Periodic Variation Solutions and Tori like Solutions for Stochastic Hamiltonian Systems

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Abstract: In this paper, we study recurrence phenomenon for Hamiltonian systems perturbed by noises, especially path-wise random periodic variation solution (RPVS) and invariant tori like solution. Concretely speaking, for linear Schrödinger equations, we completely clarify when RPVS exists, and for nearly integrable Hamiltonian systems perturbed by noises we prove that the existence of invariant tori like solutions is related to the involution property of multi component driven Hamiltonian functions.

Key words: Random system; Hamiltonian system; Recurrence phenomenon; Invariant tori

CLC Number: O211.63
AMS(2010)Subject Classification: 60H10
Document code: A
Article ID: 1001-9847(2021)02-0477-12

1. Introduction

We first consider finite dimensional random dynamical systems. Let $M$ be a $2d$ dimensional symplectic manifold with symplectic form $\omega$. Given a Hamiltonian function $H$ on $M$, the associated Hamiltonian vector field is denoted by $X_H$. Given a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $Z_t$ be an $\mathbb{R}^l$-valued driving semi-martingale, $Y_0 \in \mathcal{F}_0$ be an $M$-valued random variable, and $\{H_\alpha\}_{\alpha=0}^l$ be Hamiltonian functions on $M$. We will study the following type stochastic differential equations which may be seen as the analogies of Hamiltonian systems of the deterministic case:

$$dY_t = \sum_{\alpha=0}^l X_{H_\alpha} \circ dZ_t^\alpha,$$

where $\circ$ refers to the Stratonovich integral and (1.1) is understood in the sense of the integral equation with initial data $Y_0$. Moreover, for simplicity we write $dt = dZ_t^0$ in (1.1) and in the following.

Let’s first consider the case $M = T^*T^d$, where $T^d = \mathbb{R}^d/2\pi\mathbb{Z}^d$ denotes the $d$-dimensional torus. In the perturbation theory, one considers the Hamiltonian functions defined in $T^*T^d$, which can be viewed as $T^d \times \mathbb{R}^d$, of the following form

$$H(p, q) = N(p) + H_1(p, q),$$

Received date: 2020-06-29
where \((p, q) \in \mathbb{R}^d \times \mathbb{R}^d\) is the action and angle variables respectively. If \(H_1 = 0\), the Hamiltonian system associated with \((1.2)\) is integrable. If \(H_1(p, q)\) is a small perturbation in some sense, the Hamiltonian system corresponding to \((1.2)\) is called nearly integrable. The classic celebrated KAM theorem\(^{[2]}\) states that the invariant tori persists under the perturbation with suitable non-degenerate conditions. The KAM theory in the deterministic case is a fundamental result of Hamiltonian systems, and it has many significant and wide applications to various problems, for instance celestial mechanics, symplectic algorithms\(^{[3,7]}\), Anderson localization etc. In the stochastic case, few results are known. Let \((\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})\) denote the canonical metric dynamical system describing \(\mathbb{R}^1\)-valued Brownian motion \(\{B_t\}_{t \in \mathbb{R}}\). Let us begin with the toy model problem:

\[
(dp, dq) = J \nabla N_0 dt + J \nabla N_1 \circ dB_t, \tag{1.3}
\]

where \(N_0, N_1\) only depend on \(p, \nabla N_i\) denotes the gradient field generated by \(N_i\), and \(J\) denotes the standard complex structure in \(\mathbb{R}^d\). The solution of \((1.3)\) can be written as

\[
p_i(t) = p_i^0, \quad q_i(t) = t\partial_{p_i} N_0(p^*) + B_t \partial_{p_i} N_1(p^*), \quad i = 1, \ldots, d. \tag{1.4}
\]

This can be seen as the stochastic version of integrable systems. Now, assume that for some \(k \in \mathbb{Z}^d\), \(\partial_{p_i} N_0(p^*) = 2\pi k_i/T\) for all \(i = 1, \ldots, d\), then it is easy to see

\[
(p(t + T, \omega), q(t + T, \omega)) = (p(t, \theta_T \omega), q(t, \theta_T \omega)) + (\xi(\omega), \zeta(\omega)) \tag{1.5}
\]

with \((\xi, \zeta) = (0, \ldots, 0, \partial_{p_i} N_1(p^*) B_T(\omega), \ldots, \partial_{p_i} N_1(p^*) B_T(\omega))\) for all \((t, \omega) \in \mathbb{R} \times \Omega\), and as a random dynamical system\(^{[1]}\) there holds

\[
\varphi(t, \theta_\omega)(p(s, \omega), q(s, \omega)) = (p(t, \omega), q(t, \omega)), \tag{1.6}
\]

where \(\varphi(t, \omega)(p, q)\) denotes the solution of \((1.3)\) with initial data \((p, q)\).

Inspired by \((1.5), (1.6)\), we introduce the notion of periodic variation solutions as follows:

**Definition 1.1** Let \(M\) be a finite or infinite dimensional linear space or a smooth manifold embedded into Euclidean spaces. Let \(\varphi : \mathbb{R} \times \Omega \times M \to M\) be the mapping which defines a measurable random dynamical system on the measurable space \((M, \mathcal{B})\) over a metric random dynamical system \((\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})\). We say \(Y(t, \omega)\) is a random periodic variation solution (RPVS) with \(\mathcal{F}_0\) measurable initial data \(Y(0, \omega)\) if there exists some \(T > 0\) and an \(M\)-valued (when \(M\) is a linear space) or \(\mathbb{R}^N\)-valued (when \(M\) is a manifold) random function \(\xi : \omega \in \Omega \to M\) or \(\xi : \omega \in \Omega \to \mathbb{R}^N\) such that for all \(t \in \mathbb{R}, \omega \in \Omega\), there holds

\[
\begin{cases}
(i) & Y(t + T, \omega) - Y(t, \theta_T \omega) = \xi(\omega), \\
(ii) & \varphi(t, \theta_\omega)(Y(s, \omega)) = Y(t + s, \omega)
\end{cases} \tag{1.7}
\]

where \((i)\) in \((1.7)\) holds in \(\mathbb{R}^N\) if \(M\) is a manifold embedded into \(\mathbb{R}^N\).

If the random dynamical system is a two parameter stochastic flow \(\varphi : \mathbb{R} \times \mathbb{R} \times \Omega \times M \to M\), \((1.7)\) is replaced by

\[
\begin{cases}
(i) & Y(t + T, \omega) - Y(t, \theta_T \omega) = \xi(\omega), \\
(ii) & \varphi(t + s, t, \omega, Y(t, \omega)) = Y(t + s, \omega)
\end{cases} \tag{1.8}
\]

for any \(t, s \in \mathbb{R}, \omega \in \Omega\).

If we require \(\xi = 0\) in \((i)\) of \((1.7)\), then solutions satisfying \((1.7)\) are called random periodic solutions (RPS), i.e.,

\[
\begin{cases}
(i) & Y(t + T, \omega) - Y(t, \theta_T \omega) = 0, \\
(ii) & \varphi(t, \theta_\omega)(Y(s, \omega)) = Y(t + s, \omega).
\end{cases} \tag{1.9}
\]

The other widely used notion of periodic solutions is the periodic Markov process solution: We say the solution of a stochastic equation is a periodic homogeneous Markov process if it is an $\mathbb{R}^m$ valued homogeneous Markov process and the joint distribution $\mathbb{P}(u_t \in A_1, \cdots, u_n \in A_n)$ satisfies

$$\mathbb{P}(u_{t+T} \in A_1, \cdots, u_{n+T} \in A_n) = \mathbb{P}(u_t \in A_1, \cdots, u_n \in A_n)$$

for some $T > 0$ and all $0 \leq t_1 < \cdots < t_n < \infty$, $n \in \mathbb{N}$, $\{A_j\}_{j=1}^n \subset \mathcal{B}(\mathbb{R}^m)$. Compared with (1.10), random periodic solutions defined by (1.9) are sometimes called pathwise periodic solutions. One can similarly define periodic non-homogeneous Markov process as well.

We remark that various notions of solutions with diverse recurrence properties have been introduced and intensively studied in the stochastic dissipative systems[3,7–10].

We summarize the existence/non-existence of RPS and RPVS for (1.3) in the following lemma. It is somewhat casual, and the precise statement can be found in Section 3.

**Proposition 1.1** (1.3) has no random periodic solutions except for some trivial cases (See Proposition 3.1). For $\mathcal{F}_0$ measurable initial data $(p(0), q(0)) = (\xi, \eta)$, the solution of (1.3) is a random periodic variation solution iff $\{\nabla N_i(\xi)\}_{i=0,1}$ are deterministic.

In general, if the frequencies $\{\partial_{\eta^i} N_0(p^*)\}_{i=1}^d$ are rationally independent, the solution of (1.3) is invariant tori like:

$$(p(t), q(t)) = (p^*, \lambda_1 t + Y_1(t, \omega), \cdots, \lambda_d t + Y_d(t, \omega)),$$

where $\{Y_j(t, \omega)\}_{j=1}^d$ are RPVSs.

If there is essentially only one driving Hamiltonian in (1.1), i.e., $H_\alpha = F_\alpha(H)$ for all $\alpha = 0, \cdots, l$, it is easy to apply the classic KAM theorem to obtain random invariant tori like solutions in the stochastic case. Generally multi driving Hamiltonian functions may lead to non-existence of invariant tori. For the special case $d = 1$, we will see there exists a neat result:

**Proposition 1.2** Let $d = 1$. Let $H_0$ be a Hamiltonian function on $\mathbb{R} \times \mathbb{T}$ which gives rise to a small analytic perturbation of integrable systems. Let $\{H_\alpha\}_{\alpha=1}^l$ be analytic functions of $p, q \in \mathbb{C}^2$. If the driving Hamiltonian functions satisfy

$$\{H_0, H_\alpha\} = 0, 1 \leq \alpha \leq l,$$

then under reasonable non-degenerate assumptions, (1.1) has random invariant tori. See Proposition 3.2 for the precise statement.

In a summary for SODEs, we remark that (i) Random periodic solutions generally do not exist for Hamiltonian type equations; (ii) The existence of random periodic variation solutions and invariant tori depends heavily on the involution property of the driving Hamiltonian functions.

Let’s consider infinite dimensional random dynamical systems. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ be the canonical complete filtered Wiener space endowed with filtration $\mathcal{F}_s := \sigma\{B_{r_1} - B_{r_2} : s \leq r_1, r_2 \leq t\}$. Denote $\mathcal{F}_t := \bigvee_{s \leq t} \mathcal{F}_s$, and $\mathcal{F}^t := \bigvee_{s \geq t} \mathcal{F}_s$. Let $\{\zeta_j(t)\}_{j \in \mathbb{Z}}$ be a sequence of independent $\mathbb{R}$-valued standard Brownian motions on $t \in \mathbb{R}$ associated to the filtration $(\mathcal{F}_t)_{t \in \mathbb{R}}$. Let $\Phi : L^2(\mathbb{T}^d; \mathbb{C}) \rightarrow L^2(\mathbb{T}^d; \mathbb{C})$ be a linear bounded operator with $Q = \Phi \Phi^*$ being a
finite trace operator in $L^2(T^d; \mathbb{C})$. Letting $\{\mathcal{e}_j\}_{j \in \mathbb{Z}}$ be the orthonormal basis for $L^2(T^d; \mathbb{C})$, we define the process $W$ to be
\[
W(t, x, \omega) = \sum_{j \in \mathbb{Z}} \zeta_j(t, \omega) \mathcal{e}_j(x), \quad t \in \mathbb{R}, x \in T^d, \omega \in \Omega. \tag{1.11}
\]
It is easy to see the series (1.11) converges in $L^2(\Omega \times T^d; \mathbb{C})$ and almost surely in $L^2(T^d; \mathbb{C})$. This process is a special case of $Q$-cylindrical Wiener process with $Q = \Phi^* \Phi$.

**Theorem 1.1**  
(i) If $\Phi \neq 0$, then the linear stochastic Schrödinger equation with additive noise
\[
idu = \Delta u dt + dW_t, \quad x \in T^d,
\]
has no RPVS.

(ii) Let us consider the linear stochastic Schrödinger equation with combined noise:
\[
idu = \Delta u dt + (\lambda u + f(t, x)) \circ dB_t, \quad x \in T^d, \tag{1.12}
\]
where $\lambda \in \mathbb{R}$, $f$ is a real valued function which is periodic in $t$, $f(t + T_1) = f(t)$, and smooth in $x \in T^d$. Then $u$ is an RPVS with $\mathcal{F}_0$ measurable initial data $u_0 \in L^2(\Omega, L^2_T)$ for (1.12) if only if $f = 0$ and $u = 0$.

In the following, we denote $\Delta_T u(t, w) = u(t + T, w) - u(t, \theta_T w)$.

2. Linear SPDEs

We divide the proof of Theorem 1.1 into two propositions. Let $\{\mathcal{e}_k\}_{k \in \mathbb{Z}^d}$ be the eigenfunctions of $\Delta$ in $T^d$ such that $\Delta \mathcal{e}_k = -|k|^2 \mathcal{e}_k$, and $\pi_k$ denote the projection onto span$\{\mathcal{e}_k\}$. We get $f : \Omega \rightarrow L^2_T$ is $(\mathcal{F}, \mathcal{B}(L^2_T))$ measurable iff $\pi_k f$ is $(\mathcal{F}, \mathcal{B}(\mathbb{C}))$ measurable for all $k \in \mathbb{Z}^d$. And similar results hold with $\mathcal{F}$ replaced by $\mathcal{F}_s^t, \mathcal{F}^t$ and $\mathcal{F}_t$ as well. These facts will be used widely in this section without emphasis.

**Proposition 2.1**  
Assume that the operator $\Phi \neq 0$ in (1.11). The linear stochastic Schrödinger equation with additive noise
\[
idu = \Delta u dt + dW_t, \quad x \in T^d, \tag{2.1}
\]
has no RPVS.

**Proof** Define
\[
\xi_k(t, \omega) = \sum_{j \in \mathbb{Z}} \zeta_j(t, \omega) \langle \Phi \mathcal{e}_j, \mathcal{e}_k \rangle, \quad t \in \mathbb{R}, k \in \mathbb{Z}^d, \omega \in \Omega, \tag{2.2}
\]
then we have $E|\xi_k(t, \omega)|^2 = |t| \beta_k^2$ where $\{\beta_k\}$ are defined by
\[
\beta_k = \left( \sum_{j \in \mathbb{Z}} |\langle \Phi \mathcal{e}_j, \mathcal{e}_k \rangle|^2 \right)^{\frac{1}{2}}, \quad k \in \mathbb{Z}^d. \tag{2.3}
\]
Applying the Fourier transform to (2.1) gives
\[
\hat{du}_k = i|k|^2 \hat{u}_k dt - i \hat{d} \xi_k.
\]
The solution is an Ornstein-Uhlenbeck process
\[
\hat{u}_k = e^{i|k|^2 t} \hat{a}_k - i \int_0^t e^{i|k|^2 (t-s)} d\xi_k(s),
\]
where $\hat{a}_k = (u_0, \mathcal{e}_k), k \in \mathbb{Z}^d$. Thus we have
\[
\hat{u}_k(T + t, \omega) - \hat{u}_k(t, \theta_T \omega) = e^{i|k|^2 (T+t)} \hat{a}_k(\omega) - e^{i|k|^2 t} \hat{a}_k(\theta_T(\omega)) - i \int_0^{T+t} e^{i|k|^2 (t+s)} d\xi_k(s) + i \int_0^T e^{i|k|^2 (t-s)} d\xi_k(s, \theta_T \omega). \tag{2.4}
\]
And by change of variables, (2.4) reduces to
\[ \hat{u}_k(T + t, \omega) - \hat{u}_k(t, \theta_T \omega) = e^{i|k|^2 T} \left( e^{i|k|^2 T} a_k(\omega) - a_k(\theta_T(\omega)) \right) \]
\[ - e^{i|k|^2 T} \int_0^T e^{i|k|^2 (T - s)} d\xi_k(s). \]
(2.5)

Thus, if \( u \) is a random periodic variation solution, then there holds
\[ a_k(\omega) - e^{-i|k|^2 T} a_k(\theta_T(\omega)) = i \int_0^T e^{-i|k|^2 s} d\xi_k(s), \forall k \in \mathbb{Z}^d. \]
(2.6)

By iteration of (2.6), there holds
\[ a_k(\omega) - e^{-i|k|^2 LT} a_k(\theta_{LT}(\omega)) = i \int_0^{LT} e^{-i|k|^2 s} d\xi_k(s), \forall L \in \mathbb{Z}_+. \]
(2.7)

Since we have by be Cauchy-Schwartz inequality and the Itô isometry formula that
\[ \mathbb{E} \left| a_k(\omega) - e^{-i|k|^2 LT} a_k(\theta_{LT}(\omega)) \right|^2 \leq 4\mathbb{E}|a_k|^2 \]
\[ \mathbb{E} \left[ i \int_0^{LT} e^{-i|k|^2 s} d\xi_k(s) \right]^2 = |\beta_k|^2 LT, \]
the contradiction follows if \( \beta_k \neq 0 \) by letting \( L \to \infty \). Therefore, \( \beta_k = 0 \) for all \( k \in \mathbb{Z}^d \), which leads to \( \Phi = 0 \), since \( \{\hat{e}_j\}_{j \in \mathbb{Z}} \) and \( \{e_k\}_{k \in \mathbb{Z}^d} \) are complete bases. Hence, no random periodic variation solution exists if \( \Phi \neq 0 \).

**Proposition 2.2** Let us consider the linear stochastic Schrödinger equation with combined noise:
\[ idu = \Delta u dt + (\lambda u + f(t,x)) \circ dB_t, \quad x \in \mathbb{T}^d, \]
(2.8)
where \( \lambda \in \mathbb{R} \), \( f \) is a real valued function which is periodic in \( t \), \( f(t + T_1) = f(t) \), and smooth in \( x \in \mathbb{T}^d \). Then \( u \) is an RPVS if \( \mathcal{F}_0 \) measurable initial data \( u_0 \in L^2(\Omega, L^2_\beta) \) for (2.8) if only if \( f = 0 \) and \( u = 0 \).

**Proof** Since (2.8) is non-autonomous, for RPVS, we use the definition in (1.8). Let us choose the eigenfunctions \( \{e_k\} \) for Laplacian to be real valued functions, e.g. \( \{\sin(j \cdot x), \cos(j \cdot x)\}_{j \in \mathbb{Z}} \). The solution is written as
\[ \hat{u}_k(t) = e^{i|k|^2 - i\lambda B_t} a_k(\omega) - i \int_t^T e^{i\lambda \int_s^t f_k(h)dh + i|k|^2 (t - \tau)} e^{-i\lambda (B_{\tau} - B_s)} \hat{f}_k(\tau) dB_\tau, \]
(2.9)
where \( a_k = (u_0, e_k), \ k \in \mathbb{Z}^d \). Then one has by change of variables that
\[ \hat{u}_k(T + t, \omega) - \hat{u}_k(t, \theta_T \omega) = e^{i|k|^2 T} \left( e^{i|k|^2 T} a_k(\omega) - a_k(\theta_T(\omega)) \right) \]
\[ - e^{i|k|^2 T} \int_0^T e^{i\lambda \int_0^s f_k(h)dh + i|k|^2 (T - \tau)} e^{i\lambda B_\tau} \hat{f}_k(\tau) dB_\tau \]
\[ - e^{i|k|^2 T} \int_0^T e^{i\lambda \int_0^s f_k(h)dh + i|k|^2 (T - \tau)} e^{i\lambda B_\tau} \hat{f}_k(\tau) dB_\tau. \]
(2.10)

Now, we prove RPVS exists iff \( f = 0 \) and \( u = 0 \). Assume that \( u \) is an RPVS, i.e. \( \Delta_T u(t) = \xi \) for some \( \mathcal{F} \)-measurable random variable \( \xi \) and all \( t \geq 0 \).

**Step 1** Taking the covariance of both sides of (2.10), we obtain by the Itô isometry formula and the Cauchy-Schwartz inequality that
\[ \sum_{k \in \mathbb{Z}^d} \int_0^{T} \left| \hat{f}_k(\tau) \right|^2 e^{i\lambda \int_0^\tau f_k(h)dh + i|k|^2 (T - \tau)} dB_\tau \leq C(T) + \mathbb{E}|\xi|^2 \]
(2.11)
for all \( t \geq 0 \). Since \( f \) is real valued and we have taken the orthogonal basis to be real functions, we see \( \{ \hat{f}_k \} \) are real for \( k \in \mathbb{Z}^d \). Then by \( f(t + T_1) = f(t) \) for all \( t \geq 0 \), one has
\[
|e^{i\lambda f^T_{t+nT_1} f_k(h)dh} - e^{i\lambda f^T_{t+nT_1} f_k(h-T)dh}|^2 = |e^{i\lambda f^T_{t} f_k(h)dh} - e^{i\lambda f^T_{t} f_k(h-T)dh}|^2,
\]
for \( n \in \mathbb{Z} \). Thus (2.11), (2.12) show
\[
C(T) \geq \int_{T}^{T+T}|\hat{f}_k|^2 |e^{i\lambda f^T_{t+nT_1} f_k(h)dh} - e^{i\lambda f^T_{t+nT_1} f_k(h-T)dh}|^2 d\tau
\]
\[
= \int_{T}^{T+T}|\hat{f}_k|^2 |e^{i\lambda f^T_{t} f_k(h)dh} - e^{i\lambda f^T_{t} f_k(h-T)dh}|^2 d\tau
\]
\[
= \sum_{0 \leq j \leq n-1} \int_{jT}^{(j+1)T}|\hat{f}_k|^2 |e^{i\lambda f^T_{t} f_k(h)dh} - e^{i\lambda f^T_{t} f_k(h-T)dh}|^2 d\tau
\]
\[
= n \int_{T}^{T+T}|\hat{f}_k|^2 |e^{i\lambda f^T_{t} f_k(h)dh} - e^{i\lambda f^T_{t} f_k(h-T)dh}|^2 d\tau,
\]
where in the last line we applied the periodicity of \( f \), (2.12) and change of variables. Thus, letting \( n \to \infty \), we get
\[
\sum_{k \geq 0} \int_{T}^{T+T}|\hat{f}_k|^2 |e^{i\lambda f^T_{t} f_k(h)dh} - e^{i\lambda f^T_{t} f_k(h-T)dh}|^2 d\tau = 0,
\]
which by the periodicity of \( f \) further shows that
\[
\hat{f}_k(t) \left( e^{i\lambda f^T_{t} f_k(h)dh} - e^{i\lambda f^T_{t} f_k(h-T)dh} \right) = 0
\]
holds for all \( t \geq 0 \). Assume that \( \hat{f}_k \) is not identically zero. Let \( (t_1, t_2) \) be any interval contained in \( \hat{f}_k^{-1}((0, \infty)) \), then, choosing \( |t_1 - t_2| \) to be sufficiently small, we have for \( t \in (t_1, t_2) \)
\[
\int_{L}^{L} \hat{f}_k(h)dh - \int_{T}^{T} \hat{f}_k(h-T)dh = 2\pi L
\]
for some \( L \in \mathbb{Z} \) which depends on \( t_1, t_2 \) and is independent of \( t \in (t_1, t_2) \). Taking derivatives to \( t \in (t_1, t_2) \) yields \( \hat{f}_k(t) = \hat{f}_k(t - T) \) for \( t \in (t_1, t_2) \).

Since \( \hat{f}_k^{-1}((0, \infty)) \) and \( \hat{f}_k^{-1}((-\infty, 0)) \) consist of countable numbers of open intervals, we have shown:

If \( f \) is nontrivial, then \( T \) is a period of \( f \).

(2.13)

Back to (2.10), we see, for all \( t \geq 0 \),
\[
\hat{u}_k(T + t, \omega) - \hat{u}_k(t, \theta_T \omega) = \mathcal{E}(e^{ik^2t-i\lambda B_{t+T}} a_k(\omega)) - \mathcal{E}(e^{ik^2t-i\lambda B_{t}} a_k(\theta_T(\omega)))
\]
\[
- i \mathcal{E}(e^{ik^2t-i\lambda B_{t+T}}) \int_{0}^{T} e^{i\lambda f^T_{t+T} f_k(h)dh} e^{ik^2(T-h)} e^{i\lambda B_{t}} \hat{f}_k(\tau)dB_{\tau}.
\]

Recall \( \Delta_{T-\omega} = \xi \). Denote \( \xi_k = \langle \xi, e_k \rangle \), \( k \in \mathbb{Z}^d \). Since \( u_0 \) is \( \mathcal{F}_0 \) measurable, the underline parts are \( \mathcal{F}_T \) measurable. Taking conditional expectation \( \mathcal{E}(.|\mathcal{F}_T) \), by the independence of \( B_{t+T} - B_{T} \) and \( \mathcal{F}_T \), we have
\[
\mathcal{E}(\xi_k|\mathcal{F}_T) = \mathcal{E}(e^{ik^2t-i\lambda B_{t+T-B_{T}}}) e^{-i\lambda B_{T}} \mathcal{E}(e^{ik^2T} a_k(\omega)) - \mathcal{E}(e^{ik^2T} a_k(\theta_T(\omega)))
\]
\[
- i \mathcal{E}(e^{ik^2t-i\lambda B_{t+T-B_{T}}}) e^{-i\lambda B_{T}} \int_{0}^{T} e^{i\lambda f^T_{t+T} f_k(h)dh} e^{ik^2(T-h)} e^{i\lambda B_{T}} \hat{f}_k(\tau)dB_{\tau}.
\]

Thus since
\[
\mathcal{E}(e^{ik^2t-i\lambda (B_{t+B_{T}})}) = e^{ik^2t} e^{-\frac{i}{2} \lambda^2 t},
\]
by taking covariance of (2.14) and the Itô formula, we get that
\[
E \left| E(\xi_k | \mathcal{F}_T) \right|^2 \leq e^{-\frac{1}{2}\lambda^2 T} \left( 2E|a_k|^2 + \int_0^T E \left| \hat{f}_k(\tau) \right|^2 \, d\tau \right),
\]
which as \( t \to \infty \) yields \( E(\xi_k | \mathcal{F}_T) = 0 \), a.s.

Inserting this to (2.14) shows
\[
a_k(\omega) - e^{i\lambda B_T - i|k|^2 T} a_k(\theta_T(\omega)) = \int_0^T e^{i\lambda f^{i+T}_k(\theta_T(\omega))} e^{-i|k|^2 T + i\lambda B_T} \hat{f}_k(\tau) \, dB_T.
\]

Then by iteration we obtain for all \( t \geq 0 \)
\[
a_k(\omega) - e^{i\lambda B_T - i|k|^2 T} a_k(\theta_{LT}(\omega)) = \sum_{0 \leq j \leq L - 1} \int_{jT}^{(j+1)T} e^{i\lambda f^{i+T}_k(\theta_T(\omega))} e^{-i|k|^2 T + i\lambda B_T} \hat{f}_k(\tau - jT) \, dB_T,
\]
where we chose \( t_j := LT - (j+1)T \) and applied the fact that \( T \) is a period of \( f \). Then by the Itô formula,
\[
4E|a_k(\omega)|^2 \geq L^2 \int_0^T E \left| \hat{f}_k(\tau) \right|^2 \, d\tau.
\]
Therefore, by letting \( L \to \infty \), we see there exists no RPVS if \( f \) is nontrivial.

Step 2: Now, it remains to consider the degenerate case when \( f \equiv 0 \). In this case, (2.10) reduces to
\[
a_k(\omega) = e^{i(\lambda B_T - i|k|^2 LT)} a_k(\theta_{LT}(\omega)),
\]
which combined with \( \theta_T^t \mathbb{P} = \mathbb{P} \) gives
\[
e^{-i|k|^2 LT} E a_k = E(e^{-i\lambda B_T} a_k) = E(e^{-i\lambda B_T}) E(a_k) = e^{-\frac{1}{2}LT\lambda^2} E(a_k),
\]
where in the second equality we used \( \sigma(B_s - B_t : s, t \leq 0) \), \( u_0 \) is \( \mathcal{F}_0 \) measurable and belongs to \( L^2(\Omega; L^2) \), we obtain by (2.17) and the representation theorem for square integrable random variables (see Theorem 1.1.3 in [11]) that there exists a unique adapted process \( M_t \) such that
\[
a_k(\omega) = \int_0^T M(t, \omega) \, dB_t.
\]
Thus, by the same reason as (2.16), we see from (2.15) that
\[
E(a_k(\theta_{LT\omega})|\mathcal{F}_0) = E(e^{-i|k|^2 LT} e^{-i\lambda B_T} a_k|\mathcal{F}_0) = e^{-i|k|^2 T} E(e^{-i\lambda B_T}) a_k = e^{-\frac{1}{2}LT\lambda^2 - i|k|^2 T} a_k(\omega).
\]

Then, applying (2.18), we arrive at
\[
E(\int_{-\infty}^0 M(t, \theta_{LT\omega}) \, dB_t(\theta_{LT\omega})|\mathcal{F}_0) = e^{-\frac{1}{2}LT\lambda^2 - i|k|^2 T} \int_{-\infty}^0 M(t, \omega) \, dB_t,
\]
which by change of variables gives
\[
\int_{-\infty}^0 M(t - T, \theta_{LT\omega}) \, dB_t = \int_{-\infty}^{0} e^{-\frac{1}{2}LT\lambda^2 - i|k|^2 T} M(t, \omega) \, dB_t.
\]
Since \( M(t - T, \theta_{LT\omega}) \) and \( M(t, \omega) \) are \( \mathcal{F}_t \) adapted and belong to \( L^2((-\infty, 0) \times \Omega) \), by the Itô formula, (2.19) shows
\[
M(t - T, \theta_{LT\omega}) = e^{-\frac{1}{2}LT\lambda^2 - i|k|^2 T} M(t, \omega), \quad \forall t \leq 0.
\]
Taking conditional expectation of both sides of (2.15) w.r.t. \( \mathcal{F}_0^T \) gives
\[
E(a_k(\theta_T \omega) | \mathcal{F}_0^T) = E(e^{-i|k|^2t}e^{-i\lambda_Br}a_k | \mathcal{F}_0^T) = e^{-i|k|^2t-\lambda_Br}E(a_k | \mathcal{F}_0^T),
\]
which combined with (2.18) shows
\[
\int_0^T M(t-T,\theta_T \omega)dB_t = e^{-i|k|^2T-\lambda_Br}E(a_k | \mathcal{F}_0^T). \tag{2.21}
\]
However, since \( \mathcal{F}_0^T \) is independent of \( \mathcal{F}_0 \), we see \( E(a_k | \mathcal{F}_0^T) = E(a_k) = 0 \).

Thus (2.21) shows
\[
\int_0^T M(t-T,\theta_T \omega)dB_t = 0,
\]
which by the Itô formula further yields
\[
M(t-T,\theta_T \omega) = 0, \forall t \in [0,T]. \tag{2.22}
\]
Since \( \theta_T \) is invertible, taking (2.22) as the starting point and doing iteration by (2.20) illustrate that
\[
M(t,\omega) = 0, \forall t \leq 0.
\]
Therefore, we conclude from (2.18) that
\[ a_k = 0, \text{ a.s. in } \Omega \text{ for all } k \in \mathbb{Z}^d. \]

3. Finite Dimensional Case

In this section, we prove the results stated in Section 1 for SODEs.

**Proposition 3.1** Let \( N_0, N_1 \) be \( C^1 \) bounded functions of \( p \in \mathbb{R}^d \). Consider the SODE for \( (p,q) \in \mathbb{R}^d \times \mathbb{T}^d \):
\[
\begin{cases}
dp = 0, \\
dq = \nabla_p N_0(p)dt + \nabla_p N_1(p)dB_t, \\
(p_0, q_0) = (\xi, \eta) \in L^2(\Omega; \mathbb{R}^{2d}) \text{ is } \mathcal{F}_0 \text{ measurable.}
\end{cases} \tag{3.1}
\]

Then we have
1) \((p,q)\) is RPVS if and only if \( \nabla_p N_0(\xi), \nabla_p N_1(\xi) \) are independent of \( \omega \);
2) \((p,q)\) is RPS if and only if \( \nabla_p N_0(\xi) = 2\pi \bar{k}/T \) with \( k \in \mathbb{Z}^d, \nabla_p N_1(\xi) = 0 \) and \( \xi, \eta \) are deterministic.

**Proof** Step 1 The solution for (3.1) is given by
\[
p(t, \omega) = \xi(\omega), \quad q(t, \omega) = \eta(\omega) + \bar{N}_0(\xi(\omega))t + \bar{N}_1(\xi(\omega))B_t,
\]
where we adopt the notation \( \bar{N}_i(p) = \nabla_p N_i(p) \) for simplicity. Then we have
\[
\Delta_T p(t, \omega) = \Delta_T \xi
\]
\[
\Delta_T q(t, \omega) = \left( \bar{N}_0(\xi(\omega)) - \bar{N}_0(\xi(\theta_T \omega)) \right) t + T\bar{N}_0(\xi(\omega))
+ \left( \bar{N}_1(\xi(\omega)) - \bar{N}_1(\xi(\theta_T \omega)) \right) B_t + \bar{N}_1(\xi(\theta_T \omega))B_T + \eta(\omega) - \eta(\theta_T \omega). \tag{3.2}
\]
The covariance of (3.2) is of order \( c_0 t^2 \) as \( t \to \infty \), if \( c_0 \) defined by
\[
c_0 := E \left| \bar{N}_0(\xi(\omega)) - \bar{N}_0(\xi(\theta_T \omega)) \right|^2
\]
does not vanish. If \((p,q)\) is an RPVS, then the covariance of \( \Delta_T q(t, \omega) \) is identical w.r.t. \( t \geq 0 \). Thus \( c_0 = 0 \). Since \( \xi \in \mathcal{F}_0 \) measurable, \( \bar{N}_0(\xi(\omega)) - \bar{N}_0(\xi(\theta_T \omega)) \) is \( \mathcal{F}_T \) measurable. Then \( B_{t+T} - B_T \) is independent of \( \Delta_T \bar{N}_1(\xi) \). Hence there holds
\[
E \left| \bar{N}_1(\xi(\omega)) - \bar{N}_1(\xi(\theta_T \omega))B_{t+T} \right|^2 = E \left| \Delta_T \bar{N}_1(\xi)(B_{t+T} - B_T) \right|^2 + O(E \left| B_T \Delta_T \bar{N}_1(\xi) \right|^2)
\]
By (3.10) and (3.11), we have

Thus by \(c_0 = 0\), the covariance of (3.2) is of order \(c_1t\) as \(t \to \infty\) if \(c_1 := E|\Delta T \tilde{N}_1(\xi)|^2\) is not zero. Hence, \(c_1 = 0\) as well. Now we summarize that if \((p, q)\) is an RPVS then

\[N_i(\omega) = \tilde{N}_i(\theta_T \omega), \forall i = 0, 1.\]

(3.3) implies \(N_i(\xi)\) is independent of \(\omega\) by similar arguments in Section 2. In fact, the representation theorem of square random variables show

\[\tilde{N}_i(\xi(\omega)) = \tilde{N}_i(\xi(\theta_T \omega))\]

for \(M_i \in L^2((-\infty, 0) \times \Omega)\) because \(\tilde{N}_i(\xi) \in L^2(\Omega)\) is \(\mathcal{F}_0\) measurable for \(i = 0, 1\). Inserting (3.4) into (3.3) yields

\[\int_0^T M_i(t, \omega) \, dB_t = \int_{-\infty}^T M_i(t - T, \theta_T \omega) \, dB_t, \forall i = 0, 1.\]

Then the Itô formula shows

\[M_i(t - T, \theta_T \omega) = 0, \quad t \in [0, T],\]
\[M_i(t - T, \theta_T \omega) = M_i(t, \omega), t \leq 0.\]

Therefore, we arrive at

\[\tilde{N}_i(\xi(\omega)) = E\tilde{N}_i(\xi), \forall i = 0, 1,\]

namely, they are independent of \(\omega\).

Step 2 If we assume furthermore that \((p, q)\) is an RPS, then by (3.5) and (3.2), for some \(\vec{k} \in \mathbb{Z}^d\) there holds

\[\Delta T \xi = 0,\]
\[B_T E\tilde{N}_1(\xi) + TE\tilde{N}_0(\xi) + \Delta T \eta = 2\vec{k} \pi, \forall \omega \in \Omega.\]

(3.6)
(3.7)

Similar arguments as Step 1 show that (3.6) implies \(\xi\) is deterministic. And we see \(TE\tilde{N}_0(\xi) = 2\vec{k} \pi\) by applying mathematical expectation to (3.7) since \(\theta_T \mathbb{P} = \mathbb{P}\). Writing \(\eta = E(\eta) + \int_{-\infty}^0 h(t) \, dB_t\) for some \(h \in L^2((-\infty, 0) \times \Omega)\), taking conditional expectation of (3.7) w.r.t. \(\mathcal{F}_0^T\), one obtains

\[B_T E\tilde{N}_1(\xi) = \int_0^T h(t - T, \theta_T \omega) \, dB_t,\]

(3.8)

And taking conditional expectation of (3.7) w.r.t. \(\mathcal{F}_0\) gives

\[\int_{-\infty}^0 h(t, \omega) \, dB_t = \int_{-\infty}^0 h(t - T, \theta_T \omega) \, dB_t,\]

(3.9)

Since \(B_T = \int_0^T \, dB_t\), the Itô formula with (3.8) shows

\[h(t - T, \theta_T \omega) = E\tilde{N}_1(\xi), t \in [0, T].\]

(3.10)

Meanwhile, (3.9) with the Itô formula yields

\[h(t - T, \theta_T \omega) = h(t, \omega), t \in (-\infty, 0).\]

(3.11)

By (3.10) and (3.11), we have

\[h(t, \omega) = E\tilde{N}_1(\xi), t \in (-\infty, 0),\]
which belongs to $L^2((-\infty, 0) \times \Omega)$ if and only if $E\tilde{N}_1(\xi) = 0$. Hence, we have deduced

$$E\tilde{N}_1(\xi) = 0, E\tilde{N}_0(\xi) = 2\tilde{k}\pi/T, \eta(\omega) = E(\eta), \xi(\omega) = E(\xi).$$

Therefore, by Step 1, we conclude that $(p, q)$ is an RPS if and only if

$$\tilde{N}_0(\xi) = 2\tilde{k}\pi/T, \tilde{k} \in \mathbb{Z}^d, \tilde{N}_1(\xi) = 0, \xi, \eta \text{ are deterministic.}$$

We have several examples:

**Example 3.1** Let $d = 2$, $N_0(p) = f(p_1)$, $N_1(p) = g(p_1)$, then $(p_0, q_0) = (c_1, \phi, \psi_1, \psi_2)$, where $c_1$ is deterministic and $\phi, \psi_1, \psi_2$ are $\mathcal{F}_0$ adapted random variables, evolves to an RPVS.

It is an RPS iff $\dot{f}(c_1) = \dot{g}(c_1) = 0$, $\psi_1, \psi_2$ are deterministic.

Given $\varrho > 0$, $p^* \in \mathbb{R}^d$, define the set $D_{p^*, \varrho}$ by

$$D_{p^*, \varrho} = \{(p, q) \in \mathbb{C}^{2d} : |p - p^*| \leq \varrho, |3q| \leq \varrho\},$$

where $\exists q = (\exists q_1, \ldots, \exists q_d)$. Let $\mathfrak{A}_{p^*, \varrho}$ be the set of complex valued continuous functions on $D_{p^*, \varrho}$ which are analytic functions in the interior, $2\pi$-periodic in $q$ (i.e. $f(p, q + 2\pi) = f(p, q)$, $\forall f \in \mathfrak{A}_{p^*, \varrho}$) and real valued for $(p, q) \in \mathbb{R}^{2d}$.

**Proposition 3.2** Let $H_0$ be a Hamiltonian function of the form $H_0 = N(p) + \epsilon H(p, q)$ with $N, \tilde{H} \in \mathfrak{A}_{p^*, \varrho}$. Denote

$$\tilde{X}_0 = (\partial_{p_1} N(p^*), \ldots, \partial_{p_d} N(p^*)), \quad (B_{ij}) = \frac{\partial^2 N(p)}{\partial p_i \partial p_j}(p^*).$$

Assume that $(B_{ij})$ is a non-singular $d \times d$ matrix and $\tilde{X}_0$ satisfies the Diophantine condition, i.e. there exists a positive constant $\gamma$ such that $\tilde{X} \in \Omega$, defined by

$$\Omega_\gamma = \{\tilde{X} \in \Omega \subset \mathbb{R}^d : |\tilde{X} \cdot \tilde{k}| \geq \gamma |\tilde{k}|^{-d}, \forall \tilde{k} \in \mathbb{Z}^d, \tilde{k} \neq 0\}.$$

- Consider the stochastic ODE (1.1) in $\mathbb{R}^1 \times \mathbb{R}$ with Hamiltonian functions $\{H_\alpha\}_{0 \leq \alpha \leq 1}$ which satisfy

$$\{H_0, H_\alpha\} = 0, \quad 0 \leq \alpha \leq 1. \quad (3.12)$$

And assume that $\{H_\alpha\}_{\alpha = 1}$ are analytic functions in $D_{p^*, \varrho}$ and continuous to the boundary as well. Then there exists a constant $\epsilon_0 > 0$ such that for given $\epsilon \in [0, \epsilon_0]$ there exists a symplectic transform $\Psi : D_{p^*, \varrho} \rightarrow D_{p^*, \varrho}$ and solution $(p(t), q(t))$ to (1.1) satisfying

$$\Psi(p(t), q(t)) = (p^*, \sum_{\alpha = 0}^{l} \lambda_\alpha Z_\alpha^\varsigma).$$

for some $\varsigma > \frac{\gamma}{2l}$, constants $\lambda_\alpha \in \mathbb{R}$, $\alpha = 1, \ldots, l$, and $\lambda_0 = \partial_{p_1} N(p^*)$.

- Let $d \geq 1$. Consider the stochastic ODE (1.1) in $\mathbb{R}^d \times \mathbb{T}^d$ with Hamiltonian functions $H_0(p, q) = N(p) + \epsilon \tilde{H}(p, q)$ and $\{F_{\beta}(H_0)\}_{1 \leq \beta \leq 1}$:

$$\langle dp, dq \rangle = J\nabla H dt + \sum_{\beta = 1}^{l} J\nabla (F_{\beta}(H)) \circ dZ^\beta_t,$$

where $\{F_{\beta}\}$ are smooth functions. Then there exists a constant $\epsilon_0 > 0$ such that for given $\epsilon \in [0, \epsilon_0]$ there exists a symplectic transform $\Psi : D_{p^*, \varrho} \rightarrow D_{p^*, \varrho}$ satisfying

$$\Psi(p(t), q(t)) = (p^*, \tilde{X}_0 t + \sum_{\beta = 1}^{l} \tilde{X}_{\beta} Z^\beta_t). \quad (3.14)$$

**Proof** Step 1 Let $\lambda = \nabla N(p^*)$ be the unperturbed frequency. By the classic KAM theorem, there exists a symplectic differmorphism $\Phi : B_{p^*, \frac{1}{2} \varrho} \rightarrow B_{p^*, \frac{1}{2} \varrho}$ such that

$$H \circ \Phi = z + \lambda \cdot (p - p^*) + R(p, q) \quad (3.15)$$

in $(p, q) \in B_{p^*, \frac{1}{2} \varrho}$ for some $z \in \mathbb{R}$ and an analytic function $R(\cdot, \cdot)$ which is of the order $|p - p^*|^2$. 

We will frequently use the following identity
\[(d\Phi)^{-1}X_H = X_{H \circ \Phi}.\]  (3.16)
Denote \(\Psi = (\Psi^1, \ldots, \Psi^{2d})\) the inverse of \(\Phi\), then
\[\Psi^k \circ \Phi(p, q) = p_k, \quad k = 1, \ldots, n,\]
\[\Psi^j \circ \Phi(p, q) = q_j, \quad j = n + 1, \ldots, 2n.\]
Assume that \((p(t), q(t))\) solve (3.13), then by Itô's formula
\[\Psi^k(p(t), q(t)) = \Psi^k(p(0), q(0)) + \int_0^t m(X_H, X_{\Phi^k})dt + \sum_{a=1}^l \int_0^t m(F_{a}(X_H), X_{\Phi^k}) \circ dZ_t, \quad k = 1, \ldots, n\]  (3.17)
\[\Psi^j(p(t), q(t)) = \Psi^j(p(0), q(0)) + \int_0^t m(X_H, X_{\Phi^j})dt + \sum_{a=1}^l \int_0^t m(F_{a}(X_H), X_{\Phi^j}) \circ dZ_t, \quad j = n + 1, \ldots, 2n.\]  (3.18)
By (3.16) and the fact \(\Phi\) is a symplectic diffeomorphism, we get
\[m(X_{F_{a}(H)}, X_{\Phi^k}) = m(d\Psi F_{a}(H), d\Psi X_{\Phi^k}) = m(X_{F_{a}(H \circ \Phi)}, X_{\Phi^k} \circ \Phi)\]
\[= \begin{cases} m(X_{F_{a}(H \circ \Phi)}, X_{p^*}), & a = 1, \ldots, n, \\ m(X_{F_{a}(H \circ \Phi)}, X_{q^*}), & a = n + a', \ a' = 1, \ldots, n \end{cases}\]
\[= \begin{cases} O(|p - p^*|^2), & a = 1, \ldots, n, \\ F'_a(z)\lambda_a + O(|p - p^*|), & a = n + a', \ a' = 1, \ldots, n \end{cases}\]
where in the last line we applied the Taylor expansion to \(F_a\) at \(z\) and used (3.15) to expand \(H \circ \Phi\).

Thus (3.17), (3.18) show
\[d[\Psi(p(t), q(t))] = (O(|p - p^*|^2), \lambda + O(|p - p^*|)) dt + \sum_{a=1}^l (O(|p - p^*|^2), F'_a(z)\lambda + O(|p - p^*|)) dZ^a_t,\]
which implies that (3.14) is a solution.

Step 2 It remains to prove the left \(S^1 \times \mathbb{R}^1\) case. Applying (3.15) to \(H_0\) gives the expansion
\[H_0 \circ \Phi = z_0 + \lambda(p - p^*) + R(p, q)\]
for some symplectic diffeomorphism \(\Phi\), \(z_0 \in \mathbb{R}\), \(\lambda = H'_0(p^*)\), and \(R\) is of order \(|p - p^*|^2\). Since \(\Phi\) keeps the symplectic form,
\[\{H_0 \circ \Phi, H_{\beta} \circ \Phi\} = \{H_0, H_{\beta}\} = 0,\]
where we applied (3.12) in the last line. Therefore, we arrive at
\[0 = \{H_0 \circ \Phi, H_{\beta} \circ \Phi\} = \{z + \lambda(p - p^*) + R(p, q), H_{\beta} \circ \Phi\}\]
for all \(\beta = 0, \ldots, l\). Denote \(H_{\beta} = H_{\beta} \circ \Phi\). Then one sees
\[\lambda \partial_q H_{\beta} + O(|p - p^*|) \partial_q H_{\beta} = O(|p - p^*|^2) \partial_p H_{\beta}.\]  (3.19)
Expand \(H_{\beta}\) into powers of \(|p - p^*|\):
\[H_{\beta} = A_0(q) + A_1(q)(p - p^*) + A_2(q)(p - p^*)^2 + \cdots\]  (3.20)
Then, comparing the coefficients of zero and one order of \( |p - p^*| \), (3.19) gives
\[
\lambda \partial_q A_0(q) = 0, \quad \lambda \partial_q A_1(q) = 0.
\]
Since \( \lambda \neq 0 \), \( A_0(q) \) and \( A_1(q) \) are constants. Thus (3.20) shows the symplectic transform \( \Phi \) transforms all \( \{ H_\alpha \} \) into the canonical form
\[
H_\beta \circ \Phi = z_\beta + \lambda_\beta (p - p^*) + R(p, q).
\]
Then the same argument in Step 1 gives the desired result.

References: