 Z, l, 0) - V(t, x, l, 0)] = (M_{Z_1}(-\gamma e^{r(T-t)}\beta_1))V(t, x, l, 0),
$$

$$
E_{P}[V(t, x + \beta_2 Z, l, 0) - V(t, x, l, 0)] = (M_{Z_2}(-\gamma e^{r(T-t)}\beta_2))V(t, x, l, 0),
$$

$$
V(t, x - \beta_\iota \zeta, l, 1) - V(t, x, l, 0) = (e^{r(g(t) - \tilde{g}(t))} - 1)\gamma e^{\beta \zeta e^{r(T-t)}} V(t, x, l, 0).
$$

Substituting (4.16) back into the equation (4.14) yields

$$
sup \inf \{\gamma(r(x + w l))e^{r(T-t)} + \tilde{g}'(t)) + (x - \rho l - e^{-\rho u})\gamma e^{r(T-t)} + ((r_0 - k_1 - k_2)x + k_1 l + k_2 o + [(\theta - \eta) \lambda k \mu_y + \eta \lambda \lambda k \mu_y - (1 + \eta)\chi(1 - \alpha)\lambda \lambda k \sigma_y] + (r_1 - r_0)\beta_1 + (r_2 - r_0)\beta_2 + \delta \beta_3 + \alpha \lambda \lambda k \sigma_y \phi_1 + \beta_1 \sigma_2 \phi_2 + \beta_2 \sigma_3 \phi_3(-\gamma e^{r(T-t)}) + \frac{1}{2}(\alpha^2 \lambda \lambda k \sigma_y + \beta_1^2 \sigma_1^2 + \beta_2^2 \sigma_2^2)(\gamma e^{2r(T-t)}) + (\lambda + \lambda_k) \phi_4(M_{Z_1}(-\gamma e^{r(T-t)}\beta_1 - 1)) + (\lambda + \lambda_k) \phi_5(M_{Z_2}(-\gamma e^{r(T-t)}\beta_2 - 1)) + h_\delta \phi_6(e^{r(g(t) - \tilde{g}(t))} - 1) - \frac{\phi_2^2 \gamma}{2\gamma} + \frac{\phi_3^2 \gamma}{2\gamma} + \frac{\phi_3^2 \gamma}{2\gamma} + (\lambda + \lambda_k) \phi_4 ln \phi_4 - \phi_4 + 1}{\phi_3} + h_\delta \phi_6 ln \phi_6 - \phi_6 + 1} = 0.
$$

(4.17)

As in the post-default case, maximizing over $\phi$ yield the following first-order condition for the maximum point $\phi^*(t) = (\phi_1^*(t), \phi_2^*(t), \phi_3^*(t), \phi_4^*(t), \phi_5^*(t), \phi_6^*(t))$:

$$
\begin{align*}
\phi_1^*(t) &= -\alpha \sqrt{\lambda \lambda k \sigma_y \phi_1 e^{r(T-t)}}, \\
\phi_2^*(t) &= -\beta_1 \sigma_1 \phi_2 e^{r(T-t)}, \\
\phi_3^*(t) &= -\beta_2 \sigma_2 \phi_3 e^{r(T-t)}, \\
\phi_4^*(t) &= \text{exp}\left\{\frac{\phi_4}{\gamma} (M_{Z_1}(-\gamma e^{r(T-t)}\beta_1 - 1))\right\}, \\
\phi_5^*(t) &= \text{exp}\left\{\frac{\phi_5}{\gamma} (M_{Z_2}(-\gamma e^{r(T-t)}\beta_2 - 1))\right\}, \\
\phi_6^*(t) &= \text{exp}\left\{\frac{\phi_6}{\gamma} (e^{r(g(t) - \tilde{g}(t))} - 1)\right\} - 1.
\end{align*}
$$

(4.18)

Plugging (4.18) into (4.17) and minimizing over $u$ yields the following first-order optimality conditions for the robust optimal strategy

$$
\begin{align*}
\alpha^*(t) &= \frac{\eta \mu_y + 2(1 + \eta)\chi \sigma_y}{2(1 + \eta)\chi \sigma_y + (\gamma + \phi_3)\sigma_y e^{r(T-t)}}, \\
\beta_1^*(t) &= \frac{e^{r(T-t)}}{\gamma + \phi_3} \text{exp}\left\{\frac{\phi_4}{\gamma} (M_{Z_1}(-\gamma e^{r(T-t)}\beta_1 - 1))\right\}, \\
\beta_2^*(t) &= \frac{e^{r(T-t)}}{\gamma + \phi_3} \text{exp}\left\{\frac{\phi_5}{\gamma} (M_{Z_2}(-\gamma e^{r(T-t)}\beta_2 - 1))\right\}, \\
\beta_3^*(t) &= \frac{e^{r(T-t)}}{\gamma + \phi_3} \text{exp}\left\{\frac{\phi_6}{\gamma} (e^{r(g(t) - \tilde{g}(t))} - 1)\right\} - 1.
\end{align*}
$$

(4.19)

**Theorem 4.2** Let $W(\phi_6) = \frac{2h}{\delta} \phi_6 ln \phi_6 + h^R \phi_6 - \frac{\delta}{\gamma}$, then $W(\phi_6)$ has a unique positive root $\phi_6$.

**Proof** Since $W'(\phi_6) = \frac{2h}{\delta} (ln \phi_6 + 1) + h^R$, it is easy to verify $W(\phi_6)$ is a decreasing function on $(0, e^{\frac{-2h}{\delta} \phi_6})$ and increases on $(e^{\frac{-2h}{\delta} \phi_6}, \infty)$. Moreover, $\lim_{\phi_6 \to 0^+} W(\phi_6) = \infty$ and $\lim_{\phi_6 \to \infty} W(\phi_6) = -\infty$. Therefore, $W(\phi_6)$ has a unique positive root $\phi_6$. Therefore, $W(\phi_6)$ has a unique positive root $\phi_6$. The root $\phi_6$ is the optimal reinsurance level.
\[ -\frac{\delta}{\varsigma}, \lim_{\phi_k \to \infty} W(\phi_k) = \infty. \text{ Therefore the function } W(\phi_k) = 0 \text{ admits a unique positive root } \phi^*_k > 0. \]

Substituting (4.18), (4.19) and into (4.17), we obtain that satisfies the following ordinary differential equation (ODE):

\[
\gamma \hat{g}'(t) - \frac{\delta_\gamma}{\varsigma} \hat{g}(t) + ((\theta - \eta)\lambda_k \mu_y - (1 + \eta)\chi \lambda_k \sigma_y)(-\gamma e^{r(T-t)}) - \frac{1}{2} (\eta \mu_y + 2(1 + \eta)\chi \lambda_k \sigma_y)\lambda_k e^{r(T-t)} \]
\[
- \gamma e^{r(T-t)}(r_1 - r_0)\beta_1^*(t) + \frac{1}{2} \gamma(\gamma + \vartheta_2)\sigma^2_\gamma(\beta_1^*(t))^2 e^{2r(T-t)} - \gamma e^{r(T-t)}(r_2 - r_0)\beta_2^*(t) \\
+ \frac{1}{2} \gamma(\gamma + \vartheta_3)\sigma^2_\gamma(\beta_2^*(t))^2 e^{2r(T-t)} + \frac{(\lambda + \lambda_1)\gamma}{\vartheta_4} \phi_4^* + \frac{(\lambda + \lambda_2)\gamma}{\vartheta_5} \phi_5^* + \frac{h^P_\gamma}{\vartheta_6} \phi_6^* - \frac{(\lambda + \lambda_2)\gamma}{\vartheta_4} \\
- \frac{\gamma}{\vartheta_5} - \frac{h^P_\gamma}{\vartheta_6} - \frac{\delta}{\varsigma} \ln\left(\frac{\delta}{\varsigma \phi_6^*} \right) + \frac{\delta_\gamma}{\varsigma} \hat{g}(t) = 0, 
\]

thus

\[
\hat{g}(t) = \frac{1}{\gamma} e^{-\frac{\gamma}{\varsigma}(T-t)} \int_t^T e^{\frac{\gamma}{\varsigma}(T-s)} [(\theta - \eta)\lambda_k \mu_y - (1 + \eta)\chi \lambda_k \sigma_y)(-\gamma e^{r(T-s)}) \\
- \frac{1}{2} (\eta \mu_y + 2(1 + \eta)\chi \lambda_k \sigma_y)\lambda_k e^{r(T-s)} - \gamma e^{r(T-s)}(r_1 - r_0)\beta_1^*(s) \\
+ \frac{1}{2} \gamma(\gamma + \vartheta_2)\sigma^2_\gamma(\beta_1^*(s))^2 e^{2r(T-s)} - \gamma e^{r(T-s)}(r_2 - r_0)\beta_2^*(s) \\
+ \frac{1}{2} \gamma(\gamma + \vartheta_3)\sigma^2_\gamma(\beta_2^*(s))^2 e^{2r(T-s)} + \frac{(\lambda + \lambda_1)\gamma}{\vartheta_4} \phi_4^*(s) + \frac{(\lambda + \lambda_2)\gamma}{\vartheta_5} \phi_5^*(s) + \frac{h^P_\gamma}{\vartheta_6} \phi_6^*(s) \\
- \frac{(\lambda + \lambda_2)\gamma}{\vartheta_4} - \frac{(\lambda + \lambda_2)\gamma}{\vartheta_5} - \frac{h^P_\gamma}{\vartheta_6} - \frac{\delta}{\varsigma} \ln\left(\frac{\delta}{\varsigma \phi_6^*} \right) + \frac{\delta_\gamma}{\varsigma} \hat{g}(s) ds. 
\]

Finally, putting the pre-default and post-default case together, we have the following solution to the HJB equation (4.1) associated with the value function \( M(t, x, l, z) \)

\[ M(t, x, l, z) = (1 - z)V(t, x, l, 0) + zV(t, x, l, 1) \]

where \( z = 1 \) or \( z = 0 \), and \( V(t, x, l, 1), V(t, x, l, 0) \) defined in (4.6), (4.15), respectively.

The robust optimal strategy is given by

\[
\begin{align*}
\alpha^*(t) &= \frac{\eta \mu_y + 2(1 + \eta)\chi \sigma_y}{2(1 + \eta)\chi \sigma_y + (\gamma + \vartheta_1)\sigma_y e^{r(T-t)}}, \\
\beta_1^*(t) &= \frac{\gamma + \vartheta_2}{\gamma + \vartheta_3} \left[ r_1 - r_0 + (\lambda + \lambda_1)\phi_4^*(t) E^P[Z_1 e^{-\gamma \beta_1^*(t) e^{r(T-t)} Z_1}] \right], \\
\beta_2^*(t) &= \frac{\gamma + \vartheta_2}{\gamma + \vartheta_3} \left[ r_2 - r_0 + (\lambda + \lambda_2)\phi_5^*(t) E^P[Z_2 e^{-\gamma \beta_2^*(t) e^{r(T-t)} Z_2}] \right], \\
\beta_3^*(t) &= \frac{\gamma + \vartheta_2}{\gamma}(\frac{\delta}{\varsigma} \ln\left(\frac{\delta}{\varsigma \phi_6^*} \right) - (g(t) - \hat{g}(t))] 1_{\{\tau > t\}}.
\end{align*}
\]

The worst-case measure \( \phi^*(t) = (\phi_1^*(t), \phi_2^*(t), \phi_3^*(t), \phi_4^*(t), \phi_5^*(t), \phi_6^*(t)) \) is given by equation (4.18). \( g(t) \) and \( \hat{g}(t) \) are given by (4.13) and (4.20). Note that \( \beta_1^*(t), \beta_2^*(t) \) and \( \beta_3^*(t) \) are unique solutions to the respective equations.

**Theorem 4.3** If there exists a function and a control policy \( (\alpha^*(t), \beta_1^*(t), \beta_2^*(t), \beta_3^*(t)) \), which satisfy the HJB equation (4.1), then \( M(t, x, l, z) \) is the corresponding value function and \( (\alpha^*(t), \beta_1^*(t), \beta_2^*(t), \beta_3^*(t)) \) is an optimal strategy.

**Proof** From [11], the above theorem will hold if \( (\alpha^*(t), \beta_1^*(t), \beta_2^*(t), \beta_3^*(t)) \) and the corresponding candidate value function \( M(t, x, l, z) \) has the following three properties:
(i) \( u^* \) is an admissible strategy;
(ii) \( E^{P^*}(\sup_{t \in [0,T]} |M(t, X^{u^*}(t), L(t), H(t))|^4) < \infty \);
(iii) \( E^{P^*}(\sup_{t \in [0,T]} |I(t)|^2) < \infty \), where
\[
I(t) = \sum_{i=1}^{3} \frac{\phi_i(t)^2}{2\rho_i(s,t, X^{u^*}(t), L(t), H(t))} + \sum_{j=1}^{2} \lambda_j \phi_j(t, X^{u^*}(t), L(t), H(t)) + h^P(1-H(t)) \phi_k(t, X^{u^*}(t), L(t), H(t)).
\]

The proof of the above three properties can be found in [11], so we omit it here.

5. Suboptimal Strategy

To evaluate the importance of taking ambiguity into account, we determine how much an insurer suffers from ignoring it. Next we investigate the utility loss of the insurer when ignoring the effects of ambiguity. The strategy of the ANI is called the suboptimal strategy. When the AAI adopts the suboptimal strategy, the corresponding value function is defined by

\[
\inf_{\bar{\nu} \in \mathcal{D}} E^{P^*}_{t,x,l,z} [-\frac{1}{\gamma} e^{-\gamma(X^{\bar{\nu}}(T)+\pi L(T))} + \int_t^T \Phi(s, X(s), L(s), H(s))ds],
\]

where \( \bar{\nu}^* \) is a optimal reinsurance and investment strategy when model ambiguity is not considered for ANI.

We can solve the problem (5.1) by using the similar HJB equation as that in Section 4.2. We find that the worst-case measure for the problem (5.1) has the similar expression as Eq. (4.18), only need to change \( g(t) \) to the function \( g_1(t) \), and \( \tilde{g}(t) \) to the function \( \tilde{g}_1(t) \). Via a calculation that is parallel to the method in Section 4, we obtain the value function for optimization problem (5.1) under the suboptimal strategy as follows:

\[
\tilde{M}(t, x, l, z) = -\{z \frac{1}{\gamma} e^{-\gamma(e^{(T-t)}(x+\pi l)-g_1(t))} + (1-z) \frac{1}{\gamma} e^{-\gamma(e^{(T-t)}(x+\pi l)-\tilde{g}_1(t))}\},
\]

where
\[
\begin{align*}
g_1(t) & = \int_t^T \left[ \bar{\nu}(x - \rho l - e^{-\rho \omega} o) - r(x + \pi l) + (r_0 - k_1 - k_2)x + k_1 l + k_2 o + (\theta - \eta)\lambda_k\mu_y \\
& - (1 + \eta)\chi_1\sigma_1 e^{(T-s)} - (\eta\mu_y + (1 + \eta)\chi_2\sigma_2)^2 \lambda_k e^{(T-s)} - e^{(T-s)}(r_1 - r_0)\beta_1^*(s) \right] ds, \\
\end{align*}
\]

\[
\begin{align*}
\tilde{g}_1(t) & = \frac{1}{\gamma} e^{-\frac{1}{\gamma}(T-t)} \int_t^T e^{\frac{1}{\gamma}(T-s)} \left[ \gamma\bar{\nu}(x + \pi l)e^{(T-s)} - \gamma \omega e^{(T-s)}(x - \rho l - e^{-\rho \omega} o) + ((r_0 \\
& - k_1 - k_2)x + k_1 l + k_2 o + (\theta - \eta)\lambda_k\mu_y - (1 + \eta)\chi_1\sigma_1 e^{(T-s)} - (\eta\mu_y + (1 + \eta)\chi_2\sigma_2)^2 \lambda_k e^{(T-s)} - e^{(T-s)}(r_1 - r_0)\beta_1^*(s) \right] ds, \\
\end{align*}
\]

\[
\begin{align*}
& + \frac{1}{2} (\chi_1 + \gamma_1)\sigma_1^2 (\beta_1^*(s))^2 e^{2r(T-s)} - e^{r(T-s)}(r_2 - r_0)\beta_2^*(s) \\
& + \frac{1}{2} (\chi_2 + \gamma_2)\sigma_2^2 (\beta_2^*(s))^2 e^{2r(T-s)} + \frac{\lambda + \lambda_1}{\theta_4} \exp\left\{ \frac{\theta_4}{\gamma} (M_{Z_1}(-\gamma e^{r(T-s)}\beta_1^*(s)) - 1) \right\} \\
& + \frac{\lambda + \lambda_2}{\theta_5} \exp\left\{ \frac{\theta_5}{\gamma} (M_{Z_2}(-\gamma e^{r(T-s)}\beta_2^*(s)) - 1) \right\} - \frac{\lambda + \lambda_1}{\theta_4} - \frac{\lambda + \lambda_2}{\theta_5} ds, \\
\end{align*}
\]

with
\[ + \frac{h^P}{\vartheta_6} \sigma_6^*(s) - \frac{\lambda + \lambda_1}{\vartheta_4} - \frac{\lambda + \lambda_2}{\vartheta_5} \frac{h^P}{\vartheta_6} - \frac{\delta}{\vartheta_4} \ln\left( \frac{\delta}{\vartheta_6 \phi_6^*(s)} \right) + \frac{\delta \gamma}{\vartheta_6} g_1(s) \] \text{d}s \text{.} \tag{5.4}

Then we define the post-default utility loss as

\[ V(t, x - K_1, l, 1) = V(t, x, l, 1) \text{.} \tag{5.5} \]

From (4.13) and (5.3), it then follows that

\[ K_1 = \frac{1}{\lambda} e^{-r(T-t)} [g(t) - g_1(t)] \text{.} \]

Similarly, we define pre-default utility loss as

\[ V(t, x - K_2, l, 0) = V(t, x, l, 0) \text{.} \tag{5.6} \]

From (4.20) and (5.4), it then follows that

\[ K_2 = \frac{1}{\lambda} e^{-r(T-t)} [\tilde{g}(t) - \tilde{g}_1(t)] \text{.} \]

The loss function \( K_1, K_2 \) measures the utility loss of the AAI when she adopts the suboptimal strategy \( \bar{u}^*(t) \).

\section{6. Sensitivity Analysis}

In this section, we analyze the impact of parameters and model uncertainty on optimal investment and reinsurance strategies by some numerical examples. We suppose that the claim sizes are exponentially distributed with the parameter \( \theta = 2 \). In the following analysis, unless otherwise stated, the basic parameters are listed below:

\[ T = 2, \eta = 0.5, \chi = 1, \mu_y = 0.5, \sigma_y = 0.5, \lambda_1 = 0.12, \lambda_2 = 0.38, \lambda = 0.2, \]

\[ \sigma_1 = 0.15, \sigma_2 = 0.2, r_1 = 0.15, r_2 = 0.12, r_0 = 0.1, \gamma = 0.2, \vartheta_1, \vartheta_2, \vartheta_3 = 1, \]

\[ \vartheta_4, \vartheta_5, \vartheta_6 = 2, \delta = 0.2, \Delta = 0.25, \varsigma = 0.4, h^P = 0.125, h^Q = 0.5. \]

At first, in Fig. 1, we know that \( \beta_1^*(t) \) decreases with the model uncertainty coefficient \( \nu_2 \) and risk aversion coefficient \( \gamma \), which indicate that the insurer will pay less money to invest in stock market when \( \nu_2 \) or \( \gamma \) becomes larger. Similarly, \( \beta_2^*(t) \) decreases with the model uncertainty coefficient \( \nu_3 \) and risk aversion coefficient \( \gamma \). From Figs. 3 and 4, we can see that the optimal portfolio strategy \( \beta_1^*(t) \) and \( \beta_2^*(t) \) are decreasing while the value of \( \lambda \) increases. This is also to be expected, because a greater value of \( \lambda \) implies a greater value of the expected jump number, the investor would rather invest less money to the risky asset. For strategy \( \beta_1 \) and \( \beta_2 \), we can find that when \( \lambda_1 < \lambda_2 \), the value of \( \beta_1 \) is larger than \( \beta_2 \), and vice versa. This implies that the values of the optimal portfolio strategy are sensitive to the counting processes.

As described in Fig. 5, the optimal wealth invested in the defaultable increases when the loss rate \( \varsigma \) decreases. On the contrary, the insurer will invest more money in the defaultable bond with higher credit spread \( \delta \). Note that \( \nu \) reflects the insurer’s attitude towards the jump risk caused by default, we can find that a large \( \nu \) induces less wealth invested in the defaultable bond, which is well depicted in Fig. 6, and also shows that the robust optimal investment strategy \( \beta_1^*(t) \) decreases with respect to the default intensity \( h^P \). In fact, the probability of default will be larger when the default intensity \( h^P \) is higher, and the counterparty risk of the defaultable bond will undermine its investment grade and thus make it less attractive to the insurer.

Next we discuss the effect of \( v_1, \gamma \) on the strategy \( \alpha(t) \) over time respectively. Since \( v_1 \) reveals the ambiguity aversion for jump risk caused by claim. As is illustrated \( \frac{\partial \alpha(t)}{\partial v_1} < 0, \frac{\partial \alpha(t)}{\partial \gamma} < 0 \), with higher \( v_1 \) and \( \gamma \), the insurer has lower risk exposure in insurance market, so as to less amount of money will be paid to purchase reinsurance. We also find that the
robust optimal reinsurance strategies under the generalized mean-variance premium are very different from that under the variance premium principle.

Fig. 1 Effect of $\gamma$ and $\upsilon_2$

Fig. 2 Effect of $\gamma$ and $\upsilon_3$

Fig. 3 Effect of $\gamma$ and $\lambda$

Fig. 4 Effect of $\gamma$ and $\lambda$

Fig. 5 Effect of $\zeta$ and $\delta$

Fig. 6 Effect of $h^P$ and $\upsilon_6$

References:


具有共同冲击相依性的跳扩散金融市场中有着延迟和违约风险的鲁棒最优再保险和投资策略

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摘要：在本文中，我们考虑跳扩散模型下具有延迟和违约风险的鲁棒最优再保险和投资问题。保险人可以投资无风险资产、可违约的债券和两个风险资产，其中两个风险资产遵循跳跃扩散模型且受到同种因素带来共同影响而相互关联。假设允许保险人购买比例再保险，特别地再保险保费利用均值方差保费原则来计算。在考虑与绩效相关的资本流入/流出下，保险公司的财富过程通过随机微分延迟方程建模。保险公司的目标是最大程度地发挥终端财富和平均绩效财富组合的预期指数效应，以分别研究违约前和违约后的情况。此外，推导了最优策略的闭式表达式和相应的价值函数。最后通过数值算例和敏感性分析，表明了各种参数对最优策略的影响。另外对于模糊厌恶投资者，忽视模型模糊性风险会带来显著的效用损失。

关键词：鲁棒最优控制；跳扩散模型；共同冲击相依性；随机微分延迟方程；违约风险