Time-Consistent Investment and Reinsurance Problems for Mean-Variance Insurers with Default Risk Under Variance Premium Principle

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Abstract: In this paper, we consider an optimal investment-reinsurance strategy problem under mean-variance criterion for an insurer, who can trade in a risk-free asset, a stock whose price process is described by Heston model and a defaultable bond. The insurer can purchase proportional reinsurance or acquire new insurance business. In particular, we assume that the premium of insurance and reinsurance are calculated through the variance premium principle. Applying game theoretic perspective, we solve the extended Hamilton-Jacobi-Bellman (HJB) equations and derive optimal time-consistent investment-reinsurance strategies for the pre-default case and the post-default case, respectively. Finally, numerical simulations are provided to analyze the impact of model parameters on the optimal reinsurance and investment strategies.

Key words: Time-consistent strategy; Mean-variance criterion; Defaultable bond; Extended HJB equation

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1. Introduction

Investment and reinsurance are two important parts of the financial market. The study of investment-reinsurance is favored by many scholars. Therefore, these problems for an insurer has been extensively studied in the literature. For example, Browne\(^1\) studied the optimal investment problem for an insurer with the diffusion risk model to maximize the utility of terminal wealth and minimize the probability of ruin. YANG and ZHANG\(^2\) added jump diffusion on the basis of Browne\(^1\). XU et al.\(^3\), LIANG et al.\(^4\) and GUAN et al.\(^5\) investigated optimal investment and reinsurance strategies in different situations. Furthermore, BI and GUO\(^6\) studied an optimal investment problem for an insurer in a jump diffusion financial...
market under the mean-variance criterion and HUANG et al.\cite{7} studied the optimal control problem for an insurer with constrained control variables.

To the best of our knowledge, many literatures are based on expected value premium principle. This principle is embodied in two risks with the same mean, and the same premium will be charged. But everyone agrees that the underlying danger may appear strongly different. Thus, one could therefore opt for the variance premium principle. SUN et al.\cite{8} investigated the optimal control problem for an insurer with constrained control variables. In this paper, both the insurance and reinsurance premium payments are calculated by using the variance premium principle which is more suitable for our case.

Although optimal investment-reinsurance problem has been extensively studied, most previous works assumed that the insurers are allowed to invest in one risk-free asset and one stock. The default risk is rarely considered in the modeling framework. Bielecki and Jang\cite{9} investigated the optimal investment for an investor who can invest in a risk-free asset, a stock and a defaultable bond. Capponi and Figueroa-López\cite{10} studied a portfolio optimization problem with a defaultable bond in a regime-switching market. BO et al.\cite{11} considered optimal investment-consumption strategy when the object function is HARA utility function in a defaultable market. ZHAO et al.\cite{12} studied the optimal time-consistent investment-reinsurance strategy for mean-variance insurers with a defaultable security in a jump-diffusion risk model.

In most of the literature mentioned above, the price processes of risky assets are assumed to follow geometric Brownian motion, which implies that the volatilities of the risky assets’ prices are constant or deterministic. Stochastic volatility is an important feature in the asset model and many empirical evidences tend to support that the volatilities of the risky assets’ prices are stochastic. WANG et al.\cite{13} studied investment strategies for an insurer and a reinsurer under the CEV model. SUN and GUO\cite{14} studied optimal mean-variance investment and reinsurance problem for an insurer with stochastic volatility. GU et al.\cite{15} and A and LI\cite{16} studied the reinsurance and investment under stochastic volatility model.

In recent years, optimal reinsurance and investment problems under the mean-variance criterion have drawn much attention. We know that dynamic mean-variance optimization problem is a time-inconsistent problem and it can not be solved by Bellman optimality principle. One approach is to study the corresponding precommitment problem, while the other is to formulate the problem in a game theoretic framework. CHEN and Yam\cite{17} investigated an optimal investment-reinsurance problem for insurers in the market with regime-switching under the mean-variance criterion. SHEN and ZENG\cite{18} investigated an optimal investment-reinsurance mean-variance problem for mean-variance insurers with square-root factor process. ZENG and LI\cite{19} and ZENG et al.\cite{20} derived the time-consistent investment and reinsurance strategies for mean-variance insurers. ZENG et al.\cite{21} derived the robust equilibrium investment and reinsurance strategies for mean-variance insurers.

In this paper, we consider the optimal investment-reinsurance problem for mean-variance insurer in a defaultable market. The insurer’s surplus process is described by a general jump process and both the insurance and reinsurance premium are assumed to be calculated via the variance premium principle. The insurer is allowed to invest in a risk-free asset, a stock and a defaultable bond. We formulate the problem in a game theoretic framework to deal with
time inconsistency. By using stochastic control theory, we solve the extented HJB equations and derive optimal investment and reinsurance strategies for the pre-default case and the post-default case, respectively.

The outline of this paper is as follows. In Section 2, we describe the model dynamics of the surplus process and the financial market. In Section 3, we propose optimal control problem under mean-variance criterion. In Section 4, we derive the closed-form expressions of the optimal investment-reinsurance strategy and the value function for the pre-default case and the post-default case, respectively. In Section 5, we present some numerical illustrations and sensitivity analysis for our results. Section 6 concludes the paper.

2. The Model

We consider a complete probability space \((\Omega, \mathcal{F}, P)\). Let \(\mathbb{F} := \{\mathcal{F}_t\}_{t \geq 0}\) be the right-continuous, \(P\)-complete filtration generated by two standard Brownian motions \(\{W_1(t)\}\) and \(\{W_2(t)\}\). Denote \(\mathbb{H} := \{\mathcal{H}_t\}_{t \geq 0}\) as the filtration of a default process \(\{H(t)\}\). \(\mathbb{G} := \{\mathcal{G}_t\}_{t \geq 0}\) is the enlarged filtration of \(\mathbb{F}\) and \(\mathbb{H}\), i.e., \(\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_t\). Let \(P\) be the real world probability measure. We assume that there exists a risk neutral measure \(Q\) equivalent to measure \(P\). All stochastic processes introduced below are assumed to be well defined and adapted processes in this space.

We suppose that the risk process of the insurer is described by the Cramèr-Lundberg (C-L) model, i.e.,

\[
dR(t) = cdt - d \sum_{i=1}^{N(t)} Y_i,
\]

where \(c\) is the premium rate, \(\sum_{i=1}^{N(t)} Y_i\) represents the aggregate claims up to time \(t\), \(\{N(t), t \geq 0\}\) is a homogeneous Poisson process with intensity \(\lambda\), and \(\{Y_i, i \geq 1\}\) is a sequence of positive independent and identically distributed random variables with common distribution \(F(y)\) and finite first and second moments \(\mu_1\) and \(\mu_2\) respectively. Assume that the premium rate \(c\) is calculated according to the variance principle, that is, \(ct = \mathbb{E}\left[\sum_{i=1}^{N(t)} Y_i\right] + \eta \text{Var}\left[\sum_{i=1}^{N(t)} Y_i\right] = \lambda(\mu_1 + \eta \mu_2)t\), where \(\eta > 0\) is the relative safety loading of the insurer, \(\mathbb{E}[]\) denotes the expectation and \(\text{Var}[]\) denotes the variance.

As an effective tool of reduce risk, we allow the insurer to purchase proportional reinsurance or acquire new business with the retention level \(q(t) \in [0, \infty)\) at time \(t\). That is, for a claim \(Y_i\) occurring at time \(t\), the insurer pays \(q(t)Y_i\), and the reinsurer or new businessmen pays the outstanding amount \((1 - q(t))Y_i\). For this reinsurance or new business, the premium has to be paid at the rate of \(\mathbb{E}\left[\sum_{i=1}^{N(t)} (Y_i - q(t)Y_i)\right] + \theta \text{Var}\left[\sum_{i=1}^{N(t)} (Y_i - q(t)Y_i)\right] = \lambda[(1 - q(t))\mu_1 + \theta(1 - q(t))^2\mu_2]t\), where \(\theta \geq \eta\) represents the relative safety loading of the reinsurer or new businessmen.

Then the surplus process of the insurer is

\[
dR(t) = [\lambda \mu_1 q(t) + \lambda \mu_2 (\eta - \theta(1 - q(t))^2)]dt - q(t)d \sum_{i=1}^{N(t)} Y_i.
\]

We assume that there are three assets available for an insurer in the financial market: one risk-free asset, one stock and one defaultable bond. The price process of the risk-free
asset is modeled by
\[
\begin{aligned}
&\frac{dB(t)}{B(t)} = rB(t)dt, \quad t \in [0, T], \\
&B(0) = b_0 > 0,
\end{aligned}
\]
where \( r > 0 \) is the interest rate of the risk-free asset. The price process of the stock is given by the Heston model
\[
\begin{aligned}
&\frac{dS(t)}{S(t)} = \left[ (r + \beta L(t))dt + \sqrt{L(t)}dW_1(t) \right], \\
&S(0) = s_0 > 0,
\end{aligned}
\]
where \( \beta, \kappa, \alpha \) and \( \sigma \) are positive constants, \( \kappa \) is the mean reversion rate, \( \alpha \) is the long-run mean, and \( \sigma \) is the volatility of the volatility. \( \{W_1(t)\} \) and \( \{W_2(t)\} \) are two one-dimensional standard Brownian motions with \( \text{Cov}(W_1(t), W_2(t)) = \rho t \), where \( \rho \in [-1, 1] \). Moreover, we assume that \( 2\alpha \geq \sigma^2 \) is required to ensure that zero is not accessible or \( L(t) \) is almost surely nonnegative.

Let \( \tau \) be a nonnegative random variable and the first jump time of a Poisson process defined on \( (\Omega, \mathcal{F}, P) \), representing the default time. For each \( t \geq 0 \), the default process \( \{H(t)\} \) is defined by \( H(t) := 1_{[\tau \leq t]} \). Furthermore, we assume that the default process \( \{H(t)\} \) has a constant intensity \( h^P > 0 \). Then the process \( \{M^P(t)\} \) is given by the following equation
\[
M^P(t) := H(t) - \int_0^t (1 - H(u^-))h^P du,
\]
which is a \((\mathcal{G}, P)\)-martingale.

Let \( \zeta \in (0, 1) \) denote the loss rate and \( 1/\Delta \geq 1 \) denote the default risk premium. The dynamics of the defautable bond price under \( P \) follows from
\[
dP(t) = P(t^-)[r dt + (1 - H(t))(1 - \Delta)\delta dt - (1 - H(t^-))\zeta dM^P(t)].
\]
The dynamics of the defautable bond price is derived in Appendix.

Let \( X^\pi(t) \) denote the insurer’s wealth at time \( t \), and \( \pi_1(t) \) and \( \pi_2(t) \) be the dollar amounts invested in the stock and the defautable bond, respectively. A trading strategy is denoted by \( \pi = (q(t), \pi_1(t), \pi_2(t))_{t \in [0, T]} \). Under strategy \( \pi \), the investor’s wealth process \( \{X^\pi(t)\}_{t \in [0, T]} \) follows from
\[
\begin{aligned}
\frac{dX^\pi(t)}{X^\pi(t)} &= \frac{X^\pi(t) - \pi_1(t) - \pi_2(t)}{S(t)} dB(t) + \frac{\pi_1(t)}{S(t)} dS(t) + \frac{\pi_2(t)}{P(t)} dP(t) + dR(t) \\
&= [rX^\pi(t) + \beta \pi_1(t)L(t) + \pi_2(t)(1 - H(t))(1 - \Delta)\delta + \lambda \mu_1 q(t)] dt + \frac{\pi_1(t)\sqrt{L(t)}dW_1(t) - \pi_2(t)\zeta dM^P(t) - q(t)d\sum_{i=1}^{N(t)} Y_i (2.1) ,}
\end{aligned}
\]
with \( X^\pi(0) = x_0, (1 - H(t^-))dM^P(t) = dM^P(t) \) and under the convention that \( 0/0 = 0 \). This convention is needed to deal with the post-default case, i.e., \( \tau \leq t \), so that \( P(t) = 0 \) and we fix \( \pi_2(t) = 0 \) afterwards.

**Definition 2.1** A trading strategy \( \pi = (q(u), \pi_1(u), \pi_2(u))_{u \in [t, T]} \) is said to be admissible if it satisfies
1) \( (q(u), \pi_1(u), \pi_2(u)) \) is \( \mathcal{F}_u \)-predictable;
2) \( q(u) \geq 0, \ E[\int_t^T (q(u)^2 + \pi_1(u)^2 + \pi_2(u)^2)du] < \infty \), for any \( u \in [t, T] \);
3) \((\pi, X^\pi)\) is the unique strong solution to the stochastic differential equation (2.1).
For any initial condition \((t, x, l, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}_+ \times \{0, 1\}\), let \(H(t, x, l, z)\) denote the set of all admissible strategies. Here \(z\) denotes the initial default state with \(z = 0\) and \(z = 1\) corresponding to the pre-default case \((r > t)\) and the post-default case \((r \leq t)\), respectively.

3. Problem Formulation and Verification Theorem

In this section, we will formulate the problem within a game theoretic framework, which is developed by Björk and Murgoci\(^ {22}\). We consider an optimization problem for the insurer purchasing reinsurance or acquiring new business and investing in the risk-free asset, the stock and the defaultable bond under the mean-variance criterion. At any time \(t \in [0, T]\), the insurer aims to maximize the expectation and minimize the variance, i.e.,
\[
\sup_{\pi \in H} J^\pi(t, x, l, z) := \sup_{\pi \in H} \left\{ E_{t,x,l,z}[X^\pi(T)] - \frac{\gamma}{2} \text{Var}_{t,x,l,z}[X^\pi(T)] \right\},
\]
where \(E_{t,x,l,z}[\cdot] = E[\cdot | X^\pi(t) = x, L(t) = t, H(t) = z], \text{Var}_{t,x,l,z}[\cdot] = \text{Var}[\cdot | X^\pi(t) = x, L(t) = l, H(t) = z]\) and \(\gamma > 0\) is the coefficient of risk aversion of the insurer. Problem (3.1) is time-inconsistent since there is a non-linear function of the expectation of terminal wealth in the variance term. We know that the time-inconsistent problem can not be solved by Bellman optimality principle. Therefore, we view our problem as a non-cooperative game. Then solving the problem (3.1) is actually equivalent to looking for Nash equilibrium points for the game.

In what follows, we provide the definition of the equilibrium strategy and the equilibrium value function for the problem (3.1).

Definition 3.1 For any fixed initial state \((t, x, l, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}_+ \times \{0, 1\}\), consider an admissible strategy \(\pi^*(t, x, l, z)\). Choose four fixed numbers \(\tilde{\pi}_1 \in \mathbb{R}, \tilde{\pi}_2 \in \mathbb{R}, \tilde{q} \in \mathbb{R}+\) and \(\epsilon \in \mathbb{R}_+\) and define the following strategy:
\[
\pi^*(u, \tilde{x}, \tilde{l}, \tilde{z}) := \begin{cases} 
(\tilde{q}, \tilde{\pi}_1, \tilde{\pi}_2), & \text{for} (u, \tilde{x}, \tilde{l}, \tilde{z}) \in [t, t + \epsilon) \times \mathbb{R} \times \mathbb{R}_+ \times \{0, 1\}, \\
\pi^*(u, \tilde{x}, \tilde{l}, \tilde{z}), & \text{for} (u, \tilde{x}, \tilde{l}, \tilde{z}) \in [t + \epsilon, T) \times \mathbb{R} \times \mathbb{R}_+ \times \{0, 1\}.
\end{cases}
\]
If
\[
\lim_{\epsilon \to 0} \inf_{t \in [0, T]} J^{\pi^*}(u, \tilde{x}, \tilde{l}, \tilde{z}) - J^{\pi^*}(t, x, l, z) \geq 0
\]
for all \((\tilde{\pi}_1, \tilde{\pi}_2, \tilde{q}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+\) and \((t, x, l, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}_+ \times \{0, 1\}\), \(\pi^*(t, x, l, z)\) is called an equilibrium strategy and the equilibrium value function \(W(t, x, l, z)\) is defined by
\[
W(t, x, l, z) := J^{\pi^*}(t, x, l, z) = E_{t,x,l,z}[X^{\pi^*}(T)] - \frac{\gamma}{2} \text{Var}_{t,x,l,z}[X^{\pi^*}(T)].
\]
Based on the definition above, equilibrium strategy is time-consistent. Therefore, the aim of the insurer is to find an equilibrium strategy and the corresponding equilibrium value function of the optimization problem (3.1).

Let \(C^{1,2,2}([0, T] \times \mathbb{R} \times \mathbb{R}_+)\) denote the space of \(\phi(t, x, l)\) and its derivatives \(\phi_t(t, x, l), \phi_x(t, x, l), \phi_{xx}(t, x, l), \phi_l(t, x, l)\) and \(\phi_u(t, x, l)\) are continuous on \([0, T] \times \mathbb{R} \times \mathbb{R}_+\) and \(\pi \in H\), define the infinitesimal generator
\[
\mathcal{A}^\pi \phi(t, x, l, z) = \begin{cases} 
\phi_t(t, x, l, 1) + \left[ \phi_x + \beta \phi_l + \lambda \mu_1 q + \lambda \mu_2 \eta - \lambda \mu_2 \theta (1 - q)^2 \right] \phi_x(t, x, l, 1) \\
+ \kappa (\alpha - l) \phi_l(t, x, l, 1) + \frac{1}{2} \sigma^2 \phi_{xx}(t, x, l, 1) + \frac{1}{2} \lambda \phi_{xx}(t, x, l, 1) + \lambda \phi_{ll}(t, x, l, 1) \end{cases},
\]
where \(\phi(t, x, l, 1) \in C^{1,2,2}([0, T] \times \mathbb{R} \times \mathbb{R}_+)\) and \(\pi \in H\), define the infinitesimal generator
\[
\mathcal{A}^\pi \phi(t, x, l, z) = \begin{cases} 
\phi_t(t, x, l, 1) + \left[ \phi_x + \beta \phi_l + \lambda \mu_1 q + \lambda \mu_2 \eta - \lambda \mu_2 \theta (1 - q)^2 + \sigma^2 \right] \phi_x(t, x, l, 1) \\
+ \kappa (\alpha - l) \phi_l(t, x, l, 1) + \frac{1}{2} \sigma^2 \phi_{xx}(t, x, l, 1) + \frac{1}{2} \lambda \phi_{xx}(t, x, l, 1) + \lambda \phi_{ll}(t, x, l, 1) \end{cases}.
\]
Then we have the following verification theorem.

**Theorem 3.1** For the post-default case \((z = 1)\) and pre-default case \((z = 0)\), if there exist two real-valued functions \(V(t, x, l, z), g(t, x, l, z) \in C^{1,2,2}(\{0, T\} \times \mathbb{R} \times \mathbb{R}_+ \times \{0, 1\})\) satisfying the following extended HJB system of equations:

\[
\sup_{\pi \in \Pi} \left\{ \mathcal{A}^V(t, x, l, z) - \mathcal{A}^g \frac{\gamma}{2} g(t, x, l, z)^2 + \gamma g(t, x, l, z) \mathcal{A}^g(t, x, l, z) \right\} = 0, \tag{3.3}
\]

\[
V(T, x, l, z) = x, \tag{3.4}
\]

\[
\mathcal{A}^V g(t, x, l, z) = 0, \tag{3.5}
\]

\[
g(T, x, l, z) = x, \tag{3.6}
\]

and

\[
\pi^* := \arg \sup_{\pi \in \Pi} \left\{ \mathcal{A}^V(t, x, l, z) - \mathcal{A}^g \frac{\gamma}{2} g(t, x, l, z)^2 + \gamma g(t, x, l, z) \mathcal{A}^g(t, x, l, z) \right\},
\]

then \(W(t, x, l, z) = V(t, x, l, z), E_{t, x, l, z}[X^{\pi^*}(T)] = g(t, x, l, z)\) and \(\pi^*\) is the optimal time-consistent strategy.

The proof of this theorem is similar to the one of Theorem 4.1 in [22], and we omit it here.

### 4. Optimal Investment and Reinsurance Strategy

In this section, we derive the optimal strategies and the corresponding value functions in the post-default case \((z = 1)\) and the pre-default case \((z = 0)\), respectively.

I Post-default case: \(z = 1\)

In the post-default case, we have \(P(t) = 0, \tau \leq t \leq T\). Thus \(\pi_2(t) = 0\) for \(\tau \leq t \leq T\). Let \(z = 1\) in (3.3). The HJB equation turns into a relatively form, which is the following partial differential equation

\[
\sup_{\pi \in \Pi} \left\{ V_t(t, x, l, 1) + \left[ \tau x + \beta l \pi_1 + \lambda \mu_1 q + \lambda \mu_2 \eta - \lambda \mu_2 \theta(1 - q)^2 \right] V_x(t, x, l, 1)
+ \kappa(\alpha - l) V_l(t, x, l, 1) + \frac{1}{2} \sigma^2 l V_{xx}(t, x, l, 1) + \frac{1}{2} \sigma^2 V_{xl}(t, x, l, 1) + \pi_1 \sigma l V_{xl}(t, x, l, 1)
- \frac{\gamma}{2} \pi_1^2 l V_{ll}(t, x, l, 1)^2 - \frac{\gamma}{2} \sigma^2 l g_l(t, x, l, 1)^2 - \gamma \pi_1 \sigma l g_l(t, x, l, 1) g_l(t, x, l, 1)
- \lambda \left[ V(t, x, l, 1) + \frac{\gamma}{2} g(t, x, l, 1)^2 \right] + \lambda \tilde{E}[V(t, x - qY, l, 1) - \frac{\gamma}{2} g(t, x - qY, l, 1)^2
+ \gamma g(t, x, l, 1) g(t, x - qY, l, 1)] \right\} = 0 \tag{4.1}
\]

with the boundary conditions (3.4) and (3.6). Let

\[
V(t, x, l, 1) = e^{(T-t)\tau} A(t)l + \tilde{A}(t), \quad A(T) = 0, \quad \tilde{A}(T) = 0,
\]

\[
g(t, x, l, 1) = e^{(T-t)\tau} a(t)l + \pi(t), \quad a(T) = 0, \quad \pi(T) = 0. \tag{4.2}
\]

Substituting (4.2) into (4.1), we obtain

\[
\sup_{\pi \in \Pi} \left\{ A_t l + \tilde{A}_t + \left[ \beta l \pi_1 + \lambda \mu_1 q + \lambda \mu_2 \eta - \lambda \mu_2 \theta(1 - q)^2 \right] e^{(T-t)\tau} + \kappa(\alpha - l) A(t) - \frac{\gamma}{2} \sigma^2 a(t)^2
- \frac{\gamma}{2} \pi_1^2 e^{2(T-t)\tau} - \gamma \pi_1 \sigma l e^{(T-t)\tau} a(t) - \lambda \mu_1 e^{(T-t)\tau} q - \frac{\gamma}{2} \lambda \mu_2 e^{2(T-t)\tau} q^2 \right\} = 0. \tag{4.3}
\]

Differentiating (4.3) w.r.t. \(\pi_1\) and \(q\) gives

\[
\pi_1^* = \frac{\beta - \gamma \rho \sigma a(t)}{\gamma e^{(T-t)\tau}}, \quad q^* = \frac{2\theta}{2\theta + \gamma e^{(T-t)\tau}}. \tag{4.4}
\]
Substituting (4.4) into (4.3) and (3.5) yields
\[ A_t + \lambda \mu_2(\eta - \theta)e^{(T-t)} + \kappa(\alpha - l)A(t) - \frac{\gamma}{2}\sigma^2l\alpha(t)^2 \]
\[ + \frac{l(\beta - \gamma\rho\alpha(t))^2}{2\gamma} + \frac{2\lambda\mu_2^2\theta^2e^{(T-t)}}{2\theta + \gamma e^{(T-t)}} = 0, \]
\[ a_t + \lambda \mu_2(\eta - \theta)e^{(T-t)} + \kappa(\alpha - l)a(t) + \frac{4\lambda\mu_2\theta^2e^{(T-t)}}{2\theta + \gamma e^{(T-t)}} \]
\[ + \frac{l(\beta - \gamma\rho\alpha(t))^2}{\gamma} - \frac{4\lambda\mu_2\theta^2e^{(T-t)}}{(2\theta + \gamma e^{(T-t)})^2} = 0. \]

By separation of variances, we obtain four differential equations. Considering the boundary conditions, we derive
\[ a(t) = \frac{\beta^2}{\gamma(\kappa + \beta\rho\sigma)}(1 - e^{-(\kappa + \beta\rho\sigma)(T-t)}), \]
\[ A(t) = e^{-\kappa(T-t)}\int_t^T e^{\kappa(s)}\left[ -\frac{\gamma}{2}\sigma^2a(s)^2 + \frac{(\beta - \gamma\rho\alpha(s))^2}{2\gamma} \right]ds, \]
\[ \overline{A}(t) = \int_t^T \left[ \lambda\mu_2(\eta - \theta)e^{(T-s)} + \kappa\alpha A(s) + \frac{2\lambda\mu_2^2\theta^2e^{(T-s)}}{2\theta + \gamma e^{(T-s)}} \right]ds, \]
\[ \pi(t) = \int_t^T \left[ \lambda\mu_2(\eta - \theta)e^{(T-s)} + \kappa\alpha a(s) + \frac{4\lambda\mu_2\theta^2e^{(T-s)}}{2\theta + \gamma e^{(T-s)}} - \frac{4\lambda\mu_2\theta^4e^{(T-s)}}{(2\theta + \gamma e^{(T-s)})^2} \right]ds. \]

II Pre-default case: \( z = 0 \)

In the pre-default case, there exist the price process of the defaultable bond. Let \( z = 0 \) in (3.3). The HJB equation turns into a relatively form which is the following partial differential equation
\[ \sup_{\pi \in \mathbb{H}} \left\{ V_t(t, x, l, 0) + [rx + \beta l\pi_1 + \lambda\mu_1q + \lambda\mu_2\eta - \lambda\mu_2\theta(1 - q)^2 + \pi_2\theta]V_x(t, x, l, 0) \right. \]
\[ + \kappa(\alpha - l) \cdot V_t(t, x, l, 0) + \frac{1}{2}\pi_1^2[V_{xx}(t, x, l, 0) + \frac{1}{2}\sigma^2l^2V_{ll}(t, x, l, 0)] + \pi_1\sigma \rho lV_{xl}(t, x, l, 0) \]
\[ - \frac{\gamma}{2}\pi_1^2l^2g_x(t, x, l, 0)^2 - \frac{\gamma}{2}\sigma^2l^2g_t(t, x, l, 0)^2 - \gamma\pi_1\sigma \rho l^2g_x(t, x, l, 0)g_l(t, x, l, 0) \]
\[ - \lambda[V(t, x, l, 0) + \frac{\gamma}{2}g(t, x, l, 0)^2] + \lambda\epsilon[V(t, x - qY, l, 0) - \frac{\gamma}{2}g(t, x - qY, l, 0)^2] \]
\[ + \gamma g(t, x, l, 0)g(t, x - qY, l, 0)] + [V(t, x - \pi_2\zeta, l, 1) - V(t, x, l, 0)]h^P \]
\[ - \frac{\gamma}{2}[g(t, x - \pi_2\zeta, l, 1) - g(t, x, l, 0)]h^P \} = 0 \]  
with the boundary conditions (3.4) and (3.6). Similar to the post-default case, let
\[ V(t, x, l, 0) = e^{(T-t)}x + B(t)l + \overline{B}(t), \quad B(T) = 0, \overline{B}(t) = 0, \]
\[ g(t, x, l, 0) = e^{(T-t)}x + b(t)l + \overline{b}(t), \quad b(T) = 0, \overline{b}(t) = 0. \]

Substituting (4.7) into (4.6), we yield the following first-order condition for the maximum point \((q^*, \pi_1^*, \pi_2^*)\)
\[ \pi_1^* = \frac{\beta - \gamma\rho\sigma b(t)}{\gamma e^{(T-t)}}, \quad q^* = \frac{2\theta}{2\theta + \gamma e^{(T-t)}}, \]
\[ \pi_2^* = \frac{\delta - \zeta h^P + \gamma\zeta h^P(a(t)l + \pi(t) - b(t)l - \overline{b}(t))}{\gamma\zeta^2h^Pe^{(T-t)}}. \]

Introducing equations (4.7) and (4.8) into the equations (4.6) and (3.5) yields
\[ \left[ B(t) - (\kappa + h^P)B(t) + A(t)h^P - \frac{\gamma}{2}\sigma^2b(t)^2 + \frac{(\beta - \gamma\rho\sigma b(t))^2}{2\gamma} + \frac{(\delta - \zeta h^P)(a(t)l - b(t))}{\zeta} \right]l \]
where
\[ b_t - \kappa b(t) + \frac{\beta (\beta - \gamma \sigma b(t))}{\gamma} + \frac{\delta (a(t) - b(t))}{\zeta} \right] = 0. \] (4.10)

By separation of variances, we obtain four differential equations. Considering the boundary

conditions, we derive
\[ b(t) = e^{-(\kappa + \beta \sigma + \frac{\lambda}{2}) (t - s)} \int_s^t e^{(\kappa + \beta \sigma + \frac{\lambda}{2}) (t - r)} \left[ \frac{\beta^2}{\gamma} + \frac{\delta}{\zeta} s, b(s) \right] ds, \]
\[ B(t) = e^{-(\kappa + \beta \sigma) (t - s)} \int_s^t e^{(\kappa + \beta \sigma) (t - r)} \left[ A(s) h^{T} - \frac{\gamma}{2} \sigma^2 b(s)^2 + F_1(s) \right] ds, \]
\[ \bar{b}(t) = e^{-\frac{\lambda}{2} (t - s)} \int_s^t e^{\frac{\lambda}{2} (t - r)} \left[ \kappa b(s) + \lambda \mu_2 (\eta - \theta) e^{r(T - t)} + \frac{\delta \pi(s)}{\zeta} + \frac{\delta - \zeta h^P}{\gamma h^{P^2} \zeta^2} + F_2(s) \right] ds, \]
\[ \bar{B}(t) = e^{-\frac{\lambda}{2} (t - s)} \int_s^t e^{\frac{\lambda}{2} (t - r)} \left[ \lambda \mu_2 (\eta - \theta) e^{r(T - t)} + \kappa B(s) + \bar{A}(s) h^{P} + F_3(s) \right] ds, \] (4.11)

where
\[ F_1(t) = \frac{\beta - \gamma \rho \sigma b(t)}{2\gamma} + \frac{(\delta - \zeta h^P)(a(t) - b(t))}{\zeta}, \]
\[ F_2(t) = \frac{4 \lambda \mu_2 \theta^2 e^{r(T - t)}}{2\theta + \gamma e^{r(T - t)}}, \]
\[ F_3(t) = \frac{2 \lambda \mu_2 \theta^2 e^{r(T - t)}}{2\theta + \gamma e^{r(T - t)}} + \frac{\delta^2}{2\gamma h^{P^2} \zeta^2} + \frac{h^P}{2\gamma} - \frac{\delta}{\zeta} + \frac{\delta \pi(t) - \bar{b}(t)}{\zeta} + \frac{\zeta h^P (\bar{b}(t) - \pi(t))}{\zeta}. \]

**Theorem 4.1** For the mean-variance problem (3.1), the optimal investment-reinsurance

strategy is given by
\[ \pi_1^* = \frac{\beta - \gamma \rho \sigma a(t)}{\gamma e^{r(T - t)}} I_{(\tau \leq t)} + \frac{\beta - \gamma \rho \sigma b(t)}{\gamma e^{r(T - t)}} I_{(\tau > t)}, \]
\[ q^* = \frac{2 \theta}{2\theta + \gamma e^{r(T - t)}}, \]

and the value functions are given by
\[ W(t, x, l, 1) = V(t, x, l, 1) = e^{r(T - t) x} + A(t) L + \bar{A}(t), \]
\[ W(t, x, l, 0) = V(t, x, l, 0) = e^{r(T - t) x} + B(t) L + \bar{B}(t), \]

for the post-default and pre-default cases, respectively, where \( A(t) \bar{A}(t) B(t) \) and \( \bar{B}(t) \) are given by (4.5) and (4.11).

**5. Numerical Illustration and Sensitivity Analysis**

In this section, we provide some numerical examples to illustrate the effects of model
parameters on the optimal reinsurance-investment strategy. For convenience, in the following

analysis, unless otherwise stated, the basic parameters are listed as follows: \( \theta = 0.4, \eta = 0.2, \beta = 0.3, \kappa = 0.3, \alpha = 0.2, \rho = 0.5, \lambda = 1, \mu_1 = 0.3, \mu_2 = 0.5, r = 0.04, \gamma = 1, \sigma = 0.2, \)
$T = 10, t = 0, x = 1, \delta = 0.02, \rho = 0.5, \zeta = 0.2$ and $h^P = 0.005$. We give the detailed analysis for the optimal investment-reinsurance strategy $\pi_1^*(t)$, $\pi_2^*(t)$ and $q^*(t)$.

Fig.1 and Fig.2 show that the optimal reinsurance strategy $q^*(t)$ increases with respect to time $t$, namely, as time elapses, the insurer should keep more insurance business by purchasing less reinsurance or acquire more new business. In addition, Fig.1 illustrates that when the safety loading of the reinsurer $\theta$ increases, the insurer will purchase less reinsurance or acquire more new business. Fig.2 illustrates that the optimal reinsurance strategy decreases w.r.t. the absolute risk aversion parameter $\gamma$, the insurer will purchase more reinsurance or acquire less new business.

Fig.3 and Fig.4 show that the optimal investment strategy $\pi_1^*(t)$ increases with respect to time $t$ for the post-default case. Fig.3 illustrates that the optimal investment strategy $\pi_1^*(t)$ decreases when the absolute risk aversion parameter $\gamma$ increases. In other words, the larger $\gamma$ is, the more risk averse the insurer is. Fig.4 illustrates that the optimal investment strategy $\pi_1^*(t)$ decreases when correlation coefficient $\rho$ increases.

Fig.5 and Fig.6 show that the optimal investment strategy $\pi_1^*(t)$ decreases with respect to $t$ for the pre-default case. Fig.5 illustrates that the optimal investment strategy $\pi_1^*(t)$ decreases when $\delta$ increases. Fig.6 illustrates that the optimal investment strategy $\pi_1^*(t)$ decreases when $\gamma$ increases, which means that the insurer is averse to risk. Fig.7 and Fig.8 show that the optimal investment strategy $\pi_2^*(t)$ decreases with respect to $\gamma$. Fig.7 illustrates that
the optimal investment strategy \( \pi^*_2(t) \) decreases when \( \gamma \) increases. Fig. 8 illustrates that the optimal investment strategy \( \pi^*_2(t) \) increases when \( \delta \) increases.

6. Conclusion

In this paper, we investigate the optimal time-consistent investment-reinsurance problem for mean-variance insurer in a defaultable market. The insurer’s surplus process is described by a general jump process and both the insurance and reinsurance premium are assumed to be calculated via the variance premium principle. The insurer is allowed to invest in a risk-free asset, a stock and a defaultable bond. By using stochastic control theory, we solve the extended HJB equations and derive optimal investment and reinsurance strategies for the pre-default case and the post-default case, respectively. Finally, we also give a numerical example and some figures to illustrate our results. We find that: 1) the default event has no effect on the optimal reinsurance strategy; 2) the default event has an impact on the optimal dollar amount invested in the stock due to the stock price follows Heston model. In future research, we can consider more complicated models, such as the mean-variance criterion with state-dependent risk aversion or the stochastic default intensity. These problems may be more complicated calculation.

Appendix

In this appendix we derive the price for a defaultable bond. Let \( Q \) be the risk-neutral
probability measure. Denote 1/Δ ≥ 1 as the default risk premium, 1 − ζ as the default recovery rate, ζ ∈ (0, 1) as the loss rate, and \( h^Q = h^P/Δ \) as the default intensity under the risk-neutral measure \( Q \).

The pre-default value of the bond at time \( t \) is given by
\[
V(t) = E^Q \left[ e^{-r(T-t)}(1-\zeta)V_T-I_{[T\leq T]}jG_t \right] I_{\{t>T\}} + I_{\{t>T\}}e^{-r(T-t)}
\]
\[
= E^Q \left[ \int_t^T e^{-r(s-t)}(1-\zeta)V_s - h^Q e^{-h^Q(s-t)}ds \| G_t \right] I_{\{t>T\}} + e^{-h^Q(T-t)}e^{-r(T-t)}I_{\{t>T\}}
\]
\[
= e^{-(h^Q+r)(T-t)} + (1-\zeta)h^Q E^Q \left[ \int_t^T e^{-(r+h^Q)(s-t)}V_s ds \| G_t \right] \text{ on } \{ \tau > t \}. \tag{7.1}
\]

Differentiating (7.1) with respect to \( t \) yields
\[
dV(t) = (h^Q + r)V(t)dt - (1-\zeta)h^QV(t)dt = (r + \zeta h^Q)V(t)dt. \tag{7.2}
\]

By some standard calculus
\[
V(t) = e^{(r+\zeta h^Q)(T-t)}. \tag{7.3}
\]

We now define the price process for a defaultable bond under \( Q \) as follows
\[
P(t) = V(t)I_{\{t>T\}} + (1-\zeta)e^{(r-\zeta)(1-T)}I_{\{\tau\leq t\}}
\]
\[
= e^{(r+\zeta h^Q)(T-t)}I_{\{t>T\}} + e^{-(r+\zeta h^Q)(T-t)}(1-\zeta)e^{(r-\zeta)(1-T)}I_{\{\tau\leq t\}}. \tag{7.4}
\]

Recall that \( H(t) = I_{\{\tau\leq t\}} \), and note that \( dH(t) = (1 - H(t))dH(t), V(t)dH(t) = V(\tau)dH(t) \) and \( e^{(r-\zeta)dH(t)} = dH(t) \). Applying Itô’s formula to (7.4), we obtain
\[
dP(t) = rP(t)dt - P(t)(1-H(t))dH(t)dt = -(1-\zeta)\zeta dH(t). \tag{7.5}
\]

Moreover, the price of \( \{P(t)\}_{t\geq0} \) under the physical probability measure \( P \) is
\[
dP(t) = P(t-)[r dt + (1-H(t))(1-\Delta)\delta dt - (1-\zeta)\zeta dM^P(t)],
\]
where \( \delta = h^Q\zeta, M^P(t) = H(t) - \int_0^t (1-H(s))h^P ds \) is a \( P \)-martingale.

References:


方差保费原则下具有违约风险的均值-方差保险者的
时间一致最优投资和再保险问题

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摘要: 本文研究在均值-方差准则下保险者的最优投资再保险策略问题。其中保险者可以投资到无风险资产, 股票和违约债券上。股票服从Heston模型。保险者可以购买比例再保险或者得到新保险业务。特别地, 保险和再保险的保费通过方差保费原则来计算。通过使用博弈论方法, 我们分别解决了违约前和违约后的扩展的HJB方程并且得到了相应的时间一致最优投资再保险策略表达式。最后, 我们用数值例子来说明模型参数对最优策略的影响。

关键词: 时间一致策略; 均值-方差准则; 违约债券; 扩展的HJB方程