Fixed Point Theory and the Existence of Two Positive Periodic Solutions of Riccati’s Equation

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Abstract: This paper deals with a class of Riccati’s equation. By the principle of contraction mapping, the existence of one positive periodic solution of Riccati’s equation is obtained; By variable transformation method, Riccati’s equation is turned into Bonulli’s equation. According to the periodic solution of Bernoulli’s equation and variable transformation, another periodic solution of Riccati’s equation is obtained. And we discuss the stability of the two positive periodic solutions, one periodic solution is attractive on some region and unstable on another region, and another periodic solution is unstable on $\mathbb{R}$.

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1. Introduction

Consider the nonlinear Riccati type first-order differential equation:
\[
\frac{dx}{dt} = a(t)x^2 + b(t)x + c(t). \tag{1.1}
\]
No matter in the classical theory of differential equation or in the branches of modern science, Riccati’s equation has important applications, and there are many studies on this equation\cite{1-6}. In [1-2], the sufficient conditions of the existence of periodic solutions of Eq.(1.1) were given, also in [1], the stability of periodic solutions of Eq.(1.1) was obtained; In [3], the author studied some special types of Riccati’s equation, and got the general solution and periodic solutions of Eq.(1.1); In [4], the author discussed Eq.(1.1) with characteristic multiplier; In [5], the author considered the high dimensional Riccati’s equation, and obtained some sufficient conditions of the existence of periodic solutions of Eq.(1.1); In [6], the author got some criteria for the existence of periodic solutions of Eq.(1.1).

Recently, Mokhtarzadeh, Pournaki and Razani\cite{7} dealt with scalar Riccati’s differential equations, and assumed that $a$, $b$, and $c$ are $\omega$-periodic continuous real functions on $\mathbb{R}$ and gave certain conditions to guarantee the existence of at least one periodic solution for Eq.(1.1);
Wilczynski\cite{9} gave a few sufficient conditions for the existence of two periodic solutions of the Riccati’s ordinary differential equation in the plane.

However, the above literatures only give the existence of periodic solutions, but few papers involve the size range and stability of periodic solutions. When \( b(t) \equiv 0 \), in \([9]\), we obtain the existence of two periodic solutions Riccati’s equation with \( \omega \)-periodic coefficients and different symbols. One of these solutions is unstable on \( \mathbb{R} \) and the other one is attractive on some region.

In this paper, we discuss a class of Riccati’s equation. By the principle of contraction mapping and variable transformation method, we obtain the existence of two positive periodic solutions of Riccati’s equation, and give the ranges of the size of the periodic solutions; further, we discuss the stability of the two periodic solutions. One periodic solution is attractive on a certain region and unstable on another region, and another periodic solution is unstable on \( \mathbb{R} \). Some conclusions of related papers (e.g. [1,9]) are generalized.

2. Preliminaries

In this section, we give a lemma which will be used later.

**Lemma 2.1\[^{10}\]** Consider the following:

\[
\frac{dx}{dt} = a(t)x + b(t), \tag{2.1}
\]

where \( a(t), b(t) \) are \( \omega \)-periodic continuous functions on \( \mathbb{R} \). If \( \int_0^\omega a(t)dt \neq 0 \), then Eq.(2.1) has a unique \( \omega \)-periodic continuous solution \( \eta(t) \), and \( \eta(t) \) can be written as follows:

\[
\eta(t) = \begin{cases} 
\int_{-\infty}^t e^{\int_{t}^{\tau} a(\tau)d\tau}b(s)ds, & \int_0^\omega a(t)dt < 0, \\
-\int_{t}^{+\infty} e^{\int_{t}^{\tau} a(\tau)d\tau}b(s)ds, & \int_0^\omega a(t)dt > 0.
\end{cases}
\tag{2.2}
\]

For the sake of convenience, suppose that \( f(t) \) is an \( \omega \)-periodic continuous function on \( \mathbb{R} \). We denote

\[
f_M = \sup_{t \in [0, \omega]} f(t), \quad f_L = \inf_{t \in [0, \omega]} f(t). \tag{2.3}
\]

The rest of the paper is arranged as follows: We will study the existence and stability property of the periodic solutions of Eq.(1.1) in the next section, we end this paper with two examples.

3. Existence and Stability of the Periodic Solutions

In this section, we obtain the existence of periodic solutions of Riccati’s equation by using the contraction mapping theorem, and discuss the stability of periodic solutions. Moreover, this proof process is suitable for a class of Bernoulli’s equation, so we can get the existence and stability of periodic solutions of the class of Bernoulli’s equation.

**Theorem 3.1** Consider Eq.(1.1), \( a(t), b(t) \) and \( c(t) \) are all \( \omega \)-periodic continuous functions on \( \mathbb{R} \). Suppose that the following conditions hold:

\( (H_1) \) \( a(t) > 0; \)
\( (H_2) \) \( b(t) < 0; \)
\( (H_3) \) \( c(t) \geq 0; (c(t) \not\equiv 0); \)
\( (H_4) \) \( b_M^2 - 4a_Mc_M > 0. \)

Then Eq.(1.1) has two positive \( \omega \)-periodic continuous solutions:
1) One positive $\omega$-periodic continuous solution is $\gamma_1(t)$,
\[
\frac{-b_L - \sqrt{b_L^2 - 4a_Lc_L}}{2a_L} \leq \gamma_1(t) \leq \frac{-b_M - \sqrt{b_M^2 - 4a_Mc_M}}{2a_M},
\]
and $\gamma_1(t)$ is attractive if given initial value on $D_1 = \{ x(t_0) | x(t_0) < \frac{1}{\zeta(t_0)} + \gamma_1(t_0) \}$, and unstable if given initial value on $D_2 = \{ x(t_0) | x(t_0) \geq \frac{1}{\zeta(t_0)} + \gamma_1(t_0) \}$. Here $x(t_0)$ is any given initial value of Eq.(1.1), and
\[
\zeta(t) = \int_{t}^{+\infty} e^{-\int_{s}^{t} [2a(\tau)\gamma_1(\tau) + b(\tau)]} d\tau a(s) ds.
\]
2) Another positive $\omega$-periodic continuous solution is $\gamma_2(t)$,
\[
\frac{-b_M + \sqrt{b_M^2 - 4a_Mc_M}}{2a_M} \leq \gamma_2(t) \leq \frac{-b_L + \sqrt{b_L^2 - 4a_Lc_L}}{2a_L},
\]
and $\gamma_2(t)$ is unstable on $\mathbb{R}$.

**Proof**
1) We prove that Eq.(1.1) has two positive $\omega$-periodic continuous solutions $\gamma_1(t)$ and $\gamma_2(t)$.

Define a set as follows:
\[
B = \{ \varphi(t) \in C(\mathbb{R}, \mathbb{R}) | \varphi(t + \omega) = \varphi(t), \frac{-b_L - \sqrt{b_L^2 - 4a_Lc_L}}{2a_L} \leq \varphi(t) \leq \frac{-b_M - \sqrt{b_M^2 - 4a_Mc_M}}{2a_M} \}.
\]

By (H$_1$), (H$_2$), (H$_3$) and (H$_4$), we have
\[
\frac{-b_L - \sqrt{b_L^2 - 4a_Lc_L}}{2a_L} \leq \frac{-b_M - \sqrt{b_M^2 - 4a_Mc_M}}{2a_M}.
\]
Thus $B \neq \emptyset$, given any $\varphi(t), \psi(t) \in B$, the distance is defined as follows:
\[
\rho(\varphi, \psi) = \sup_{t \in [0, \omega]} |\varphi(t) - \psi(t)|.
\]

Thus ($B, \rho$) is a complete metric space.

Given any $\varphi(t) \in B$, consider the following equation:
\[
\frac{dx}{dt} = \left[ a(t)\varphi(t) + b(t) \right] x + c(t).
\]

Since $a(t), b(t), c(t)$ and $\varphi(t)$ are $\omega$-periodic functions on $\mathbb{R}$, and $a(t)\varphi(t) + b(t)$ is an $\omega$-periodic function on $\mathbb{R}$, by (3.1), (H$_1$) and (H$_2$), we get that
\[
\frac{b_L - \sqrt{b_L^2 - 4a_Lc_L}}{2} = a(t)\varphi(t) + b(t) \leq \frac{b_M - \sqrt{b_M^2 - 4a_Mc_M}}{2} < 0.
\]

Hence we have
\[
\frac{b_L - \sqrt{b_L^2 - 4a_Lc_L}}{2} \leq \int_{0}^{\omega} [a(t)\varphi(t) + b(t)] dt \leq \frac{b_M - \sqrt{b_M^2 - 4a_Mc_M}}{2} < 0.
\]

According to Lemma 2.1, Eq.(3.3) has a unique $\omega$-periodic continuous solution as follows:
\[
\eta(t) = \int_{-\infty}^{t} e^{\int_{s}^{t} [a(\tau)\varphi(\tau) + b(\tau)] d\tau} c(s) ds.
\]

By (H$_3$), (3.4) and (3.5), we get
\[
\int_{-\infty}^{t} e^{\int_{s}^{t} \frac{b_L - \sqrt{b_L^2 - 4a_Lc_L}}{2} (t-s) c_L ds} \leq \eta(t) = \int_{-\infty}^{t} e^{\int_{s}^{t} a(\tau)\varphi(\tau) + b(\tau) d\tau} c(s) ds.
\]
\[
\begin{align*}
\leq \int_{-\infty}^{t} e^{b_M - \sqrt{b_M^2 - 4a_M c_M}}(t-s) c_M ds.
\end{align*}
\]

Thus we have
\[
- \frac{c_L}{b_L - \sqrt{b_L^2 - 4a_L c_L}} \leq \eta(t) \leq - \frac{c_M}{b_M - \sqrt{b_M^2 - 4a_M c_M}},
\]
that is,
\[
- \frac{b_L - \sqrt{b_L^2 - 4a_L c_L}}{2a_L} \leq \eta(t) \leq - \frac{b_M - \sqrt{b_M^2 - 4a_M c_M}}{2a_M}.
\]

Hence, \( \eta(t) \in B \).

Define a mapping as follows:
\[
(T\varphi)(t) = \eta(t) = \int_{-\infty}^{t} e^{\int_{s}^{t} [a(\tau)\varphi(\tau) + b(\tau)] d\tau} c(s) ds.
\]

Thus if given any \( \varphi(t) \in B \), then \( (T\varphi)(t) \in B \). Hence \( T : B \to B \), Given any \( \varphi(t), \psi(t) \in B \), we have
\[
\begin{align*}
\left| (T\varphi)(t) - (T\psi)(t) \right| &= \left| \int_{-\infty}^{t} \left( e^{\int_{s}^{t} [a(\tau)\varphi(\tau) + b(\tau)] d\tau} - e^{\int_{s}^{t} [a(\tau)\psi(\tau) + b(\tau)] d\tau} \right) c(s) ds \right| \\
&= \left| \int_{-\infty}^{t} e^{t} \left( \int_{s}^{t} [a(\tau)\varphi(\tau) - \psi(\tau)] d\tau \right) c(s) ds \right| \\
&\leq \int_{-\infty}^{t} e^{t} \left( \int_{s}^{t} a(\tau) d\tau \right) c(s) ds \left| \rho(\varphi, \psi) \right|,
\end{align*}
\]

here, \( \xi \) is between \( \int_{s}^{t} [a(\tau)\varphi(\tau) + b(\tau)] d\tau \) and \( \int_{s}^{t} [a(\tau)\psi(\tau) + b(\tau)] d\tau \). By (3.4), it follows
\[
\xi \leq \frac{b_M - \sqrt{b_M^2 - 4a_M c_M}}{2}(t - s).
\]

Hence we have
\[
\left| (T\varphi)(t) - (T\psi)(t) \right| \leq \int_{-\infty}^{t} e^{b_M - \sqrt{b_M^2 - 4a_M c_M}}(t-s)(t-s) a_M c_M ds \rho(\varphi, \psi) \\
= \frac{a_M c_M}{(b_M - \sqrt{b_M^2 - 4a_M c_M})^2} \rho(\varphi, \psi).
\]

Therefore, we can get
\[
\rho(T\varphi, T\psi) \leq \frac{a_M c_M}{(b_M - \sqrt{b_M^2 - 4a_M c_M})^2} \rho(\varphi, \psi).
\]

By (H2) and (H4), it follows
\[
a_M c_M < \frac{1}{4} b_M^2 \leq \left( \frac{b_M - \sqrt{b_M^2 - 4a_M c_M}}{2} \right)^2.
\]

Hence we have
\[
0 < \frac{a_M c_M}{(b_M - \sqrt{b_M^2 - 4a_M c_M})^2} < 1.
\]

By (3.8) and (3.9), \( T \) is a contraction mapping. According to the contraction mapping principle, \( T \) has a fixed point on \( B \). The fixed point is the positive \( \omega \)-periodic continuous solution \( \gamma_1(t) \) of Eq. (1.1), and
\[
- \frac{b_L - \sqrt{b_L^2 - 4a_L c_L}}{2a_L} \leq \gamma_1(t) \leq - \frac{b_M - \sqrt{b_M^2 - 4a_M c_M}}{2a_M}.
\]
Thus we have

$$y(t) = x(t) - \gamma_1(t),$$  \hspace{1cm} (3.11)

here \(x(t)\) is the unique solution of Eq.(1.1) with initial value \(x(t_0) = x_0\), and \(\gamma_1(t)\) is the periodic solution of Eq.(1.1). Differentiating both sides of (3.11) along the solution of Eq.(1.1), we get

$$\frac{dy}{dt} = \frac{dx}{dt} - \frac{d\gamma_1}{dt} = a(t)(x^2(t) - \gamma_1^2(t)) + b(t)(x(t) - \gamma_1(t))$$

$$= a(t)(x(t) + \gamma_1(t))(x(t) - \gamma_1(t)) + b(t)(x(t) - \gamma_1(t))$$

$$= a(t)(x(t) - \gamma_1(t) + 2\gamma_1(t))(x(t) - \gamma_1(t)) + b(t)(x(t) - \gamma_1(t))$$

$$= [2a(t)\gamma_1(t) + b(t)](x(t) - \gamma_1(t)) + a(t)(x(t) - \gamma_1(t))^2$$

$$= [2a(t)\gamma_1(t) + b(t)]y + a(t)y^2.$$  \hspace{1cm} (3.12)

This is Bernoulli’s equation. Let

$$u(t) = \frac{1}{y(t)},$$  \hspace{1cm} (3.13)

then Eq.(3.12) can be turned into the following equation:

$$\frac{du}{dt} = -[2a(t)\gamma_1(t) + b(t)]u - a(t).$$  \hspace{1cm} (3.14)

Note that

$$-\sqrt{b_L^2 - 4a_Lc_L} \leq 2a(t)\gamma_1(t) + b(t) \leq -\sqrt{b_M^2 - 4a_Mc_M} < 0,$$  \hspace{1cm} (3.15)

thus we have

$$0 < \sqrt{b_M^2 - 4a_Mc_M} \leq -[2a(t)\gamma_1(t) + b(t)] \leq \sqrt{b_L^2 - 4a_Lc_L}. $$  \hspace{1cm} (3.16)

According to Lemma 2.1, Eq.(3.14) has a unique \(\omega\)-periodic continuous solution as follows:

$$\zeta(t) = \int_t^{t+\omega} e^{-\int_s^t [2a(\tau)\gamma_1(\tau) + b(\tau)]d\tau} a(s)ds.$$  \hspace{1cm} (3.17)

By the transformations (3.11) and (3.13), it follows that Eq.(1.1) has another \(\omega\)-periodic continuous solution \(\gamma_2(t)\) as follows:

$$\gamma_2(t) = \frac{1}{\zeta(t)} + \gamma_1(t).$$  \hspace{1cm} (3.18)

Suppose that \(t^*\) is the extreme point of function \(\gamma_2(t)\), then we have

$$\frac{d\gamma_2(t^*)}{dt^*} = a(t^*)\gamma_2^2(t^*) + b(t^*)\gamma_2(t^*) + c(t^*)$$

$$= a(t^*) \left( \gamma_2(t^*) + \frac{b(t^*) + \sqrt{b^2(t^*) - 4a(t^*)c(t^*)}}{2a(t^*)} \right) \left( \gamma_2(t^*) + \frac{b(t^*) - \sqrt{b^2(t^*) - 4a(t^*)c(t^*)}}{2a(t^*)} \right)$$

$$= 0.$$  \hspace{1cm} (3.19)

Then we can get that all extreme points \(t^*\) of function \(\gamma_2(t)\) satisfy

$$\gamma_2(t^*) = \frac{-b(t^*) + \sqrt{b^2(t^*) - 4a(t^*)c(t^*)}}{2a(t^*)}.$$  \hspace{1cm} (3.20)

Thus we have

$$\frac{-b_M + \sqrt{b_M^2 - 4a_Mc_M}}{2a_M} \leq \gamma_2(t) \leq \frac{-b_L + \sqrt{b_L^2 - 4a_Lc_L}}{2a_L}.$$  \hspace{1cm} (3.21)

2) We prove the stability of two periodic solutions \(\gamma_1(t)\) and \(\gamma_2(t)\) of Eq.(1.1).

Firstly, we prove the stability of the periodic solution \(\gamma_1(t)\) of Eq.(1.1).
Easy to know, the unique solution \( u(t) \) of Eq. (3.14) with initial value \( u(t_0) = u_0 \) is as follows:

\[
u(t) = e^{-\int_{t_0}^{t} \frac{2a(s)(\gamma_1(s)+b(s))ds}{a(s)} - \int_{t_0}^{t} e^{-\int_{t_0}^{\tau} \frac{2a(\tau)(\gamma_1(\tau)+b(\tau))d\tau}{a(\tau)}} a(s)ds} u_0 - \int_{t_0}^{t} e^{-\int_{t_0}^{\tau} \frac{2a(\tau)(\gamma_1(\tau)+b(\tau))d\tau}{a(\tau)}} a(s)ds + \int_{t}^{+\infty} e^{-\int_{t_0}^{\tau} \frac{2a(\tau)(\gamma_1(\tau)+b(\tau))d\tau}{a(\tau)}} a(s)ds \]

Thus, \( u(t) \) can be rewritten as follows:

\[
u(t) = e^{-\int_{t_0}^{t} \frac{2a(s)(\gamma_1(s)+b(s))ds}{a(s)}} \left[ \frac{1}{x(t_0) - \gamma_1(t_0)} - \zeta(t_0) \right] + \zeta(t).
\]

By (3.11), it follows

\[
y(t) = e^{-\int_{t_0}^{t} \frac{2a(s)(\gamma_1(s)+b(s))ds}{a(s)}} \left[ \frac{1}{x(t_0) - \gamma_1(t_0)} - \zeta(t_0) \right] + \zeta(t).
\]

By (3.11), we have

\[
| x(t) - \gamma_1(t) | = \left| \frac{1}{e^{-\int_{t_0}^{t} \frac{2a(s)(\gamma_1(s)+b(s))ds}{a(s)}} \left[ \frac{1}{x(t_0) - \gamma_1(t_0)} - \zeta(t_0) \right] + \zeta(t)} | u(t) | \right| = \frac{1}{| u(t) |}.
\]

By (3.16), it follows

\[
e^{-\int_{t_0}^{t} \frac{2a(s)(\gamma_1(s)+b(s))ds}{a(s)}} \rightarrow +\infty (t \rightarrow +\infty).
\]

We will discuss the sign of \( \frac{1}{x(t_0) - \gamma_1(t_0)} - \zeta(t_0) \) in three cases.

(i) If \( \frac{1}{x(t_0) - \gamma_1(t_0)} - \zeta(t_0) > 0 \), that is, \( x(t_0) < \frac{1}{\zeta(t_0)} + \gamma_1(t_0) \), by (3.17), (3.27) and (3.28), it follows

\[
| x(t) - \gamma_1(t) | \rightarrow 0, (t \rightarrow +\infty).
\]

(ii) If \( \frac{1}{x(t_0) - \gamma_1(t_0)} - \zeta(t_0) < 0 \), by (3.17), (3.28), (3.30) and (3.25), we have

\[
u(+) \rightarrow -\infty.
\]

Under the condition of (3.30), now, we discuss \( u(t_0) \) in two cases.

(I) If \( \frac{1}{x(t_0) - \gamma_1(t_0)} < 0 \), then \( x(t_0) < \gamma_1(t_0) \), by (3.24), we have

\[
u(t_0) = u_0 < 0.
\]
From (3.32), (3.22), when \( t > t_0 \), it follows
\[
 u(t) < 0. \tag{3.33}
\]
By (3.31), (3.33) and (3.27), we have
\[
 \left| x(t) - \gamma_1(t) \right| \to 0, (t \to +\infty). \tag{3.34}
\]
Thus the \( \omega \)-periodic solution \( \gamma_1(t) \) of Eq.(1.1) is attractive if given initial value
\[
x(t_0) < \gamma_1(t_0). \tag{3.35}
\]
By (i) and (I) of (ii), the \( \omega \)-periodic solution \( \gamma_1(t) \) of Eq.(1.1) is attractive if given initial value
\[
x(t_0) \in D_1 = \left\{ x(t_0) \mid x(t_0) < \frac{1}{\zeta(t_0)} + \gamma_1(t_0) \right\}. \tag{3.36}
\]
(II) If \( \frac{1}{x(t_0) - \gamma_1(t_0)} > 0 \), then
\[
x(t_0) > \gamma_1(t_0). \tag{3.37}
\]
Thus \( u(t_0) = \frac{1}{x(t_0) - \gamma_1(t_0)} > 0 \). Combined by (3.31), according to zero point theorem, there exists a \( t^* > t_0 \), such that
\[
u(t^*) = e^{-\int_{t_0}^{t^*} \left[ b(x(t)) + b(x(t)) \right] \, dx} \left[ \frac{1}{x(t_0) - \gamma_1(t_0)} - \zeta(t_0) \right] + \zeta(t^*) = 0. \tag{3.38}
\]
Therefore, when \( t \to t^* \), we have
\[
e^{-\int_{t_0}^{t^*} \left[ b(x(t)) + b(x(t)) \right] \, dx} \left[ \frac{1}{x(t_0) - \gamma_1(t_0)} - \zeta(t_0) \right] + \zeta(t) \to 0.
\]
By (3.27) and the formula above, it follows
\[
\left| x(t) - \gamma_1(t) \right| \to +\infty, (t \to t^*). \tag{3.39}
\]
Combining (3.30) and (3.37), we can get
\[
x(t_0) > \frac{1}{\zeta(t_0)} + \gamma_1(t_0). \tag{3.40}
\]
By (II) of (ii), the periodic solution \( \gamma_1(t) \) of Eq.(1.1) is unstable if given initial value
\[
x(t_0) > \frac{1}{\zeta(t_0)} + \gamma_1(t_0). \tag{3.41}
\]
(iii) If \( \frac{1}{x(t_0) - \gamma_1(t_0)} - \zeta(t_0) = 0 \), that is, \( x(t_0) = \frac{1}{\zeta(t_0)} + \gamma_1(t_0) \), at this time, the unique solution of Eq.(1.1) with initial value \( x(t_0) = \frac{1}{\zeta(t_0)} + \gamma_1(t_0) \) is just the periodic solution \( \gamma_2(t) \),
\[
\left| \gamma_2(t) - \gamma_1(t) \right| = \frac{1}{\zeta(t_0)} > 0, \text{ not tend to } 0.
\]
Thus by (II) of (ii) and (iii), we get that if given initial value
\[
x(t_0) \in D_2 = \left\{ x(t_0) \mid x(t_0) \geq \frac{1}{\zeta(t_0)} + \gamma_1(t_0) \right\}, \tag{3.41}
\]
\( \gamma_1(t) \) is unstable.

Next, we prove the stability of the periodic solutions \( \gamma_2(t) \) of Eq.(1.1).

By (3.18), it follows
\[
\left| x(t) - \gamma_2(t) \right| = \left| x(t) - \gamma_1(t) - \frac{1}{\zeta(t)} \right| \geq \frac{1}{\zeta(t)} - \left| x(t) - \gamma_1(t) \right|. \tag{3.42}
\]
where \( x(t) \) is the unique solution of Eq.(1.1) with initial value \( x(t_0) = x_0 \). From the above proof, by (3.29), we know, when \( x(t_0) \in D_1 \), \( \left| x(t) - \gamma_1(t) \right| \to 0, (t \to +\infty) \), that is to say, given any \( \varepsilon > 0 \), there is a \( T > 0 \), such that \( \left| x(t) - \gamma_1(t) \right| < \varepsilon \) as \( t \geq t_0 + T \). So, when \( t \geq t_0 + T \), we have
\[
\left| x(t) - \gamma_2(t) \right| > \frac{1}{\zeta(t)} - \varepsilon. \tag{3.43}
\]
Because of the arbitrariness of $\varepsilon$, it follows
\[ |x(t) - \gamma_2(t)| \geq \frac{1}{\zeta(t)}, \]  \hspace{1cm} (3.44)

Note that $\zeta(t)$ is bounded and positive on $\mathbb{R}$, thus $\gamma_2(t)$ is unstable if $x(t_0) \in D_1$.

When $x(t_0) \in D_2$, by (3.39), there exists a $t^* > t_0$, such that
\[ x(t) - \gamma_1(t) \rightarrow +\infty (t \rightarrow t^*). \]  \hspace{1cm} (3.45)

Since $\zeta(t)$ is bounded and positive on $\mathbb{R}$, we have
\[ |x(t) - \gamma_2(t)| \geq |x(t) - \gamma_1(t)| - \frac{1}{\zeta(t)} \rightarrow +\infty (t \rightarrow t^*). \]  \hspace{1cm} (3.46)

Thus $\gamma_2(t)$ is unstable if $x(t_0) \in D_2$.

Therefore, the $\omega$-periodic solution $\gamma_2(t)$ of Eq.(1.1) is unstable on $D_1 \cup D_2 = \mathbb{R}$.

This is the end of the proof of Theorem 3.1.

**Theorem 3.2** Under the conditions of Theorem 3.1, Eq.(1.1) has exactly two $\omega$-periodic continuous solutions: $\gamma_1(t)$ and $\gamma_2(t)$.

**Proof** The proof of the existence of $\gamma_1(t)$ and $\gamma_2(t)$ is given in Theorem 3.1. Now, we prove that Eq.(1.1) has exactly two $\omega$-periodic continuous solutions: $\gamma_1(t)$ and $\gamma_2(t)$.

We know if $x(t_0) = \gamma_1(t_0)$, the unique solution of Eq.(1.1) is $\gamma_1(t)$, and if $x(t_0) = \gamma_2(t_0) = \frac{1}{\zeta(t_0)} + \gamma_1(t_0)$, the unique solution of Eq.(1.1) is $\gamma_2(t)$.

(i) If $x(t_0) > \gamma_2(t_0) = \frac{1}{\zeta(t_0)} + \gamma_1(t_0)$, by (3.46), the unique solution $x(t)$ of Eq.(1.1) satisfies
\[ x(t) - \gamma_2(t) \rightarrow +\infty, t \rightarrow t^*. \]

Thus $x(t)$ cannot be a periodic solution.

(ii) If $x(t_0) < \gamma_2(t_0) = \frac{1}{\zeta(t_0)} + \gamma_1(t_0)$, by (3.36), we know that $\gamma_1(t)$ is attractive, thus the unique solution $x(t)$ of Eq.(1.1) satisfies
\[ x(t) - \gamma_1(t) \rightarrow 0, t \rightarrow +\infty. \]

Hence $x(t)$ cannot be a periodic solution.

Therefore, Eq.(1.1) has exactly two $\omega$-periodic continuous solutions: $\gamma_1(t)$ and $\gamma_2(t)$.

This is the end of the proof of Theorem 3.2.

**Remark 3.1** From the proof of Theorem 3.1, if $c(t) \equiv 0$, (3.9) is not satisfied, but $B = \{ \varphi(t)|\varphi(t) = 0 \}$, $T$ also has a fixed point on $B$. The fixed point on $B$ is the periodic solution $\gamma_1(t) = 0$, thus, it is easy for us to draw the following corollary on Bernoulli’s equation.

**Corollary 3.1** Consider the following Bernoulli’s equation:
\[ \frac{dx}{dt} = a(t)x^2 + b(t)x, \]  \hspace{1cm} (3.47)

and $a(t), b(t)$ are both $\omega$-periodic continuous functions on $\mathbb{R}$. Suppose that the following conditions hold:

$(H_5)$ $a(t) > 0$;

$(H_6)$ $b(t) < 0$.

Then Eq.(3.47) has exactly two $\omega$-periodic continuous solutions:

1) One $\omega$-periodic continuous solution is $\gamma_1(t) = 0$, and $\gamma_2(t)$ is attractive if given initial value on $D_1 = \{ x(t_0) | x(t_0) < \frac{1}{\zeta(t_0)} \}$, and unstable if given initial value on $D_2 = \{ x(t_0) | x(t_0) \geq \frac{1}{\zeta(t_0)} \}$, here $x(t_0)$ is any given initial value of Eq.(3.47), and
\[ \zeta(t) = \int_t^{+\infty} e^{-\int_0^t b(\tau)d\tau} a(s)ds. \]
2) Another \(\omega\)-periodic continuous solution is \(\gamma_2(t)\),

\[
\gamma_2(t) = \frac{1}{\zeta(t)}, \quad -\frac{b_L}{a_L} \leq \gamma_2(t) \leq -\frac{b_M}{a_M},
\]

and \(\gamma_2(t)\) is unstable on \(\mathbb{R}\).

**Theorem 3.3** Consider Eq.(1.1), \(a(t), b(t)\) and \(c(t)\) are all \(\omega\)-periodic continuous functions on \(\mathbb{R}\). Suppose that the following conditions hold:

\(\text{(H}_7\text{)}\) \(a(t) < 0\);
\(\text{(H}_8\text{)}\) \(b(t) > 0\);
\(\text{(H}_9\text{)}\) \(c(t) \leq 0; (c(t) \neq 0)\);
\(\text{(H}_{10}\text{)}\) \(b_L^2 - 4a_Lc_L > 0\).

Then Eq.(1.1) has two positive continuous \(\omega\)-periodic solutions.

1) One positive \(\omega\)-periodic solution is \(\gamma_1(t)\),

\[
-\frac{b_M + \sqrt{b_M^2 - 4a_Mc_M}}{2a_M} \leq \gamma_1(t) \leq -\frac{b_L + \sqrt{b_L^2 - 4a_Lc_L}}{2a_L},
\]

and \(\gamma_1(t)\) is unstable on \(\mathbb{R}\);  
2) Another positive \(\omega\)-periodic solution is \(\gamma_2(t)\),

\[
-\frac{b_M - \sqrt{b_M^2 - 4a_Mc_M}}{2a_M} \leq \gamma_2(t) \leq -\frac{b_L - \sqrt{b_L^2 - 4a_Lc_L}}{2a_L},
\]

and \(\gamma_2(t)\) is attractive if given initial value on \(D_1 = \{x(t_0)|x(t_0) > \gamma_1(t_0)\}\), and unstable if given initial value on \(D_2 = \{x(t_0)|x(t_0) \leq \gamma_1(t_0)\}\), here \(x(t_0)\) is any given initial value of Eq.(1.1), and

\[
\zeta(t) = -\int_{-\infty}^{t} e^{-\int_{s}^{t} [2a_0(\tau)\gamma_1(\tau) + b(\tau)]d\tau}a(s)ds.
\]

**Proof** Consider the following equation:

\[
\frac{dx}{dt} = \tilde{a}(t)x^2 + \tilde{b}(t)x + \tilde{c}(t), \tag{3.48}
\]

here \(\tilde{a}(t) = -a(t) > 0, \tilde{b}(t) = -b(t) < 0, \tilde{c}(t) = -c(t) \geq 0, b_M^2 - 4a_M\tilde{c}_M = b_L^2 - 4a_Lc_L\). The result stated in Theorem 3.3 can be directly obtained by applying Theorem 3.1 to the equation (3.48), thus we omit it here.

**Theorem 3.4** Under the conditions of Theorem 3.3, Eq.(1.1) has exactly two \(\omega\)-periodic continuous solutions: \(\gamma_1(t)\) and \(\gamma_2(t)\).

**Proof** Theorem 3.4 also follows by applying Theorem 3.2 to the equation (3.48).

Similar to Corollary 3.1, we have

**Corollary 3.2** Consider Bernoulli’s Eq.(3.47), and \(a(t), b(t)\) are both \(\omega\)-periodic continuous functions on \(\mathbb{R}\). Suppose that the following conditions hold:

\(\text{(H}_{11}\text{)}\) \(a(t) < 0\);
\(\text{(H}_{12}\text{)}\) \(b(t) > 0\).

Then Eq.(3.47) has exactly two \(\omega\)-periodic continuous solutions:

1) One \(\omega\)-periodic continuous solution is \(\gamma_1(t) = 0\), and \(\gamma_1(t)\) is unstable on \(\mathbb{R}\);
2) Another \(\omega\)-periodic continuous solution is

\[
\gamma_2(t) = -\frac{1}{\int_{-\infty}^{t} e^{-\int_{s}^{t} b(\tau)d\tau}a(s)ds}, \quad \frac{b_M}{a_M} \leq \gamma_2(t) \leq -\frac{b_L}{a_L}.
\]
and $\gamma_2(t)$ is attractive if given initial value on $D_1 = \left\{ x(t_0) \mid x(t_0) > 0 \right\}$, and unstable if given initial value on $D_2 = \left\{ x(t_0) \mid x(t_0) \leq 0 \right\}$, here $x(t_0)$ is any given initial value of Eq.(3.47).

4. Examples

The following examples show the feasibility of our main results.

**Example 4.1** Consider the following equation:

$$\frac{dx}{dt} = (2 + \sin t)x^2 + (\sin t - 8)x + 2 - \cos t.$$  \hspace{1cm} (4.1)

It is easy to calculate that $a_M = 3, a_L = 1, b_M = -7, b_L = -9, c_M = 3, c_L = 1,$ and $b_M^2 - 4a_M c_M = 13 > 0$.

Clearly, the conditions (H1)-(H4) of Theorem 3.1 are satisfied. It follows from Theorem 3.1 that Eq.(4.1) has two positive $2\pi$-periodic continuous solutions $\gamma_1(t)$ and $\gamma_2(t)$, and

$$0.11252 \approx \frac{9 - \sqrt{77}}{2} = -\frac{b_L + \sqrt{b_L^2 - 4a_L c_L}}{2a_L} \leq \gamma_1(t) \leq \frac{b_M + \sqrt{b_M^2 - 4a_M c_M}}{2a_M} = \frac{7 - \sqrt{13}}{6} \approx 0.56574,$$

and $\gamma_1(t)$ is attractive on $D_1 = \left\{ x(t_0) \mid x(t_0) < \frac{1}{\zeta(t_0)} + \gamma_1(t_0) \right\}$, and unstable on $D_2 = \left\{ x(t_0) \mid x(t_0) \geq \frac{1}{\zeta(t_0)} + \gamma_1(t_0) \right\}$, here $x(t_0)$ is any given initial value of Eq.(4.1), and

$$\zeta(t) = \int_t^\infty e^{-s} \left[ 2(2 + \sin \tau)\gamma_1(\tau) + (\sin \tau - s) \right] d\tau (2 + \sin s) ds.$$

Another $2\pi$-periodic continuous solution is $\gamma_2(t)$,

$$\gamma_2(t) = \frac{1}{\zeta(t)} + \gamma_1(t),$$

$$2.14011 \approx \frac{\sqrt{37}}{3} - \frac{-9 + \sqrt{77}}{2} \leq \gamma_2(t) \leq \sqrt{77} - \frac{-7 + \sqrt{13}}{6} \approx 9.3407,$$

and $\gamma_2(t)$ is unstable on $\mathbb{R}$.

From this example, using matlab, we can deduce the value $3.541 < \gamma_2(0) = \frac{1}{\zeta(0)} + \gamma_1(0) < 3.542$. When initial value $x(0) \leq 3.541$, the solution curve of Eq.(4.1) tends to the curve of the periodic solution $\gamma_1(t)$ as $t$ achieves at some value (see Fig. 4.1); When initial value $x(0) \geq 3.542$, the solution curve of Eq.(4.1) arrives at enough large $(+\infty)$ at some time $t^*$ (see Fig. 4.2).

**Example 4.2** Consider the following equation:

$$\frac{dx}{dt} = (-2 + \sin t)x^2 + (\sin t + 8)x - 2 + \cos t.$$  \hspace{1cm} (4.2)

It is easy to calculate that $a_M = -1, a_L = -3, b_M = 9, b_L = 7, c_M = -1, c_L = -3$, and $b_L^2 - 4a_L c_L = 13 > 0$.

Clearly, the conditions (H7)-(H10) of Theorem 3.3 are satisfied. It follows from Theorem 3.3 that Eq.(4.2) has two positive $2\pi$-periodic continuous solutions $\gamma_1(t)$ and $\gamma_2(t)$, and

$$0.11252 \approx \frac{9 - \sqrt{77}}{2} = -\frac{b_M + \sqrt{b_M^2 - 4a_M c_M}}{2a_M} \leq \gamma_1(t) \leq -\frac{b_L - \sqrt{b_L^2 - 4a_L c_L}}{2a_L} = \frac{7 - \sqrt{13}}{6} \approx 0.56574,$$
and $\gamma_1(t)$ is unstable on $\mathbb{R}$.

Another $2\pi$-periodic continuous solution is $\gamma_2(t)$,

$$\gamma_2(t) = \frac{1}{\zeta(t)} + \gamma_1(t),$$

$$1.3144 \approx \frac{\sqrt{13}}{3} + \frac{9 - \sqrt{77}}{2} \leq \gamma_2(t) \leq \sqrt{77} + \frac{7 - \sqrt{13}}{6} \approx 9.3407,$$

and $\gamma_2(t)$ is attractive on $D_1 = \{x(t_0)|x(t_0) > \gamma_1(t_0)\}$, and unstable on $D_2 = \{x(t_0)|x(t_0) \leq \gamma_1(t_0)\}$, here $x(t_0)$ is any given initial value of Eq.(4.2), and

$$\zeta(t) = -\int_{-\infty}^{t} e^{-t_s} \left[2(-2+\sin \tau)\gamma_1(\tau)+\sin \tau +8\right] \text{d}\tau \left(\sin s-2\right) \text{d}s.$$

From this example, using matlab, we can deduce the value $1.28 < \gamma_1(0) < 1.29$. When initial value $x(0) \geq 1.29$, the solution curve of Eq.(4.2) tends to the curve of the periodic solution $\gamma_2(t)$ as $t$ achieves at some value (see Fig. 4.3); When initial value $x(0) \leq 1.28$, the solution curve of Eq.(4.2) arrives at enough large ($-\infty$) at some time $t'$ (see Fig. 4.4).

Fig. 4.1 The curve of the solution of Eq.(4.1) with initial value $x(0) = 3.541$

Fig. 4.2 The curve of the solution of Eq.(4.1) with initial value $x(0) = 3.542$

Fig. 4.3 The curve of the solution of Eq.(4.2) with initial value $x(0) = 0.129$

Fig. 4.4 The curve of the solution of Eq.(4.2) with initial value $x(0) = 0.128$

References:
不动点理论与里卡提方程两个正周期解的存在性

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摘要：利用压缩映射原理，得到里卡提方程一个正周期解的存在性；利用变量变换方法，将里卡提方程转化为伯努利方程。根据伯努利方程的周期解和变量变换，得到里卡提方程的另一个周期解，并讨论了两个正周期解的稳定性。一个周期解在某个区间上是吸引的，另一个周期解在R上是不稳定的。

关键词：里卡提方程；压缩映射；周期解；稳定性