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Numerical Treatment for a Class of Partial Integro-Differential Equations with a Weakly Singular Kernel Using Chebyshev Wavelets

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Abstract: In this paper, a numerical method based on fourth kind Chebyshev wavelet collocation method is applied for solving a class of partial integro-differential equations (PIDEs) with a weakly singular kernel under three types of boundary conditions. Fractional integral formula of a single Chebyshev wavelet in the Riemann-Liouville sense is derived by means of shifted Chebyshev polynomials of the fourth kind. By implementing fractional integral formula and two-dimensional fourth kind Chebyhev wavelets together with collocation method, PIDEs with a weakly singular kernel are converted into system of algebraic equation. The convergence analysis of two-dimensional fourth kind Chebyhev wavelets is investigated. Some numerical examples are included for demonstrating the efficiency of the proposed method.

Key words: Partial integro-differential equation; Weakly singular kernel; Fourth kind Chebyshev wavelet; Collocation method; Fractional integral
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1. Introduction

In science and engineering, many of the problems can be modeled as mathematical equations, including partial differential equations, integro-differential equations and partial integro-differential equations with weakly singular kernels and others. Recently, fourth-order problems have become the focus of many scholars. For example, airplane wings,bridge slabs, floor systems, and window glasses etc. are modeled by fourth-order partial differential equations. Many numerical methods are proposed for the fourth-order partial differential problems in the last few decades.[1−8] In the present work, we consider the following fourth-order partial integro-differential equations (PIDEs) with a weakly singular kernel

$$\frac{\partial u(x,t)}{\partial t}+\int_0^t (t-s)^{-\alpha}u_{xxxx}(x,s)\mathrm{d}s=f(x,t), x\in[0,1], 0<t\leq 1, \tag{1.1}$$

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subject to the initial condition

$$u(x,0)=g(x), x\in[0,1], \tag{1.2}$$

and the following three types of boundary conditions are considered as follows:

(I)

$$\begin{cases} u(0,t)=u(1,t)=0,\\ u_x(0,t)=u_x(1,t)=0, 0<t\le 1;\end{cases}$$

(II)

$$\begin{cases} u(0,t)=u(1,t)=0,\\ u_{xx}(0,t)=u_{xx}(1,t)=0, 0<t\le 1;\end{cases} \tag{1.3}$$

(III)

$$\begin{cases} u(0,t)=u(1,t)=0,\\ u_{xx}(0,t)-pu_x(0,t)=u_{xx}(1,t)-pu_x(1,t)=0, 0<t\le 1,\end{cases}$$

where $0<\alpha<1$ and p are constants.

Owing to the memory effect of integral term, it is difficult to design fast and accurate algorithms. Therefore, developing efficient numerical method for PIDEs with weakly singular kernel is still a challenge and has attracted much attention.

Many scholars have proposed different methods for partial differential equations with weakly singular kernel. Yanik and Fairweather[9] applied finite element methods to obtain the solution of a parabolic type integro-differential equations. In [10], Galerkin finite method is used to solve a parabolic integro-differential equation with a weakly singular kernel. TANG[11] solved a second-order partial integro-differential equations by finite difference method. LI and XU[12] adopted finite central difference and finite element approximation for the numerical solution of parabolic integro-differential equation. ZHANG et al.[13] proposed quintic B-spline collocation method for the numerical solution of fourth-order PIDEs with a weakly singular kernel. Also the quintic B-spline methods are used to solve time fractional fourth-order partial differential equations[14]. In [15], FD-RBF method is proposed for solving PIDE with a weakly singular kernel. A compact difference scheme is presented for PIDEs with a weakly singular kernel in [16]. Recently, quasi-wavelets numerical methods are proposed to solve PIDEs and the time-dependent fractional partial differential equation[17−18].

It is worth noting that the above methods adopted the finite difference schemes to approximate time variable. Spectral methods are widely used in seeking numerical solutions for different types of differential equations, owing to their excellent error properties and exponential rates of convergence for smooth problems. If the spectral method is used for space discretization then the finite difference schemes are used to deal with time variable, the accuracy of the numerical of smooth problems would be limited by the finite difference schemes. Wavelets, as another basis set and very well-localized functions, are considerably useful for solving differential and integral equations. According to the localization properties of wavelets, it can analyze the local characteristic of functions, which makes wavelet-based method be an effective tool in dealing with the sharp transitions caused by the singularities of the kernel. Methods based on Chebyshev wavelets have gained much attention during the last decade. It is well known that there are four kinds of Chebyshev wavelets[19]. In the literature, there is a great concentration on the first and the second kinds of Chebyshev wavelets and their various uses in numerous applications. However, there are few articles that concentrate on the third and fourth kinds.

Inspired and motivated by the work mentioned above, the main purpose of this paper is to propose an effective method based on frouth kind Chebyshev wavelets method collocation, which is applied both in space and time directions to obtain numerical solution of fourth-order PIDE with a weakly singular kernel and further extend the application of Chebyshev wavelets. The rest of the paper is organized as follows. Section 2 describes some necessary definitions and preliminaries of calculus. Section 3 gives some properties of fourth kind Chebyshev wavelets and proves the convergence analysis of two-dimensional fourth kind Chebyshev wavelets. Section 4 is devoted to deriving the fractional integral of a single Cheyshev wavelet in the sense of Riemann-Liouville fractional integral. The proposed method is described for solving PIDEs in Section 5. In Section 6, the numerical results are presented. Finally, a brief conclusion is stated in Section 7.

2. Definitions and Preliminaries

In this section, we present some necessary definitions and preliminaries of the fractional calculus theory which will be used later.

Definition 2.1 A real function $h(t)$, $t>0$, is said to be in the space C_σ, $\sigma \in \mathbb{R}$, if there is a real number ρ with $\rho>\sigma$ such that $h(t)=t^\rho h_0(t)$, where $h_0(t)\in C[0,\infty)$, and $h(t)\in C_\sigma^n$ if $h^{(n)}(t)\in C_\sigma$, $n\in\mathbb{N}$.

Definition 2.2 The Riemann-Liouville fractional integral operator I^α of order α ($\alpha \geq 0$) for a function $h(t)\in C_\sigma$ ($\sigma\geq -1$) is defined as in [20]

$$I^\alpha h(t)=\begin{cases}\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)\mathrm{d}s, & \alpha>0;\\ h(t), & \alpha=0.\end{cases}$$

Definition 2.3 The Caputo fractional derivative operator D^α of order α ($\alpha\geq 0$) for a function $h(t)\in C_1^n$ is defined as in [20]

$$D^\alpha h(t)=\begin{cases} h^{(n)}(t), & \alpha=n\in\mathbb{N};\\ \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{h^{(n)}(s)}{(t-s)^{\alpha+1-n}}\mathrm{d}s, & n-1<\alpha<n.\end{cases}$$

Some important properties of the operator I^α and D^α are needed in this paper, we only mention the following properties

1) $I^{\alpha_1}I^{\alpha_2}h(t)=I^{\alpha_1+\alpha_2}h(t)$ for $\alpha_1,\alpha_2>0$;

2) $D^\alpha I^\alpha h(t)=h(t), D^\beta I^\alpha h(t)=I^{\alpha-\beta}h(t), \alpha>\beta$;

3) $D^\alpha t^n=\frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}t^{n-\alpha}, n\geq\lceil\alpha\rceil; D^\alpha t^n=0, n\geq\lceil\alpha\rceil, n\in\mathbb{N}$;

4) $I^\alpha D^\alpha h(t)=h(t)-\sum\limits_{k=0}^{\lceil\alpha\rceil-1}h^{(k)}(0^+)\frac{t^k}{k!}$, $t>0$, where we use the ceiling function $\lceil\alpha\rceil$ to denote the smallest integer than or equal to α. For more details about fractional calculus and its properties, see [20].

3. The Fourth Kind Chebyshev Wavelets and Their Properties

The fourth kind Chebyshev wavelets defined on the interval $[0,1)$ has the following form

$$\psi_{n,m}(t)=\begin{cases}2^{\frac{k}{2}}\widetilde{W}_m(2^k t-2n+1), & \frac{n-1}{2^{k-1}}\leq t<\frac{n}{2^{k-1}},\\ 0, & \text{otherwise},\end{cases}$$

where $n = 1, 2, 3, \cdots, 2^{k-1}$ and

$$\widetilde{W}_m(t) = \sqrt{\frac{1}{\pi}} W_m(t), m = 0, 1, 2, \cdots. \tag{3.1}$$

The coefficient in Eq.(3.1) is for orthonormality. Here $W_m(t)$ are the fourth kind Chebyshev polynomials of degree m which are orthogonal with respect to the weight function $\omega(t) = \sqrt{1-t}/\sqrt{1+t}$ on the interval $[-1, 1]$ and satisfy the following recursive formula:

$$W_0(t) = 1,\ W_1(t) = 2t + 1,$$
$$W_{m+1}(t) = 2tW_m(t) - W_{m-1}(t),\ m = 1, 2, 3, \cdots.$$

Note that when dealing with the fourth kind Chebyshev wavelets the weight function has to be dilated and translated as

$$\omega_n(t) = \omega(2^k t - 2n + 1).$$

A function $f(x) \in L^2(\mathbb{R})$ defined on $[0, 1)$ may be expanded by the fourth kind Chebyshev wavelets as

$$f(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \psi_{n,m}(x), \tag{3.2}$$

where

$$c_{n,m} = \langle f(x), \psi_{n,m}(x) \rangle_{L^2_\omega[0,1)} = \int_0^1 f(x) \psi_{n,m}(x) \omega_n(x) \mathrm{d}x,$$

in which $\langle \cdot, \cdot \rangle_{L^2_\omega[0,1)}$ denotes the inner product in $L^2_\omega[0, 1)$. If the infinite series in Eq.(3.2) is truncated, then it can be written as

$$f(x) \cong \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}(x) = C^{\mathrm{T}} \Psi(x),$$

where C and $\Psi(x)$ are $2^{k-1}M \times 1$ matrices given by

$$C = \begin{pmatrix} c_{1,0} & c_{1,1} & \cdots & c_{1,M-1} & c_{2,0} & c_{2,1} & \cdots & c_{2,M-1} & \cdots & c_{2^{k-1},0} & c_{2^{k-1},1} & \cdots & c_{2^{k-1},M-1} \end{pmatrix}^{\mathrm{T}},$$

$$\Psi(x) = \begin{pmatrix} \psi_{1,0} & \psi_{1,1} & \cdots & \psi_{1,M-1} & \psi_{2,0} & \psi_{2,1} & \cdots & \psi_{2,M-1} & \cdots & \psi_{2^{k-1},0} & \psi_{2^{k-1},1} & \cdots & \psi_{2^{k-1},M-1} \end{pmatrix}^{\mathrm{T}}. \tag{3.3}$$

The two-dimensional fourth kind Chebyshev wavelets of the fourth kind are defined as

$$\psi_{n_1,m_1,n_2,m_2}(x, y) = \begin{cases} 2^{\frac{k_1+k2}{2}} \widetilde{W}_{m_1}(2^{k_1}x - 2n_1 + 1) \widetilde{W}_{m_2}(2^{k_2}y - 2n_2 + 1), \\ \dfrac{n_1 - 1}{2^{k_1-1}} \le x < \dfrac{n_1}{2^{k_1-1}}, \dfrac{n_2 - 1}{2^{k_2-1}} \le y < \dfrac{n_2}{2^{k_2-1}}, \\ 0, \qquad \text{otherwise}, \end{cases} \tag{3.4}$$

where n_1 and n_2 are defined similaryly to n, k_1 and k_2 are any positive integers, m_1 and m_2 are the orders of fourth kind Chebyshev polynomials and $\psi_{n_1,m_1,n_2,m_2}(x, y)$ forms a basis for $L^2_\omega([0, 1) \times [0, 1))$. A function $\mu(x, y)$ defined on $[0, 1) \times [0, 1)$ can be expanded by two-dimensional fourth kind Chebyshev wavelets as follows

$$\mu(x, y) = \sum_{n_1=1}^{\infty} \sum_{m_1=0}^{\infty} \sum_{n_2=1}^{\infty} \sum_{m_2=0}^{\infty} d_{n_1,m_1,n_2,m_2} \psi_{n_1,m_1,n_2,m_2}(x, y). \tag{3.5}$$

If the infinite series in Eq.(3.5) is truncated, then it can be written as

$$\mu(x, y) \cong \sum_{n_1=1}^{2^{k_1-1}} \sum_{m_1=0}^{M_1-1} \sum_{n_2=1}^{2^{k_2-1}} \sum_{m_2=0}^{M_2-1} d_{n_1,m_1,n_2,m_2} \psi_{n_1,m_1,n_2,m_2}(x, y) = \Psi^{\mathrm{T}}(x) D \Psi(y), \tag{3.6}$$

where $\Psi(x)$ and $\Psi(t)$ are $2^{k_1-1}M_1 \times 1$ and $2^{k_2-1}M_2 \times 1$ matrices and are defined in (3.3). Moreover, D is $2^{k_1-1}M_1 \times 2^{k_2-1}M_2$ matrix and its elements can be calculated from the formula

$$d_{n_1,m_1,n_2,m_2} = \int_0^1 \int_0^1 \mu(x,y)\psi_{n_1,m_1}(x)\psi_{n_2,m_2}(y)\omega_{n_1}(x)\omega_{n_2}(y)\mathrm{d}x\mathrm{d}y,$$

where $n_1 = 1, \cdots, 2^{k_1-1}, m_1 = 0, \cdots, M_1 - 1, n_2 = 1, \cdots, 2^{k_2-1}, m_2 = 0, \cdots, M_2 - 1$. In the calculation, we usually take $k = k_1 = k_2$ and $M = M_1 = M_2$.

We investigate the convergence analysis of two-dimensional fourth kind Chebyhev wavelets in the following theorem.

Theorem 3.1 Suppose that $\mu(x,y) \in L^2(\mathbb{R}^2)$ is a continuous function defined on $[0,1)\times[0,1)$, and satisfies $|\frac{\partial\mu(x,y)}{\partial x}| \le B$, $|\frac{\partial\mu(x,y)}{\partial y}| \le B$, $|\frac{\partial^2\mu(x,y)}{\partial x^2}| \le B$, $|\frac{\partial^2\mu(x,y)}{\partial y^2}| \le B$, $|\frac{\partial^2\mu(x,y)}{\partial x\partial y}| \le B$, $|\frac{\partial^3\mu(x,y)}{\partial x^2\partial y}| \le B$, $|\frac{\partial^3\mu(x,y)}{\partial x\partial y^2}| \le B$, $|\frac{\partial^4\mu(x,y)}{\partial x^2\partial y^2}| \le B$ for some positive constant B. Then the series

$$\sum_{n_1=1}^{\infty}\sum_{m_1=0}^{\infty}\sum_{n_2=1}^{\infty}\sum_{m_2=0}^{\infty} d_{n_1,m_1,n_2,m_2}\psi_{n_1,m_1}(x)\psi_{n_2,m_2}(y)$$

converges to $\mu(x,y)$, where $d_{n_1,m_1,n_2,m_2} = \langle \mu(x,y), \psi_{n_1,m_1}(x)\psi_{n_2,m_2}(y)\rangle_{L^2_\omega([0,1)\times[0,1))}$.

Proof

$$\begin{aligned} d_{n_1,m_1,n_2,m_2} &= \langle \mu(x,y), \psi_{n_1,m_1}(x)\psi_{n_2,m_2}(y)\rangle_{L^2_\omega([0,1)\times[0,1))} \\ &= \int_0^1\int_0^1 \mu(x,y)\psi_{n_1,m_1}(x)\psi_{n_2,m_2}(y)\omega_{n_1}(x)\omega_{n_2}(y)\mathrm{d}x\mathrm{d}y \\ &= \int_0^1 \psi_{n_2,m_2}(y)\omega_{n_2}(y)\Big(\int_0^1 \mu(x,y)\psi_{n_1,m_1}(x)\omega_{n_1}(x)\mathrm{d}x\Big)\mathrm{d}y \\ &= \int_0^1 \psi_{n_2,m_2}(y)\omega_{n_2}(y)A_{n_1,m_1}(y)\mathrm{d}y, \end{aligned}$$

where $A_{n_1,m_1}(y) = \int_0^1 \mu(x,y)\psi_{n_1,m_1}(x)\omega_{n_1}(x)\mathrm{d}x$.

$$\begin{aligned} A_{n_1,m_1}(y) &= \int_0^1 \mu(x,y)\psi_{n_1,m_1}(x)\omega_{n_1}(x)\mathrm{d}x \\ &= \int_{\frac{n_1-1}{2^{k-1}}}^{\frac{n_1}{2^{k-1}}} \mu(x,y)2^{\frac{k}{2}}\sqrt{\frac{1}{\pi}}W_{m_1}(2^kx - 2n_1 + 1)\omega(2^kx - 2n_1 + 1)\mathrm{d}x. \end{aligned}$$

Let $2^kx - 2n_1 + 1 = t$ and $\mathrm{d}x = \frac{1}{2^k}\mathrm{d}t$. It follows

$$A_{n_1,m_1}(y) = \sqrt{\frac{1}{\pi}}2^{-\frac{k}{2}}\int_{-1}^1 \mu(\frac{t+2n_1-1}{2^k}, y)W_{m_1}(t)\omega(t)\mathrm{d}t.$$

By putting $t = \cos\theta$, together with the definition of the fourth kind Chebyshev polynomial, we get

$$\begin{aligned} A_{n_1,m_1}(y) &= \sqrt{\frac{1}{\pi}}2^{-\frac{k}{2}}\int_0^\pi \mu(\frac{\cos\theta+2n_1-1}{2^k}, y)\frac{\sin(m_1+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\sin\theta\mathrm{d}\theta \\ &= \sqrt{\frac{1}{\pi}}2^{-\frac{k}{2}}\int_0^\pi \mu(\frac{\cos\theta+2n_1-1}{2^{k_1}}, y)\big(\cos m_1\theta - \cos(m_1+1)\theta\big)\mathrm{d}\theta \\ &= \sqrt{\frac{1}{\pi}}2^{-\frac{k}{2}}\Big(\int_0^\pi \mu(\frac{\cos\theta+2n_1-1}{2^k}, y)\cos m_1\theta\mathrm{d}\theta \\ &\quad - \int_0^\pi \mu(\frac{\cos\theta+2n_1-1}{2^k}, y)\cos(m_1+1)\theta\mathrm{d}\theta\Big). \end{aligned}$$

Applying the integration by parts, we have

$$\begin{aligned} A_{n_1,m_1}(y) &= \sqrt{\frac{1}{\pi}}2^{-\frac{3}{2}k}\Big(\frac{1}{m_1}\int_0^{\pi}\frac{\partial\mu(\frac{\cos\theta+2n_1-1}{2^k},y)}{\partial\theta}\sin m_1\theta\sin\theta\mathrm{d}\theta \\ &\quad -\frac{1}{m_1+1}\int_0^{\pi}\frac{\partial\mu(\frac{\cos\theta+2n_1-1}{2^k},y)}{\partial\theta}\sin(m_1+1)\theta\sin\theta\mathrm{d}\theta\Big) \\ &= \sqrt{\frac{1}{\pi}}2^{-\frac{3}{2}k}(I_1-I_2), \end{aligned}$$

where

$$I_1 = \frac{1}{m_1}\int_0^{\pi}\frac{\partial\mu(\frac{\cos\theta+2n_1-1}{2^k},y)}{\partial\theta}\sin(m_1\theta)\sin\theta\mathrm{d}\theta,$$

$$I_2 = \frac{1}{m_1+1}\int_0^{\pi}\frac{\partial\mu(\frac{\cos\theta+2n_1-1}{2^k},y)}{\partial\theta}\sin(m_1+1)\theta\sin\theta\mathrm{d}\theta.$$

Using the integration by parts again, we get

$$\begin{aligned} I_1 =&\frac{2^{-k}}{2m_1(m_1-1)(m_1+1)}\int_0^{\pi}\frac{\partial^2\mu(\frac{\cos\theta+2n_1-1}{2^k},y)}{\partial\theta^2}\big((m_1+1)\sin(m_1-1)\theta\sin\theta \\ &-(m_1+1)\sin(m_1+1)\theta\sin\theta\big)\mathrm{d}\theta \end{aligned}$$

and

$$\begin{aligned} I_2 =&\frac{2^{-k}}{2m_1(m_1+1)(m_1+2)}\int_0^{\pi}\frac{\partial^2\mu(\frac{\cos\theta+2n_1-1}{2^k},y)}{\partial\theta^2}\big((m_1+2)\sin m_1\theta\sin\theta \\ &-m_1\sin(m_1+2)\theta\sin\theta\big)\mathrm{d}\theta. \end{aligned}$$

Therefore

$$A_{n_1,m_1}(y) = \frac{2^{-\frac{5k}{2}}}{2\sqrt{\pi}}\frac{1}{m_1^2(m_1-1)(m_1+1)^2(m_1+2)}\int_0^{\pi}\frac{\partial^2\mu(\frac{\cos\theta+2n_1-1}{2^k},y)}{\partial\theta^2}\tau_{m_1}(\theta)\mathrm{d}\theta,$$

where $\tau_{m_1}(\theta) = m_1(m_1+1)^2(m_1+2)\sin(m_1-1)\theta\sin\theta - m_1(m_1-1)(m_1+1)(m_1+2)\sin(m_1+1)\theta\sin\theta - m_1(m_1-1)(m_1+1)(m_1+2)\sin m_1\theta\sin\theta + m_1^2(m_1-1)(m_1+1)\sin(m_1+2)\theta\sin\theta$.

So, for $m_1>1$,$m_2>1$,

$$\begin{aligned} &d_{n_1,m_1,n_2,m_2} \\ =&\int_0^1\psi_{n_2,m_2}(y)\omega_{n_2}(y)A_{n_1,m_1}(y)\mathrm{d}y \\ =&\frac{1}{2\sqrt{\pi}}2^{-\frac{5}{2}k}\frac{1}{m_1^2(m_1-1)(m_1+1)^2(m_1+2)}\int_0^1\psi_{n_2,m_2}(y)\omega_{n_2}(y)\int_0^{\pi}\frac{\partial^2\mu(\frac{\cos\theta+2n_1-1}{2^k},y)}{\partial\theta^2}\tau_{m_2}(\theta)\mathrm{d}\theta\mathrm{d}y \\ =&\frac{1}{2\sqrt{\pi}}2^{-\frac{5}{2}k}\frac{1}{m_1^2(m_1-1)(m_1+1)^2(m_1+2)}\int_0^{\pi}\tau_{m_1}(\theta)\Big(\int_0^1\frac{\partial^2\mu(\frac{\cos\theta+2n_1-1}{2^k},y)}{\partial\theta^2}\psi_{n_2,m_2}(y)\omega_{n_2}(y)\mathrm{d}y\Big)\mathrm{d}\theta \\ =&\frac{1}{4\pi}2^{-(\frac{5}{2}k+\frac{5}{2}k)}\frac{1}{m_1^2(m_1-1)(m_1+1)^2(m_1+2)}\frac{1}{m_2^2(m_2-1)(m_2+1)^2(m_2+2)} \\ &\cdot\int_0^{\pi}\int_0^{\pi}\frac{\partial^4\mu(\frac{\cos\theta+2n_1-1}{2^k},\frac{\cos\eta+2n_2-1}{2^k})}{\partial\theta^2\partial\eta^2}\tau_{m_1}(\theta)\tau_{m_2}(\eta)\mathrm{d}\theta\mathrm{d}\eta. \end{aligned}$$

A simple computation shows that

$$\int_0^{\pi}|\tau_{m_1}(\theta)|\mathrm{d}\theta \le \sqrt{2}\pi m_1(m_1+1)^2(m_1+2).$$

Hence, for $m_1>1$ and $m_2>1$, we have

$$|d_{n_1,m_1,n_2,m_2}| \le \frac{\pi B}{2}2^{-(\frac{5}{2}k+\frac{5}{2}k)}\frac{1}{m_1(m_1-1)}\cdot\frac{1}{m_2(m_2-1)}$$

$$\leq \frac{\pi B}{2^6}\frac{1}{n_1^{\frac{5}{2}}(m_1-1)^2}\cdot\frac{1}{n_2^{\frac{5}{2}}(m_2-1)^2}, \tag{3.7}$$

since $n_1 \leq 2^{k-1}$ and $n_2 \leq 2^{k-1}$. For $m_1 > 1$ and $m_2 = 1$,

$$\begin{aligned}
&d_{n_1,m_1,n_2,1}\\
=&\int_0^1 \psi_{n_2,1}(y)\omega_{n_2}(y)A_{n_1,m_1}(y)\mathrm{d}y\\
=&\frac{1}{2\sqrt{\pi}}2^{-\frac{5}{2}k}\frac{1}{m_1^2(m_1-1)(m_1+1)^2(m_1+2)}\int_0^{\pi}\tau_{m_1}(\theta)\int_0^1\frac{\partial^2\mu(\frac{\cos\theta+2n_1-1}{2^k},y)}{\partial\theta^2}\psi_{n_2,1}(y)\omega_{n_2}(y)\mathrm{d}y\mathrm{d}\theta\\
=&\frac{1}{2\sqrt{\pi}}2^{-\frac{5}{2}k}\frac{1}{m_1^2(m_1-1)(m_1+1)^2(m_1+2)}\int_0^{\pi}\tau_{m_1}(\theta)\frac{1}{\sqrt{\pi}}2^{-\frac{3}{2}k}\\
&\cdot\Big(\int_0^{\pi}\frac{\partial^3\mu(\frac{\cos\theta+2n_1-1}{2^k},\frac{\cos\eta+2n_2-1}{2^k})}{\partial\theta^2\partial\eta}(\sin^2\eta-\frac{1}{2}\sin 2\eta\sin\eta)\mathrm{d}\eta\Big)\mathrm{d}\theta.
\end{aligned}$$

So

$$\begin{aligned}
|d_{n_1,m_1,n_2,1}| &\leq \frac{1}{2\pi}2^{-4k}\frac{1}{m_1^2(m_1-1)(m_1+1)^2(m_1+2)}\frac{3\pi B}{2}\int_0^{\pi}|\tau_{m_1}(\theta)|\mathrm{d}\theta\\
&\leq \frac{1}{2\pi}2^{-4k}\frac{1}{m_1^2(m_1-1)(m_1+1)^2(m_1+2)}\frac{3\pi B}{2}\sqrt{2}\pi m_1(m_1+1)^2(m_1+2)\\
&\leq \frac{3\sqrt{2}\pi B}{2^6}\frac{1}{n_1^2(m_1-1)^2}\frac{1}{n_2^2}.
\end{aligned}\tag{3.8}$$

Similarly, for $m_1 = 1$ and $m_2 > 1$,

$$|d_{n_1,1,n_2,m_2}| \leq \frac{3\sqrt{2}\pi B}{2^6}\frac{1}{n_1^2(m_2-1)^2}\frac{1}{n_2^2}; \tag{3.9}$$

for $m_1 > 1$ and $m_2 = 0$,

$$|d_{n_1,m_1,n_2,0}| \leq \frac{\sqrt{2}\pi^{\frac{3}{2}}B}{2^4}\frac{1}{n_1^{\frac{3}{2}}(m_1-1)^2}\frac{1}{n_2^{\frac{3}{2}}}; \tag{3.10}$$

for $m_1 = 0$ and $m_2 > 1$,

$$|d_{n_1,0,n_2,m_2}| \leq \frac{\sqrt{2}\pi^{\frac{3}{2}}B}{2^4}\frac{1}{n_1^{\frac{3}{2}}(m_2-1)^2}\frac{1}{n_2^{\frac{3}{2}}}; \tag{3.11}$$

for $m_1 = 1$ and $m_2 = 1$,

$$|d_{n_1,1,n_2,1}| \leq \frac{9\pi B}{2^5}\frac{1}{n_1^{\frac{3}{2}}}\frac{1}{n_2^{\frac{3}{2}}}; \tag{3.12}$$

for $m_1 = 1, 0$ and $m_2 = 0, 1$

$$|d_{n_1,1,n_2,0}| \leq \frac{3\pi^{\frac{3}{2}}B}{2}\frac{1}{n_1^2}\frac{1}{n_2^2}. \tag{3.13}$$

Hence, by relations (3.7)-(3.13), the series $\sum\limits_{n_1=1}^{\infty}\sum\limits_{m_1=0}^{\infty}\sum\limits_{n_2=1}^{\infty}\sum\limits_{m_2=0}^{\infty} d_{n_1,m_1,n_2,m_2}\psi_{n_1,m_1}(x)\psi_{n_2,m_2}(y)$ converges to $\mu(x,y)$ uniformly. The proof is completed.

4. The Fractional Integral of a Single Chebyshev Wavelet

In Section 5, multiple integrals on $\psi_{n,m}(x)$ will be performed from 0 to x. In this section, fractional integral formula of a single Chebyshev wavelet in the Riemann-Liouville sense is derived by means of shifted Chebyshev polynomials of the fourth kind, which plays an important role in solving PIDEs with weakly singular kernels.

Theorem 4.1 The fractional integral of a Chebyshev wavelet defined on the interval $[0,1]$ with compact support $\left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right]$ is given by

$$I^\alpha \psi_{n,m}(x) = \begin{cases} 0, & 0 < x < \frac{n-1}{2^{k-1}}, \\ \frac{1}{\Gamma(\alpha)} 2^{\frac{k}{2}} \sqrt{\frac{1}{\pi}} \sum\limits_{r=0}^{m} \sum\limits_{i=r}^{m} \sum\limits_{j=0}^{r} W_{i,i-r}^{m,n,k} \frac{(-1)^j}{j+\alpha} C_r^j (x - \frac{n-1}{2^{k-1}})^{j+\alpha}, & \\ & \frac{n-1}{2^{k-1}} \le x \le \frac{n}{2^{k-1}}, \\ \frac{1}{\Gamma(\alpha)} 2^{\frac{k}{2}} \sqrt{\frac{1}{\pi}} \sum\limits_{r=0}^{m} \sum\limits_{i=r}^{m} \sum\limits_{j=0}^{r} W_{i,i-r}^{m,n,k} \frac{(-1)^j}{j+\alpha} C_r^j x^{r-j} \big[(x - \frac{n-1}{2^{k-1}})^{j+\alpha} - (x - \frac{n}{2^{k-1}})^{j+\alpha}\big], & \\ & 1 \ge x > \frac{n}{2^{k-1}}, \end{cases}$$

where $W_{i,i-r}^{m,n,k} = (-1)^{m-r} 2^{2i} 2^{r(k-1)} (n-1)^{i-r} \frac{\Gamma(m+i+1)}{\Gamma(m-i+1)\Gamma(2i+1)} \frac{i!}{(i-r)!r!}, C_r^j = \frac{r!}{j!(r-j)!}$.

Proof The analytical form of the shifted Chebyshev polynomials of the fourth kind W_m^* of degree m is given by [19]

$$W_m^*(x) = \sum_{i=0}^{m} (-1)^{m-i} 2^{2i} \frac{\Gamma(m+i+1)}{\Gamma(2i+1)\Gamma(m-i+1)} x^i. \tag{4.1}$$

According to the relation between $W_m^*(x)$ and $W_m(x)$, it gives that

$$\begin{aligned} & W_m(2^k x - 2n + 1) \\ = & \sum_{i=0}^{m} (-1)^{m-i} 2^{2i} \frac{\Gamma(m+i+1)}{\Gamma(m-i+1)\Gamma(2i+1)} \Big(2^{k-1}x - (n-1)\Big)^i \\ = & \sum_{i=0}^{m} (-1)^{m-i} 2^{2i} \frac{\Gamma(m+i+1)}{\Gamma(m-i+1)\Gamma(2i+1)} 2^{i(k-1)} \sum_{r=0}^{i} C_i^r x^{i-r} \Big(-\frac{n-1}{2^{k-1}}\Big)^r \\ = & \sum_{i=0}^{m} \sum_{r=0}^{i} (-1)^{m-i+r} 2^{2i} 2^{(i-r)(k-1)} (n-1)^r \frac{\Gamma(m+i+1)}{\Gamma(m-i+1)\Gamma(2i+1)} \frac{i!}{r!(i-r)!} x^{i-r}. \end{aligned}$$

By interchanging the summation and substituting $i-r$ with r, $W_m(2^k x - 2n + 1)$ can be written as

$$W_m(2^k x - 2n + 1) = \sum_{r=0}^{m} \sum_{i=r}^{m} (-1)^{m-r} 2^{2i} 2^{r(k-1)} (n-1)^{i-r} \frac{\Gamma(m+i+1)}{\Gamma(m-i+1)\Gamma(2i+1)} \frac{i!}{(i-r)!r!} x^r. \tag{4.2}$$

Let $W_{i,i-r}^{m,n,k} = (-1)^{m-r} 2^{2i} 2^{r(k-1)} (n-1)^{i-r} \frac{\Gamma(m+i+1)}{\Gamma(m-i+1)\Gamma(2i+1)} \frac{i!}{(i-r)!r!}$, then

$$W_m(2^k x - 2n + 1) = \sum_{r=0}^{m} \sum_{i=r}^{m} W_{i,i-r}^{m,n,k} x^r. \tag{4.3}$$

Therefore

$$\psi_{n,m}(x) = \begin{cases} 2^{\frac{k}{2}} \sqrt{\frac{1}{\pi}} \sum\limits_{r=0}^{m} \sum\limits_{i=r}^{m} W_{i,i-r}^{m,n,k} x^r, & \dfrac{n-1}{2^{k-1}} \le x < \dfrac{n}{2^{k-1}}, \\ 0, & \text{otherwise}. \end{cases} \tag{4.4}$$

So, when $\frac{n-1}{2^{k-1}} \le x \le \frac{n}{2^{k-1}}$,let $u = x - t$, then

$$\begin{aligned} I^\alpha x^r &= \frac{1}{\Gamma(\alpha)} \int_{\frac{n-1}{2^{k-1}}}^{x} (x-t)^{\alpha-1} t^r \mathrm{d}t \\ &= \frac{1}{\Gamma(\alpha)} \int_0^{x - \frac{n-1}{2^{k-1}}} u^{\alpha-1} (x-u)^r \mathrm{d}u \\ &= \frac{1}{\Gamma(\alpha)} \int_0^{x - \frac{n-1}{2^{k-1}}} u^{\alpha-1} \sum_{j=0}^{r} C_r^j x^{r-j} (-u)^j \mathrm{d}u \end{aligned}$$

$$= \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{r} \frac{(-1)^j}{j+\alpha} C_r^j x^{r-j} (x - \frac{n-1}{2^{k-1}})^{j+\alpha}.$$

In a similar way, when $x > \frac{n}{2^{k-1}}$,we have

$$\begin{aligned} I^\alpha x^r &= \frac{1}{\Gamma(\alpha)} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} (x-t)^{\alpha-1} t^r \mathrm{d}t \\ &= \frac{1}{\Gamma(\alpha)} \int_{x-\frac{n}{2^{k-1}}}^{x-\frac{n-1}{2^{k-1}}} u^{\alpha-1} (x-u)^r \mathrm{d}u \\ &= \frac{1}{\Gamma(\alpha)} \int_{x-\frac{n}{2^{k-1}}}^{x-\frac{n-1}{2^{k-1}}} u^{\alpha-1} \sum_{j=0}^{r} C_r^j x^{r-j} (-u)^j \mathrm{d}u \\ &= \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{r} \frac{(-1)^j}{j+\alpha} C_r^j x^{r-j} \big[(x - \frac{n-1}{2^{k-1}})^{j+\alpha} - (x - \frac{n}{2^{k-1}})^{j+\alpha}\big]. \end{aligned}$$

Applying the Riemann-Liouville fractional integral of order α with respect to x on $\psi_{n,m}(x)$, we obtain

$$\begin{aligned} I^\alpha \psi_{n,m}(x) &= \begin{cases} 0, & 0 < x < \frac{n-1}{2^{k-1}}, \\ \frac{1}{\Gamma(\alpha)} \int_{\frac{n-1}{2^{k-1}}}^{x} (x-t)^{\alpha-1} \psi_{n,m}(t) \mathrm{d}t, & \frac{n-1}{2^{k-1}} \le x \le \frac{n}{2^{k-1}}, \\ \frac{1}{\Gamma(\alpha)} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} (x-t)^{\alpha-1} \psi_{n,m}(t) \mathrm{d}t, & 1 \ge x > \frac{n}{2^{k-1}} \end{cases} \\ &= \begin{cases} 0, & 0 < x < \frac{n-1}{2^{k-1}}, \\ \frac{1}{\Gamma(\alpha)} 2^{\frac{k}{2}} \sqrt{\frac{1}{\pi}} \sum_{r=0}^{m} \sum_{i=r}^{m} W_{i,i-r}^{m,n,k} \int_{\frac{n-1}{2^{k-1}}}^{x} (x-t)^{\alpha-1} t^r \mathrm{d}t, & \frac{n-1}{2^{k-1}} \le x \le \frac{n}{2^{k-1}}, \\ \frac{1}{\Gamma(\alpha)} 2^{\frac{k}{2}} \sqrt{\frac{1}{\pi}} \sum_{r=0}^{m} \sum_{i=r}^{m} W_{i,i-r}^{m,n,k} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} (x-t)^{\alpha-1} t^r \mathrm{d}t, & 1 \ge x > \frac{n}{2^{k-1}} \end{cases} \\ &= \begin{cases} 0, & 0 < x < \frac{n-1}{2^{k-1}}, \\ \frac{1}{\Gamma(\alpha)} 2^{\frac{k}{2}} \sqrt{\frac{1}{\pi}} \sum_{r=0}^{m} \sum_{i=r}^{m} W_{i,i-r}^{m,n,k} \sum_{j=0}^{r} \frac{(-1)^j}{j+\alpha} C_r^j x^{r-j} (x - \frac{n-1}{2^{k-1}})^{j+\alpha}, & \\ & \frac{n-1}{2^{k-1}} x \le \frac{n}{2^{k-1}}, \\ \frac{1}{\Gamma(\alpha)} 2^{\frac{k}{2}} \sqrt{\frac{1}{\pi}} \sum_{r=0}^{m} \sum_{i=r}^{m} W_{i,i-r}^{m,n,k} \sum_{j=0}^{r} \frac{(-1)^j}{j+\alpha} C_r^j x^{r-j} \big[(x - \frac{n-1}{2^{k-1}})^{j+\alpha} - (x - \frac{n}{2^{k-1}})^{j+\alpha}\big], & \\ & 1 \ge x > \frac{n}{2^{k-1}}. \end{cases} \end{aligned}$$

Thus, we have

$$I^\alpha \psi_{n,m}(x) = \begin{cases} 0, & 0 < x < \frac{n-1}{2^{k-1}}, \\ \frac{1}{\Gamma(\alpha)} 2^{\frac{k}{2}} \sqrt{\frac{1}{\pi}} \sum_{r=0}^{m} \sum_{i=r}^{m} \sum_{j=0}^{r} W_{i,i-r}^{m,n,k} \frac{(-1)^j}{j+\alpha} C_r^j x^{r-j} (x - \frac{n-1}{2^{k-1}})^{j+\alpha}, & \\ & \frac{n-1}{2^{k-1}} \le x \le \frac{n}{2^{k-1}}, \\ \frac{1}{\Gamma(\alpha)} 2^{\frac{k}{2}} \sqrt{\frac{1}{\pi}} \sum_{r=0}^{m} \sum_{i=r}^{m} \sum_{j=0}^{r} W_{i,i-r}^{m,n,k} \frac{(-1)^j}{j+\alpha} C_r^j x^{r-j} \big[(x - \frac{n-1}{2^{k-1}})^{j+\alpha} - (x - \frac{n}{2^{k-1}})^{j+\alpha}\big], & \\ & 1 \ge x > \frac{n}{2^{k-1}}. \end{cases}$$

The proof is completed.

For example, in the case of $k=2, M=3, x=0.65, \alpha=4$, we obtain

$$I^4\Psi_{6\times 1}(0.65)=\begin{pmatrix} 0.00836881215595839 \\ 0.000258586892459386 \\ -0.00281311195130054 \\ 2.3801748055921\times 10^{-5} \\ -1.80893285224999\times 10^{-5} \\ 8.94945726902637\times 10^{-6} \end{pmatrix},$$

where $\Psi_{6\times 1}(x)=\begin{pmatrix} \psi_{1,0}(x) & \psi_{1,1}(x) & \psi_{1,2}(x) & \psi_{2,0}(x) & \psi_{2,1}(x) & \psi_{2,2}(x) \end{pmatrix}^{\mathrm{T}}$.

5. Description of the Proposed Method

To solve the partial integro-differential equations given in (1.1) under the three conditions (I)-(III), first we assume that the mixed derivative of the solution $u(x,t)$ in (1.1) is approximated by Chebyshev wavelets as

$$\frac{\partial^5 u}{\partial x^4 \partial t}=\Psi^{\mathrm{T}}(x)U\Psi(t), \tag{5.1}$$

where U is an unknown matrix which should be determined. By integrating (5.1) with respect to t and combining the initial condition (1.2), we obtain:

$$\frac{\partial^4 u}{\partial x^4}=g^{(4)}(x)+\Psi^{\mathrm{T}}(x)U(I\Psi(t)). \tag{5.2}$$

Also by integrating (5.2) four times with respect to x, we get the following equations

$$\frac{\partial^3 u}{\partial x^3}-\frac{\partial^3 u}{\partial x^3}\Big|_{x=0}=g^{(3)}(x)-g^{(3)}(0)+(I\Psi(x))^{\mathrm{T}}U(I\Psi(t)), \tag{5.3}$$

$$\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial x^2}\Big|_{x=0}-x\frac{\partial^3 u}{\partial x^3}\Big|_{x=0}=g^{(2)}(x)-g^{(2)}(0)-xg^{(3)}(0)+(I^2\Psi(x))^{\mathrm{T}}U(I\Psi(t)), \tag{5.4}$$

$$\begin{aligned}&\frac{\partial u}{\partial x}-\frac{\partial u}{\partial x}\Big|_{x=0}-x\frac{\partial^2 u}{\partial x^2}\Big|_{x=0}-\frac{1}{2}x^2\frac{\partial^3 u}{\partial x^3}\Big|_{x=0}\\ =&g^{(1)}(x)-g^{(1)}(0)-xg^{(2)}(0)-\frac{1}{2}x^2g^{(3)}(0)+(I^3\Psi(x))^{\mathrm{T}}U(I\Psi(t)),\end{aligned} \tag{5.5}$$

$$\begin{aligned}&u(x,t)-x\frac{\partial u}{\partial x}\Big|_{x=0}-\frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2}\Big|_{x=0}-\frac{1}{6}x^3\frac{\partial^3 u}{\partial x^3}\Big|_{x=0}\\ =&u(0,t)+g(x)-g(0)-xg^{(1)}(0)-\frac{1}{2}x^2g^{(2)}(0)-\frac{1}{6}x^3g^{(3)}(0)+(I^4\Psi(x))^{\mathrm{T}}U(I\Psi(t)).\end{aligned} \tag{5.6}$$

We introduce the following notations

$$F(t)=g(1)-g(0)-g^{(1)}(0)-\frac{1}{2}g^{(2)}(0)-\frac{1}{6}g^{(3)}(0)+(I^4\Psi(1))^{\mathrm{T}}U(I\Psi(t)), \tag{5.7}$$

$$G(t)=g^{(1)}(1)-g^{(1)}(0)-g^{(2)}(0)-\frac{1}{2}g^{(3)}(0)+(I^3\Psi(1))^{\mathrm{T}}U(I\Psi(t)), \tag{5.8}$$

$$H(t)=g^{(2)}(1)-g^{(2)}(0)-g^{(3)}(0)+(I^2\Psi(1))^{\mathrm{T}}U(I\Psi(t)). \tag{5.9}$$

And we suppose that $\frac{\partial u}{\partial x}\Big|_{x=0}$, $\frac{\partial^2 u}{\partial x^2}\Big|_{x=0}$ and $\frac{\partial^3 u}{\partial x^3}\Big|_{x=0}$ are unknown functions with respect to t. These functions can be expressed by U ,$\Psi(x)$ and $\Psi(t)$ with the aid of boundary conditions. We take the problem with the first type boundary condition as an example to describe the collocation method in detail.

For the first type of boundary condition (I), by putting $x=1$ into Eqs.(5.5),(5.6), we obtain a linear system of $\frac{\partial^2 u}{\partial x^2}\Big|_{x=0}$ and $\frac{\partial^3 u}{\partial x^3}\Big|_{x=0}$, which can be written as

$$-\frac{\partial^2 u}{\partial x^2}\Big|_{x=0}-\frac{1}{2}\frac{\partial^3 u}{\partial x^3}\Big|_{x=0}=G(t), \quad -\frac{1}{2}\frac{\partial^2 u}{\partial x^2}\Big|_{x=0}-\frac{1}{6}\frac{\partial^3 u}{\partial x^3}\Big|_{x=0}=F(t).$$

Solve the above equations. $\frac{\partial^2 u}{\partial x^2}\Big|_{x=0}$ and $\frac{\partial^3 u}{\partial x^3}\Big|_{x=0}$ can be expressed as follows

$$\frac{\partial^2 u}{\partial x^2}\Big|_{x=0} = 2G(t) - 6F(t), \quad \frac{\partial^3 u}{\partial x^3}\Big|_{x=0} = -6G(t) + 12F(t).$$

By putting $\frac{\partial^2 u}{\partial x^2}\Big|_{x=0}$, $\frac{\partial^3 u}{\partial x^3}\Big|_{x=0}$ into (5.6), we get

$$\begin{aligned}u(x,t) =&\frac{1}{2}x^2(2G(t) - 6F(t)) + \frac{1}{6}x^3(-6G(t) + 12F(t)) + g(x) - g(0)\\ &- g^{(1)}(x) - xg^{(1)}(0) - \frac{1}{2}x^2 g^{(2)}(0) - \frac{1}{6}x^3 g^{(3)}(0) + (I^4\Psi(x))^{\mathrm{T}}U(I\Psi(t)).\end{aligned}$$

Therefore, we have

$$\begin{aligned}u(x,t) =&(x^2 - x^3)(g^{(1)}(1) - g^{(1)}(0) - g^{(2)}(0) - \frac{1}{2}g^{(3)}(0) + (I^3\Psi(1))^{\mathrm{T}}U(I\Psi(t)))\\ &+ (2x^3 - 3x^2)(g(1) - g(0) - g^{(1)}(0) - \frac{1}{2}g^{(2)}(0) - \frac{1}{6}g^{(3)}(0) + (I^4\Psi(1))^{\mathrm{T}}U(I\Psi(t)))\\ &+ g(x) - g(0) - g^{(1)}(x) - xg^{(1)}(0) - \frac{1}{2}x^2g^{(2)}(0) - \frac{1}{6}x^3g^{(3)}(0) + (I^4\Psi(x))^{\mathrm{T}}U(I\Psi(t)).\end{aligned} \tag{5.10}$$

By applying the first order derivative with respect to t on $u(x,t)$, we have

$$\frac{\partial u}{\partial t} = (x^2 - x^3)(I^3\Psi(1))^{\mathrm{T}}U\Psi(t) + (2x^3 - 3x^2)(I^4\Psi(1))^{\mathrm{T}}U\Psi(t) + (I^4\Psi(x))^{\mathrm{T}}U\Psi(t). \tag{5.11}$$

Next we calculate the integral term $\int_0^t (t-s)^{-\alpha} u_{xxxx}(x,s)\mathrm{d}s$.

Note that

$$u_{xxxx}(x,t) = \frac{\partial(\int_0^t u_{xxxx}(x,s)\mathrm{d}s)}{\partial t}.$$

According the definition of fractional derivative in the Caputo sense, we have

$$D_t^\alpha\Big(\int_0^t u_{xxxx}(x,s)\mathrm{d}s\Big) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}u_{xxxx}(x,s)\mathrm{d}s.$$

Therefore

$$\begin{aligned}\int_0^t (t-s)^{-\alpha}u_{xxxx}(x,s)\mathrm{d}s =&\Gamma(1-\alpha)D_t^\alpha\Big(\int_0^t u_{xxxx}(x,s)\mathrm{d}s\Big)\\ =&\Gamma(1-\alpha)D_t^\alpha\Big(tg^{(4)}(x) + (\Psi(x))^{\mathrm{T}}U(I^2\Psi(t))\Big)\\ =&\Gamma(1-\alpha)\Big(\frac{1}{\Gamma(2-\alpha)}t^{1-\alpha}g^{(4)}(x) + (\Psi(x))^{\mathrm{T}}U(I^{2-\alpha}\Psi(t))\Big).\end{aligned} \tag{5.12}$$

In order to calculate the unknown matrix U in (5.1), the following collocation points are considered,

$$x_i = \frac{2i-1}{2^kM}, t_j = \frac{2j-1}{2^kM}, i,j = 1,2,\cdots,2^{k-1}M. \tag{5.13}$$

Substituting (5.11),(5.12) into Eq.(1.1) and substituting the collocation points into Eq. (1.1), we obtain the following linear system of algebraic equations:

$$\begin{aligned}&(x_i^2 - x_i^3)(I^3\Psi(1))^{\mathrm{T}}U\Psi(t_j) + (2x_i^3 - 3x_i^2)(I^4\Psi(1))^{\mathrm{T}}U\Psi(t_j) + (I^4\Psi(x_i))^{\mathrm{T}}U\Psi(t_j)\\ &+ \Gamma(1-\alpha)\Big(\frac{1}{\Gamma(2-\alpha)}t_j^{1-\alpha}g^{(4)}(x_i) + (\Psi(x_i))^{\mathrm{T}}U(I^{2-\alpha}\Psi(t_j))\Big) = f(x_i,t_j).\end{aligned} \tag{5.14}$$

Letting $f_1(x,t) = f(x,t) - \frac{\Gamma(1-\alpha)}{\Gamma(2-\alpha)}t^{1-\alpha}g^{(4)}(x)$ and by means of the tensor product of matrix, the above system can be expressed as

$$\Big[(F_1 + F_2 + F_3) \otimes F_5 + \Gamma(1-\alpha)F_4 \otimes F_6\Big]\hat{\mathbf{u}} = \hat{\mathbf{F}},$$

where

$$\begin{aligned}
F_1 =&[(x_1^2-x_1^3)I^3\Psi(1),(x_2^2-x_2^3)I^3\Psi(1),\cdots,(x_{2^{k-1}M}^2-x_{2^{k-1}M}^3)I^3\Psi(1)],\\
F_2 =&[(2x_1^3-3x_1^2)I^4\Psi(1),(2x_2^3-3x_2^2)I^4\Psi(1),\cdots,(2x_{2^{k-1}M}^3-3x_{2^{k-1}M}^2)I^4\Psi(1)],\\
F_3 =&[I^4\Psi(x_1),I^4\Psi(x_2),\cdots,I^4\Psi(2^{k-1}M)], F_4=[\Psi(x_1),\Psi(x_2),\cdots,\Psi(t_{2^{k-1}M})],\\
F_5 =&[\Psi(t_1),\Psi(t_2),\cdots,\Psi(t_{2^{k-1}M})], F_6=[I^{2-\alpha}\Psi(t_1),I^{2-\alpha}\Psi(t_2),\cdots,I^{2-\alpha}\Psi(t_{2^{k-1}M})],\\
\hat{\mathbf{F}} =&[f_1(x_1,t_1),f_1(x_1,t_2),\cdots,f_1(x_1,t_{2^{k-1}M}),f_1(x_2,t_1),f_1(x_2,t_2),\cdots,f_1(x_2,t_{2^{k-1}M}),\\
&\cdots,f_1(x_{2^{k-1}M},t_1),f_1(x_{2^{k-1}M},t_2),\cdots,f_1(x_{2^{k-1}M},t_{2^{k-1}M})],\\
\hat{\mathbf{u}} =&[\hat{u}_{1,1}^n,\hat{u}_{1,2}^n,\cdots,\hat{u}_{1,2^{k-1}M}^n,\hat{u}_{2,1}^n,\hat{u}_{2,2}^n,\cdots,\hat{u}_{2^{k-1}M,2^{k-1}M}^n]^{\mathrm{T}}.
\end{aligned}$$

By solving this system and determining U, we get the numerical solution of the problem by substituting U into (5.10).

In a similar way, we can deal with the problems under the other two types of boundary conditions. For simplicity, we also only give the derivation of $u(x,t)$ and $\frac{\partial u}{\partial t}$ under the following two types of boundary conditions.

For the second type of boundary condition (II), by putting $x=1$ into Eqs. (5.4),(5.6), we can easily get $\frac{\partial u}{\partial x}\Big|_{x=0}=\frac{1}{6}H(t)-F(t)$, $\frac{\partial^3 u}{\partial x^3}\Big|_{x=0}=-H(t)$. By putting $\frac{\partial u}{\partial x}\Big|_{x=0}$, $\frac{\partial^3 u}{\partial x^3}\Big|_{x=0}$ into (5.6) and combining the boundary condition, we obtain

$$\begin{aligned}
u(x,t) =&(\frac{1}{6}x-\frac{1}{6}x^3)(g^{(2)}(1)-g^{(2)}(0)-g^{(3)}(0)+(I^2\Psi(1))^{\mathrm{T}}U(I\Psi(t)))\\
&-x(g(1)-g(0)-g^{(1)}(0)-\frac{1}{2}g^{(2)}(0)-\frac{1}{6}g^{(3)}(0)+(I^4\Psi(1))^{\mathrm{T}}U(I\Psi(t)))\\
&+g(x)-g(0)-g^{(1)}(x)-xg^{(1)}(0)-\frac{1}{2}x^2g^{(2)}(0)-\frac{1}{6}x^3g^{(3)}(0)\\
&+(I^4\Psi(x))^{\mathrm{T}}U(I\Psi(t)).
\end{aligned}\tag{5.15}$$

and

$$\frac{\partial u}{\partial t}=(\frac{1}{6}x-\frac{1}{6}x^3)(I^2\Psi(1))^{\mathrm{T}}U\Psi(t)-x(I^4\Psi(1))^{\mathrm{T}}U\Psi(t)+(I^4\Psi(x))^{\mathrm{T}}U\Psi(t).\tag{5.16}$$

For the third type of boundary condition (III), by putting $x=1$ into Eqs. (5.4)-(5.6) and using boundary conditions, we obtain a linear system of $\frac{\partial u}{\partial x}\Big|_{x=0}$, $\frac{\partial^2 u}{\partial x^2}\Big|_{x=0}$ and $\frac{\partial^3 u}{\partial x^3}\Big|_{x=0}$, which can be written as

$$\begin{gathered}
\frac{\partial u}{\partial x}\Big|_{x=0}+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}\Big|_{x=0}+\frac{1}{6}\frac{\partial^3 u}{\partial x^3}\Big|_{x=0}+F(t)=0,\\
-p\frac{\partial u}{\partial x}\Big|_{x=0}+\frac{\partial^2 u}{\partial x^2}\Big|_{x=0}=0,\\
-p\frac{\partial u}{\partial x}\Big|_{x=0}+(1-p)\frac{\partial^2 u}{\partial x^2}\Big|_{x=0}+(1-\frac{1}{2}p)\frac{\partial^3 u}{\partial x^3}\Big|_{x=0}-pG(t)+H(t)=0.
\end{gathered}$$

The matrix form of the above linear equations is given as follows

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{6}\\ -p & 1 & 0\\ -p & 1-p & 1-\frac{1}{2}p \end{pmatrix}\begin{pmatrix} \frac{\partial u}{\partial x}\Big|_{x=0}\\ \frac{\partial^2 u}{\partial^2 x}\Big|_{x=0}\\ \frac{\partial^3 u}{\partial^3 x}\Big|_{x=0} \end{pmatrix}=\begin{pmatrix} -F(t)\\ 0\\ pG(t)-H(t) \end{pmatrix}.$$

Solving this equations, we have

$$\frac{\partial u}{\partial x}\Big|_{x=0}=\frac{1}{p^2-12}\left[(12-6p)F(t)+2(pG(t)-H(t)\right],$$

$$\frac{\partial^2 u}{\partial x^2}\Big|_{x=0} = \frac{1}{p^2-12}\left[(12p-6p^2)F(t)+2p(pG(t)-H(t))\right],$$
$$\frac{\partial^3 u}{\partial x^3}\Big|_{x=0} = \frac{1}{p^2-12}\left[12p^2F(t)-(12+6p)(pG(t)-H(t))\right].$$

Substituting $\frac{\partial u}{\partial x}\Big|_{x=0}, \frac{\partial^2 u}{\partial x^2}\Big|_{x=0}$ and $\frac{\partial^3 u}{\partial x^3}\Big|_{x=0}$ into (5.6) and rearranging the expression, we obtain

$$\begin{aligned} u(x,t) =& \frac{1}{p^2-12}[(12-6p)x+(6p-3p^2)x^2+2p^2x^3]F(t) \\ &+ \frac{1}{p^2-12}[2px+p^2x^2-(2+p)px^3]G(t)+\frac{1}{p^2-12}[-2x-px^2+(2+p)x^3]H(t) \\ &+ g(x)-g(0)-g^{(1)}(x)-xg^{(1)}(0)-\frac{1}{2}x^2g^{(2)}(0)-\frac{1}{6}x^3g^{(3)}(0)+(I^4\Psi(x))^{\mathrm{T}}U(I\Psi(t)). \end{aligned} \tag{5.17}$$

Therefore we have

$$\begin{aligned} u(x,t) =& \frac{1}{p^2-12}[(12-6p)x+(6p-3p^2)x^2+2p^2x^3](g(1)-g(0)-g^{(1)}(0)-\frac{1}{2}g^{(2)}(0) \\ &- \frac{1}{6}g^{(3)}(0)+(I^4\Psi(1))^{\mathrm{T}}U(I\Psi(t)))+\frac{1}{p^2-12}[2px+p^2x^2-(2+p)px^3](g^{(1)}(1) \\ &- g^{(1)}(0)-g^{(2)}(0)-\frac{1}{2}g^{(3)}(0)+(I^3\Psi(1))^{\mathrm{T}}U(I\Psi(t)))+\frac{1}{p^2-12}[-2x-px^2 \\ &+ (2+p)x^3](g^{(2)}(1)-g^{(2)}(0)-g^{(3)}(0)+(I^2\Psi(1))^{\mathrm{T}}U(I\Psi(t)))+g(x)-g(0) \\ &- g^{(1)}(x)-xg^{(1)}(0)-\frac{1}{2}x^2g^{(2)}(0)-\frac{1}{6}x^3g^{(3)}(0)+(I^4\Psi(x))^{\mathrm{T}}U(I\Psi(t)), \end{aligned} \tag{5.18}$$

and

$$\begin{aligned} \frac{\partial u}{\partial t} =& \frac{1}{p^2-12}[(12-6p)x+(6p-3p^2)x^2+2p^2x^3](I^4\Psi(1))^{\mathrm{T}}U(\Psi(t))) \\ &+ \frac{1}{p^2-12}[2px+p^2x^2-(2+p)px^3](I^3\Psi(1))^{\mathrm{T}}U(\Psi(t))) \\ &+ \frac{1}{p^2-12}[-2x-px^2+(2+p)x^3](I^2\Psi(1))^{\mathrm{T}}U(\Psi(t))) \\ &+ g(x)-g(0)-g^{(1)}(x)-xg^{(1)}(0)-\frac{1}{2}x^2g^{(2)}(0)-\frac{1}{6}x^3g^{(3)}(0) \\ &+ (I^4\Psi(x))^{\mathrm{T}}U(I\Psi(t)). \end{aligned} \tag{5.19}$$

Actually, the proposed method can be applied to solve the partial integro-differential equations with multiple weakly singular kernels. We consider the following problem

$$\frac{\partial u(x,t)}{\partial t} - \int_0^t [(t-s)^{-\alpha}u_{xx}(x,s)-(t-s)^{-\beta}u_{xxxx}(x,s)]\mathrm{d}s = f(x,t),$$
$$0<\alpha,\beta<1, x\in[0,1], 0<t\le 1, \tag{5.20}$$

subject to the initial condition (1.2) and the second type boundary condition (II). We can handle the problem in previous way. Here we only derive the expression of $\int_0^t (t-s)^{-\alpha}u_{xx}(x,s)\mathrm{d}s$. By applying two times differentiation of (5.15) with respect to x, we get

$$\begin{aligned} u_{xx}(x,t) =& (-x)(g^{(2)}(1)-g^{(2)}(0)-g^{(3)}(0)+(I^2\Psi(1))^{\mathrm{T}}U(I\Psi(t))) \\ &+ (g^{(2)}(x)-g^{(2)}(0)-xg^{(3)}(0))+(I^2\Psi(x))^{\mathrm{T}}U(I\Psi(t)). \end{aligned} \tag{5.21}$$

Thus

$$\int_0^t (t-s)^{-\alpha}u_{xx}(x,s)\mathrm{d}s$$

$$
\begin{aligned}
=&\Gamma(1-\alpha)D_t^\alpha\Big(\int_0^t u_{xx}(x,s)\mathrm{d}s\Big)\\
=&\Gamma(1-\alpha)D_t^\alpha\Big((-x)t(g^{(2)}(1)-g^{(2)}(0)-g^{(3)}(0))+(I^2\Psi(1))^{\mathrm{T}}U(I^2\Psi(t))\\
&+t(g^{(2)}(x)-g^{(2)}(0)-xg^{(3)}(0))+(I^2\Psi(x))^{\mathrm{T}}U(I^2\Psi(t))\Big)\\
=&\Gamma(1-\alpha)\Big((-x)\frac{1}{\Gamma(2-\alpha)}t^{1-\alpha}(g^{(2)}(1)-g^{(2)}(0)-g^{(3)}(0)+(I^2\Psi(1))^{\mathrm{T}}U(I^{2-\alpha}\Psi(t)))\\
&+\frac{1}{\Gamma(2-\alpha)}t^{1-\alpha}(g^{(2)}(x)-g^{(2)}(0)-xg^{(3)}(0))+(I^2\Psi(x))^{\mathrm{T}}U(I^{2-\alpha}\Psi(t)))\Big). \qquad (5.22)
\end{aligned}
$$

6. Numerical Examples

In this section, we give some numerical examples for the fourth-order PIDEs with three types of boundary conditions to demonstrate the efficiency and reliability of the proposed method and the PIDE with two weakly singular kernels is also considered.

Example 1 Consider the following partial integro-differential equation

$$\frac{\partial u(x,t)}{\partial t}+\int_0^t (t-s)^{-\alpha}u_{xxxx}(x,s)\mathrm{d}s=f(x,t), x\in[0,1], 0<t\leq 1,$$

subject to the initial condition

$$u(x,0)=\sin^2(\pi x), x\in[0,1],$$

and the first type of boundary condition (I),

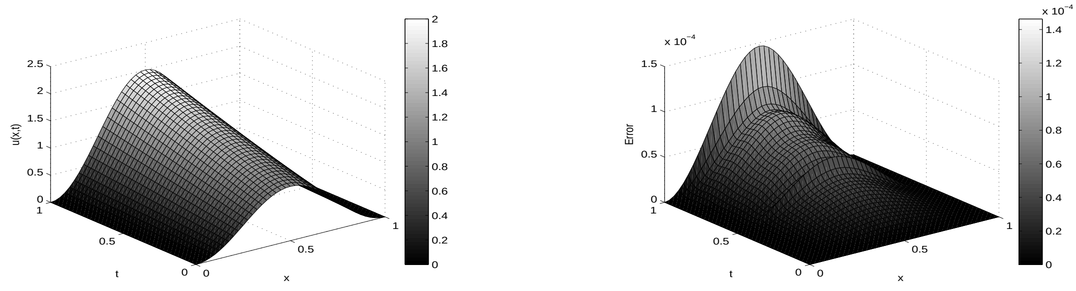
$$u(0,t)=u(1,t)=0,\quad u_x(0,t)=u_x(1,t)=0, 0<t\leq 1,$$

where $f(x,t)=\sin^2(\pi x)-\Gamma(1-\alpha)8\pi^2\cos 2\pi x\Big(\frac{t^{2-\alpha}}{\Gamma(3-\alpha)}+\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\Big)$. $f(x,t)=\sin^2(\pi x)-\Gamma(1-\alpha)8\pi^4\sin\pi x\Big(\frac{t^{2-\alpha}}{\Gamma(3-\alpha)}+\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\Big)$.

The exact solution of the problem is

$$u(x,t)=(t+1)\sin^2(\pi x).$$

The problem is solved with different values of α. Fig.1 shows the approximate solution and the absolute error of this problem in the case of $\alpha=0.3, k=2$ and $M=6$. Tab.1 lists the absolute error for different values of α at different points with $k=2, M=6$. Tab. 2 and Tab. 3 give the maximum absolute error (L_∞) obtained by the proposed method for different choices of M and α at the points (x_i,t_j), where $x_i=i/40, t_j=j/40, i,j=0,1,2,\cdots,40$.



(a) (b)

Fig. 1 Approximate solution (a) and absolute error (b) for Example 1 with $\alpha=0.3$, $k=2$ and $M=6$

Tab. 1 The absolute error of Example 1 for different α at some different points

(x,t)	$\alpha=0.1$	$\alpha=0.3$	$\alpha=0.5$	$\alpha=0.7$	$\alpha=0.9$
(0.1,0.1)	1.76743e-6	9.96690e-7	9.71224e-7	9.82030e-7	9.91631e-7
(0.2,0.2)	1.10508e-5	6.54779e-6	6.60447e-6	6.63202e-6	6.65436e-6
(0.3,0.3)	1.70898e-5	1.84326e-5	1.83014e-5	1.83021e-5	1.83309e-5
(0.4,0.4)	3.27028e-5	3.29952e-5	3.35597e-5	3.37228e-5	3.37864e-5
(0.5,0.5)	5.51068e-5	5.76723e-5	4.98918e-5	4.76658e-5	4.72514e-5
(0.6,0.6)	1.33313e-4	5.11462e-5	5.05459e-5	5.06226e-5	5.06897e-5
(0.7,0.7)	1.05417e-4	4.23211e-5	4.26857e-5	4.27615e-5	4.27913e-5
(0.8,0.8)	2.58550e-5	2.70425e-5	2.67029e-5	2.66367e-5	2.66432e-5
(0.9,0.9)	6.33376e-6	8.49517e-6	8.84158e-6	8.93422e-6	8.94815e-6

Tab. 2 Maximum absolute error of Example 1 with various choices of M,α and $k=2$

M	$\alpha=0.1$	$\alpha=0.3$	$\alpha=0.5$	$\alpha=0.7$	$\alpha=0.9$
3	1.30616e-1	7.49763e-2	5.83501e-2	5.49544e-2	5.45021e-2
4	1.12283e-2	6.75874e-3	4.84077e-3	4.44521e-3	4.39793e-3
5	4.09630e-3	3.88567e-3	2.74590e-3	2.47251e-3	2.44065e-3
6	8.67768e-4	1.46285e-4	1.07750e-4	9.59128e-5	9.45036e-5
7	5.96147e-5	8.59729e-5	6.68252e-5	5.92723e-5	5.83125e-5
8	1.60781e-6	2.00386e-6	1.52439e-6	1.35627e-6	1.33284e-6

Tab. 3 Maximum absolute error of Example 1 with various choices of M,α and $k=3$

M	$\alpha=0.1$	$\alpha=0.3$	$\alpha=0.5$	$\alpha=0.7$	$\alpha=0.9$
3	5.15735e-3	4.88303e-3	3.90068e-3	2.41774e-3	2.29140e-3
4	1.98810e-3	4.12869e-4	5.25232e-4	3.58104e-4	3.34869e-4
5	3.99359e-4	3.88900e-5	3.32672e-5	2.75819e-5	2.56466e-5
6	8.68424e-5	5.15677e-6	2.87459e-6	2.76553e-6	2.57823e-6
7	1.01470e-5	4.21375e-7	1.61828e-7	1.63933e-7	1.53962e-7
8	1.40944e-6	5.34189e-8	1.22156e-8	1.22384e-8	1.16408e-8

Example 2 Consider the following partial integro-differential equation

$$\frac{\partial u(x,t)}{\partial t}+\int_0^t (t-s)^{-\alpha}u_{xxxx}(x,s)\mathrm{d}s=f(x,t), x\in[0,1], 0<t\leq 1,$$

subject to the initial condition

$$u(x,0)=\sin(\pi x), x\in[0,1],$$

and the second type of boundary condition (II),

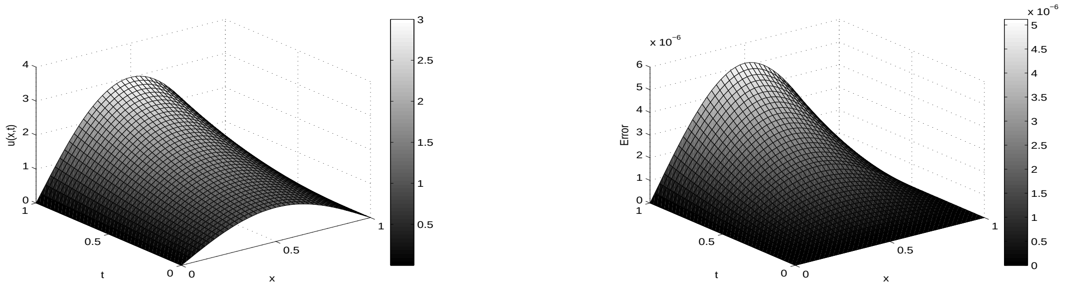
$$u(0,t)=u(1,t)=0,$$
$$u_{xx}(0,t)=u_{xx}(1,t)=0, 0<t\leq 1,$$

where $f(x,t)=(2t+1)\sin(\pi x)+\Gamma(1-\alpha)\pi^4\sin\pi x\left(\frac{2t^{3-\alpha}}{\Gamma(4-\alpha)}+\frac{t^{2-\alpha}}{\Gamma(3-\alpha)}+\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\right)$.

The exact solution of the problem is

$$u(x,t)=(t^2+t+1)\sin(\pi x).$$

The problem is solved with different values of α. Fig. 2 plots the approximate solution and the absolute error of this problem in the case of $\alpha=0.5, k=2$ and $M=6$. Tab. 4 shows the absolute error for different values of α at different points with $k=2, M=6$. Tab. 5 gives the maximum absolute error (L_∞) obtained by the proposed method for different choices of M and α at some points (x_i,t_j).



(c) (d)

Fig. 2 Approximate solution (c) and absolute error (d) of Example 2 with $\alpha = 0.5$, $k = 2$ and $M = 6$

Tab. 4 The absolute error of Example 2 for some different α at some different points

(x,t)	$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$
(0.1,0.1)	1.63615e-8	3.73476e-8	6.76366e-8	7.74003e-8	8.21272e-8
(0.2,0.2)	2.17512e-7	3.14321e-7	3.33734e-7	3.39854e-7	3.47286e-7
(0.3,0.3)	7.56195e-7	7.89293e-7	7.65672e-7	7.79559e-7	7.89867e-7
(0.4,0.4)	1.45035e-6	1.31649e-6	1.32623e-6	1.33972e-6	1.35468e-6
(0.5,0.5)	1.98938e-6	1.85327e-6	1.94620e-6	1.95058e-6	1.93073e-6
(0.6,0.6)	2.24180e-6	2.28933e-6	2.28852e-6	2.30748e-6	2.32508e-6
(0.7,0.7)	2.28079e-6	2.39428e-6	2.39212e-6	2.40366e-6	2.41639e-6
(0.8,0.8)	2.03374e-6	2.06830e-6	2.07587e-6	2.08534e-6	2.09473e-6
(0.9,0.9)	1.30154e-6	1.27999e-6	1.28435e-6	1.28803e-6	1.29434e-6

Tab. 5 Maximum absolute error of Example 2 with various choices of M and α

M	$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$
3	4.49663e-3	4.71098e-3	4.81947e-3	4.62984e-3	4.57729e-3
4	6.70310e-4	6.61212e-4	6.86610e-4	6.76394e-4	6.68879e-4
5	5.19839e-5	5.07648e-5	5.14501e-5	5.17173e-5	5.12233e-5
6	5.26043e-6	5.10912e-6	5.11876e-6	5.18527e-6	5.14905e-6
7	3.15451e-7	3.05134e-7	3.04982e-7	3.08800e-7	3.07460e-7
8	2.41261e-8	2.30893e-8	2.30787e-8	2.32903e-8	2.32483e-8

Example 3 Consider the following partial integro-differential equation

$$\frac{\partial u(x,t)}{\partial t} + \int_0^t (t-s)^{-\alpha} u_{xxxx}(x,s)\mathrm{d}s = f(x,t), x \in [0,1], 0 < t \le 1,$$

subject to the initial condition

$$u(x,0) = \pi^5 \sin(\pi x) + \frac{1}{\pi^5}\cos(\pi x) - \frac{1}{\pi^5}\cos(3\pi x), x \in [0,1],$$

and the third type of boundary condition (III),

$$u(0,t) = u(1,t) = 0,$$

$$u_{xx}(0,t) - \frac{8}{\pi^9}u_x(0,t) = u_{xx}(1,t) - \frac{8}{\pi^9}u_x(1,t) = 0, 0 < t \le 1,$$

where $f(x,t) = \pi^5\sin\pi x + \frac{1}{\pi^5}\cos\pi x - \frac{1}{\pi^5}\cos 3\pi x + \Gamma(1-\alpha)\Big(\frac{t^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\Big)\Big(\pi^9\sin\pi x + \frac{1}{\pi}\cos\pi x - \frac{81}{\pi}\cos 3\pi x\Big)$.

The exact solution of the problem is

$$u(x,t) = (t+1)(\pi^5 \sin(\pi x) + \frac{1}{\pi^5}\cos(\pi x) - \frac{1}{\pi^5}\cos(3\pi x)).$$

The problem is solved with different values of α. The graph of the approximate solution and the absolute error of this problem in the case of $\alpha = 0.7, k = 2$ and $M = 6$ is shown is Fig. 3. Tab. 6 shows the absolute error for different values of α at different points with $k = 2, M = 6$. Tab. 7 gives the maximum absolute error (L_∞) obtained by the proposed method for different choices of M and α at some points (x_i, t_j).

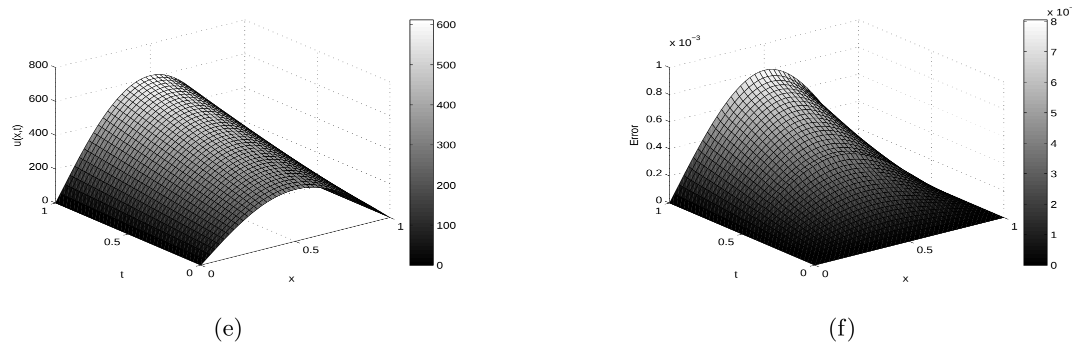


Fig. 3 Approximate solution (e) and absolute error (f) for Example 3 with $\alpha = 0.7$, $k = 2$ and $M = 6$

Tab. 6 The absolute error of Example 3 for some different α at some different points

(x,t)	$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$
(0.1,0.1)	4.67054e-6	1.07546e-5	1.92247e-5	2.15917e-5	2.27561e-5
(0.2,0.2)	5.95068e-5	8.45425e-5	8.66330e-5	8.68885e-5	8.82987e-5
(0.3,0.3)	1.93491e-4	1.93038e-4	1.82152e-4	1.83964e-4	1.85566e-4
(0.4,0.4)	3.40344e-4	2.92434e-4	2.92499e-4	2.93474e-4	2.95859e-4
(0.5,0.5)	4.20555e-4	3.81031e-4	4.06634e-4	4.03537e-4	3.94755e-4
(0.6,0.6)	4.27651e-4	4.45410e-4	4.40870e-4	4.43238e-4	4.45787e-4
(0.7,0.7)	4.06322e-4	4.38868e-4	4.35063e-4	4.35406e-4	4.36638e-4
(0.8,0.8)	3.50759e-4	3.56418e-4	3.56598e-4	3.57112e-4	3.57881e-4
(0.9,0.9)	2.16854e-4	2.08859e-4	2.09113e-4	2.08955e-4	2.09696e-4

Tab. 7 Maximum absolute error of Example 3 with various choices of M and α

M	$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$
3	6.87344e-1	7.54116e-1	7.81443e-1	7.20880e-1	7.01659e-1
4	1.04482e-1	1.01647e-1	1.09097e-1	1.05337e-1	1.02538e-1
5	8.20941e-3	7.82928e-3	8.01174e-3	8.04457e-3	7.85290e-3
6	8.35145e-4	7.89189e-4	7.88680e-4	8.04649e-4	7.89389e-4
7	5.02709e-5	4.71367e-5	4.68746e-5	4.77989e-5	4.71361e-5
8	5.25140e-6	3.56932e-6	3.54983e-6	3.59603e-6	3.56407e-6

Example 4 Consider the following partial integro-differential equation with two weakly singular kernels

$$\frac{\partial u(x,t)}{\partial t} - \int_0^t [(t-s)^{-\alpha} u_{xx}(x,s) - (t-s)^{-\beta} u_{xxxx}(x,s)] \mathrm{d}s = f(x,t), x \in [0,1], 0 < t \leq 1,$$

subject to the initial condition

$$u(x,0)=\sin(\pi x), x\in[0,1],$$

and the second type of boundary condition (II),

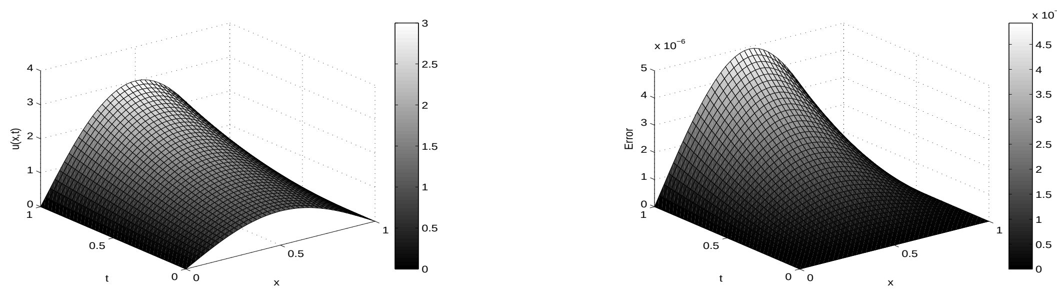
$$u(0,t)=u(1,t)=0,$$
$$u_{xx}(0,t)=u_{xx}(1,t)=0, 0<t\le 1,$$

where $f(x,t)=(2t+1)\sin\pi x+\Gamma(1-\alpha)\pi^2\sin\pi x\left(\frac{2t^{3-\alpha}}{\Gamma(4-\alpha)}+\frac{t^{2-\alpha}}{\Gamma(3-\alpha)}+\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\right)+\Gamma(1-\beta)\pi^4\sin\pi x\left(\frac{2t^{3-\beta}}{\Gamma(4-\beta)}\right.$ $\left.+\frac{t^{2-\beta}}{\Gamma(3-\beta)}+\frac{t^{1-\beta}}{\Gamma(2-\beta)}\right)$.

The exact solution of the problem is

$$u(x,t)=(t^2+t+1)\sin(\pi x).$$

This problem is solved in the case of different values of (α,β). Fig. 4 demonstrates the approximate solution and the absolute error of this problem in the case of $\alpha=4/5,\beta=9/10,k=2$ and $M=6$. Tab. 8 shows the absolute error for different values of (α,β) at different points with $k=2,M=6$. Tab. 9 gives the maximum absolute error (L_∞) obtained by the proposed method for different choices of M and (α,β) at some points.



(g) (h)

Fig. 4 Approximate solution (g) and absolute error (h) for Example 4 with $\alpha=4/5,\beta=9/10$, $k=2$ and $M=6$

Tab. 7 The absolute error of Example 4 for different α and β at some points

(x,t)	$(1/5,1/4)$	$(1/3,1/2)$	$(1/2,1/2)$	$(1/2,2/3)$	$(4/5,9/10)$
(0.1,0.1)	3.00479e-8	6.56853e-8	6.29856e-8	7.34955e-8	7.91628e-8
(0.2,0.2)	2.80001e-7	3.17570e-7	3.03382e-7	3.23637e-7	3.33920e-7
(0.3,0.3)	7.39479e-7	7.25648e-7	6.97030e-7	7.41391e-7	7.58665e-7
(0.4,0.4)	1.22427e-6	1.25348e-6	1.20678e-6	1.27222e-6	1.30049e-6
(0.5,0.5)	1.69763e-6	1.84456e-6	1.78241e-6	1.86028e-6	1.85224e-6
(0.6,0.6)	2.09254e-6	2.15340e-6	2.08086e-6	2.18526e-6	2.22863e-6
(0.7,0.7)	2.19936e-6	2.24684e-6	2.17297e-6	2.27318e-6	2.31341e-6
(0.8,0.8)	1.89514e-6	1.94512e-6	1.88251e-6	1.96871e-6	2.00329e-6
(0.9,0.9)	1.16778e-6	1.20123e-6	1.16343e-6	1.21392e-6	1.23675e-6

Tab. 8 Maximum absolute error for Example 4 with various choices of (α, β) and M

M	$(1/5, 1/4)$	$(1/3, 1/2)$	$(1/2, 1/2)$	$(1/2, 2/3)$	$(4/5, 9/10)$
3	4.26158e-3	4.53866e-3	3.29228e-3	4.25534e-3	4.37987e-3
4	6.06759e-4	6.47020e-4	6.28570e-4	5.83844e-4	6.40031e-4
5	4.65507e-5	4.83762e-5	4.70518e-5	4.43048e-5	4.90129e-5
6	4.68339e-6	4.80198e-6	4.66813e-6	4.46405e-6	4.92726e-6
7	2.80038e-7	2.85745e-7	2.77494e-7	2.66372e-7	2.94214e-7
8	2.12344e-8	2.16228e-8	2.09877e-8	2.19935e-8	2.22476e-8

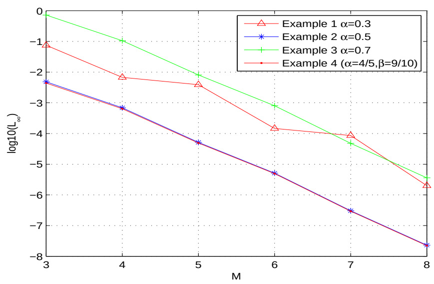


Fig. 5 Error as a function of the polynomial degree for various values of α and β

Remark 6.1 Finally, we present the errors as a function of the polynomial degree for the all examples with different values of α and β in Fig. 5, where a logarithmic scale is used for the error axis. It is obviously that the errors appear in an exponential decay. Since one can obverse that the error variations are essentially linear versus the polynomial degrees in this semi-log axis.

7. Conclusion

In this paper, wavelet collocation method based on the fourth kind Chebyshev wavelets has been applied to a class of PIDEs with a weakly singular kernel. We derived the fractional integral formula of a single Chebyshev wavelet in the Riemann-Liouville sense via the shifted Chebyshev polynomials. The convergence analysis of two-dimensional fourth kind Chebyhev wavelets was studied. By applying fractional integral formula and fourth kind Chebyhev wavelets together with collocation method, PIDEs with a weakly singular kernel are reduced to a system of algebraic equation. The proposed method is very convenient for solving PIDEs, since the initial and boundary conditions are taken into account automatically when constructing the expression of the approximate solution to the problem. Several numerical examples are given to show the applicability and accuracy of the proposed method.

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一类具有弱奇异核的偏积分微分方程的Chebyshev小波数值方法

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摘要: 本文提出一种基于第四类Chebyshev小波配置法, 求解了一类具有弱奇异核的偏积分微分方程数值解. 利用第四类移位Chebyshev多项式,在Riemann-Liouville分数阶积分意义下, 导出Chebyshev的分数次积分公式. 通过利用分数次积分公式和二维的第四类Chebyshev小波结合配置法, 将具有弱奇异核的偏积分微分方程转化为代数方程组求解. 给出了第四类Chebyshev小波的收敛性分析. 数值例子证明了本文方法的有效性.

关键词: 偏积分微分方程; 弱奇异核; 第四类Chebyshev小波; 配置法; 分数次积分