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A Modification for the Viscosity Approximation Method for Fixed Point Problems in Hilbert Spaces

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Abstract: In this paper, we present a modification for the viscosity approximation method for fixed point problems of a nonexpansive mapping in Hilbert spaces. The modification removes a control condition of the viscosity approximation method. We establish a strong convergence theorem for the modified algorithm.
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1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H. Recall that a self-mapping $f : C \to C$ is an α-contraction if there exists a constant $\alpha \in [0, 1)$ such that

$$\|f(x) - f(y)\| \le \alpha\|x - y\|, \forall x, y \in C.$$

Π_C denotes the set of all contractions on C. Note that f has a unique fixed point in C.

A mapping T of C into itself is called nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. Denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in C : Tx = x\}$.

Construction of fixed points of nonexpansive mappings is an important subject in the theory of nonexpansive mappings and its applications in a number of applied areas, in particular, in image recovery and signal processing (see, e.g., [1-2]).

An important approximation method for nonexpansive mappings is to consider the sequence $\{x_n\}$ generated by the algorithm:

$$x_{n+1} = t_n x_0 + (1 - t_n)Tx_n, \quad n \ge 0, \tag{1.1}$$

where the initial point $x_0 \in C$ is taken arbitrarily and $\{t_n\}_{n=0}^{\infty}$ is a sequence in the interval $[0, 1]$. We call the iteration process (1.1) the Halpern iteration algorithm because of the introduction by Halpern[3].

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The algorithm (1.1) has been proved to be strongly convergent in both Hilbert spaces[3−4] and uniformly smooth Banach spaces[5−7] unless the sequence $\{t_n\}$ satisfies the following conditions:

(A1) $t_n \to 0$; (A2) $\sum_{n=0}^{\infty} t_n = \infty$; (A3) either $\sum_{n=0}^{\infty} |t_n - t_{n+1}| < \infty$ or $\lim\limits_{n\to\infty}(\frac{t_n}{t_{n+1}}) = 1$.

As mentioned in [8], the algorithm (1.1) has slow convergence due to the restriction of condition (A2). Moreover, it is shown in [3] that the conditions (A1) and (A2) are necessary, in the sense that if the algorithm (1.1) is always strongly convergent for all nonexpansive mappings from C into C, the conditions (A1) and (A2) must hold. So to improve the convergence rate of the algorithm (1.1), some additional steps have to be performed. For this purpose, Yanes and XU[8] proposed the following CQ method:

$$\begin{cases} x_0 \in C, \text{arbitrarily}, y_n = t_n x_0 + (1-t_n)Tx_n, \\ C_n = \{v \in C : \|y_n - v\|^2 \le \|x_n - v\|^2 + t_n(\|x_0\|^2 + 2\langle x_n - x_0, v\rangle)\}, \\ Q_n = \{v \in C : \langle x_n - v, x_n - x_0\rangle \le 0\}, x_{n+1} = P_{C_n \bigcap Q_n} x_0, \quad n \ge 0. \end{cases} \tag{1.2}$$

They proved the algorithm (1.2) has strong convergence under the condition (A1) only, which enhances the rate of convergence of the algorithm (1.1). More precisely, they proved the following theorem:

Theorem 1.1[8] Let H be a real Hilbert space, and C be a closed convex subset of H. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \varnothing$. Assume that $\{t_n\} \subset (0,1)$ satisfying (A1). Then the sequence $\{x_n\}$ generated by (1.2) converges strongly to $P_{F(T)}x_0$.

In 2004, XU[9] studied the viscosity approximation algorithm for a nonexpansive mapping in a Hilbert space. More precisely, he proved the following theorem.

Theorem 1.2[9] Let H be a real Hilbert space, and C be a closed convex subset of H. Assume that $T : C \to C$ is a nonexpansive mapping with $F(T) \neq \varnothing$ and $f \in \Pi_C$. Let $\{x_n\}$ be given by

$$x_0 \in C, \quad x_{n+1} = t_n f(x_n) + (1-t_n)Tx_n, \quad n \ge 0. \tag{1.3}$$

Then under the conditions (A1)-(A3), $x_n \to x^*$, where x^* is the unique solution to the following variational inequality:

$$\langle (I-f)x^*, x - x^*\rangle \ge 0, \forall x \in F(T). \tag{1.4}$$

Clearly, It is an important property of the viscosity approximation algorithm to select a particular fixed point of a given nonexpansive mapping which is the unique solution to the variational inequality (1.4).

It is obvious that if $f(x) \equiv x_0$ for all $x \in C$, then the viscosity approximation method (1.4) reduces to the Halpern iteration method (1.1). We note that the conditions on the control sequence $\{t_n\}$ in (1.3) are the same as those in (1.1). Hence, according to the analysis about the Halpern iteration method (1.1) in [8], the viscosity approximation algorithm (1.3) has slow convergence due to the restriction of the condition (A2).

For this reason, we will modify the viscosity approximation algorithm and construct a new iterative algorithm by the idea of (1.2) to remove the the restriction of the condition (A2).

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$, and let C be a nonempty, closed and convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges strongly to x. We use $\omega_w(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$ to denote the weak ω-limit set of $\{x_n\}$.

For each point $x \in H$, there exists a unique nearest point denoted by $P_C(x)$ in C, that is, $\|x - P_C(x)\| \leq \|x - y\|, \ \ \forall y \in C$.

The mapping $P_C : H \to C$ is called the metric projection of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C, i.e.,

$$\|P_C(x) - P_C(y)\| \leq \|x - y\|, \quad \forall x, y \in H.$$

We recall some well known results which will be used in our analysis.

Lemma 2.1[10] For given $x \in H$ and $y \in C$:

(i) $y = P_C(x)$ if and only if $\langle x - y, z - y\rangle \leq 0, \ \ \forall z \in C$;

(ii) $\|P_C(x) - z\|^2 \leq \|x - z\|^2 - \|x - P_C(x)\|^2, \ \ \forall z \in C$.

Lemma 2.2[11] Let $\{x_k\}$ and $\{y_k\}$ be two bounded sequences in H, $\{\beta_k\}$ in [0, 1]. Suppose that

$$\begin{cases} 0 < \liminf\limits_{k\to\infty} \beta_k \leq \limsup\limits_{k\to\infty} \beta_k < 1, x_{k+1} = (1 - \beta_k) y_k + \beta_k x_k, \\ \limsup\limits_{k\to\infty} (\|y_{k+1} - y_k\| - \|x_{k+1} - x_k\|) \leq 0. \end{cases}$$

Then, $\lim\limits_{k\to\infty} \|y_k - x_k\| = 0$.

3. A Modification of the Viscosity Approximation Method

Inspired by the results of [10-11], we propose the following Algorithm 3.1 to remove the restriction of the condition (A2) for finding a fixed point of a nonexpansive mapping.

Algorithm 3.1 Initialization. Choose $u \in C$, $x_0 \in C$, positive sequences $\{t_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\epsilon_n\}$ such that

$$t_n + \beta_n + \gamma_n = 1, \ \lim_{n\to\infty} t_n = 0, \ \lim_{n\to\infty} \beta_n = \eta \in (0, 1), \ \text{ and } \sum_{n=0}^{\infty} \epsilon_n < \infty. \tag{3.1}$$

Step 1 Set $x_{n,1} := f(x_n)$.

Step 2 Inner loop $j = 1, 2, ...$

Compute

$$\begin{cases} y_{n,j} := t_j x_{n,1} + (1 - t_j) T(x_{n,j}), \\ C_{n,j} = \{v \in C : \|y_{n,j} - v\|^2 \leq \|x_{n,j} - v\|^2 + t_j(\|x_{n,1}\|^2 + 2\langle x_{n,j} - x_{n,1}, v\rangle)\}, \\ Q_{n,j} = \{v \in C : \langle x_{n,j} - v, x_{n,j} - x_{n,1}\rangle \leq 0\}, \\ x_{n,j+1} = P_{C_{n,j} \bigcap Q_{n,j}}(x_{n,1}). \end{cases} \tag{3.2}$$

If $\|x_{n,j+1} - P_{F(T)}(x_{n,1})\| \leq \epsilon_n$, then set $h_n = x_{n,j+1}$ and go to Step 3. Otherwise, increase j by 1 and repeat the inner loop Step 2.

Step 3 Set $x_{n+1} := t_n u + \beta_n x_n + \gamma_n h_n$. Then increase n by 1 and go to Step 1.

Remark 3.1 By Theorem 1.1, we have $\{x_{n,j}\}$ generated by Step 2 of Algorithm 3.1 converges strongly to the point $P_{F(T)}(f(x_n))$ as $j \to \infty$. Consequently, we have that the inner loop in Algorithm 3.1 terminates after a finite number of steps.

Theorem 3.1 Let H be a real Hilbert space, and C be a closed convex subset of H. Let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \varnothing$. Assume that $f : C \to C$ is an α-contraction. Then the sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to $x^* \in F(T)$, where x^* is the unique solution to the variational inequality (1.4).

Proof By Lemma 2.1, we have that $x^* = P_{F(T)}f(x^*)$. Since P_C is nonexpansive and f is contractive, we have x^* is unique. Firstly, we observe that

$$\begin{aligned}
&\|x_{n+1} - x^*\| \le t_n\|u - x^*\| + \beta_n\|x_n - x^*\| + \gamma_n\|h_n - x^*\| \\
&\le t_n\|u - x^*\| + \beta_n\|x_n - x^*\| + \gamma_n\|h_n - P_{F(T)}(f(x_n))\| + \gamma_n\|P_{F(T)}(f(x_n)) - P_{F(T)}f(x^*)\| \\
&\le t_n\|u - x^*\| + \beta_n\|x_n - x^*\| + \gamma_n\epsilon_n + \gamma_n\alpha\|x_n - x^*\| \\
&\le t_n\|u - x^*\| + \beta_n\|x_n - x^*\| + \gamma_n\epsilon_n + \gamma_n\|x_n - x^*\| \\
&= t_n\|u - x^*\| + (1 - t_n)\|x_n - x^*\| + \gamma_n\epsilon_n \\
&\le \max\{\|u - x^*\|, \|x_0 - x^*\|\} + \sum_{n=0}^{\infty}\epsilon_n < \infty,
\end{aligned}$$

which implies that $\{x_n\}$ is bounded, and hence, $\{h_n\}$ is also bounded. Next, we prove that $\lim\limits_{n\to\infty}\|x_{n+1} - x_n\| = 0$. We denote $x_{n+1} = \beta_n x_n + (1 - \beta_n)\xi_n$. Then, we have

$$\begin{aligned}
\xi_{n+1} - \xi_n =& \frac{t_{n+1}u + \gamma_{n+1}h_{n+1}}{1 - \beta_{n+1}} - \frac{t_n u + \gamma_n h_n}{1 - \beta_n} \\
=& (\frac{t_{n+1}}{1 - \beta_{n+1}} - \frac{t_n}{1 - \beta_n})u + (\frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n})h_n + \frac{\gamma_{n+1}}{1 - \beta_{n+1}}(h_{n+1} - h_n).
\end{aligned}$$

Thus, we get

$$\begin{aligned}
&\|\xi_{n+1} - \xi_n\| - \|x_{n+1} - x_n\| \\
&\le |\frac{t_{n+1}}{1 - \beta_{n+1}} - \frac{t_n}{1 - \beta_n}|\|u\| + |\frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n}|\|h_n\| + \frac{\gamma_{n+1}}{1 - \beta_{n+1}}\|h_{n+1} - h_n\| - \|x_{n+1} - x_n\| \\
&\le |\frac{t_{n+1}}{1 - \beta_{n+1}} - \frac{t_n}{1 - \beta_n}|\|u\| + |\frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n}|(\|P_{F(T)}f(z_n)\| + \epsilon_n) - \|x_{n+1} - x_n\| \\
&+ \frac{\gamma_{n+1}}{1 - \beta_{n+1}}(\|h_{n+1} - P_{F(T)}f(z_{n+1})\| + \|P_{F(T)}f(z_{n+1}) - P_{F(T)}f(z_n)\| + \|P_{F(T)}f(z_n) - h_n\|) \\
&\le |\frac{t_{n+1}}{1 - \beta_{n+1}} - \frac{t_n}{1 - \beta_n}|\|u\| + |\frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n}|(\|P_{F(T)}f(z_n)\| + \epsilon_n) \\
&\quad + \frac{\gamma_{n+1}}{1 - \beta_{n+1}}(\epsilon_{n+1} + \epsilon_n) + \frac{\gamma_{n+1}}{1 - \beta_{n+1}}\|x_{n+1} - x_n\| - \|x_{n+1} - x_n\| \\
&\le |\frac{t_{n+1}}{1 - \beta_{n+1}} - \frac{t_n}{1 - \beta_n}|\|u\| + |\frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n}|(\|P_{F(T)}f(z_n)\| + \epsilon_n) \\
&\quad - \frac{t_{n+1}}{1 - \beta_{n+1}}\|x_{n+1} - x_n\| + \frac{\gamma_{n+1}}{1 - \beta_{n+1}}(\epsilon_{n+1} + \epsilon_n).
\end{aligned} \tag{3.3}$$

From the assumptions in (3.1) and the boundness of $\{x_n\}$, it follows that

$$\limsup_{n\to\infty}(\|\xi_{n+1} - \xi_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Now applying Lemma 2.2, we have

$$\lim_{n\to\infty}\|\xi_n - x_n\| = 0. \tag{3.4}$$

By $x_{n+1} = \beta_n x_n + (1 - \beta_n)\xi_n$, we obtain that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1 - \beta_n)\|\xi_n - x_n\| = 0. \tag{3.5}$$

We observe that

$$\begin{aligned}
\|P_{F(T)}f(x_n) - x_{n+1}\| &\le t_n\|P_{F(T)}f(x_n) - u\| + \beta_n\|P_{F(T)}f(x_n) - x_n\| + \gamma_n\|P_{F(T)}f(x_n) - h_n\| \\
&\le t_n\|P_{F(T)}f(x_n) - u\| + \beta_n\|P_{F(T)}f(x_n) - x_n\| + \gamma_n\epsilon_n.
\end{aligned} \tag{3.6}$$

Consequently, from (3.6), we have

$$\|P_{F(T)}f(x_n)-x_n\| \le \|P_{F(T)}f(x_n)-x_{n+1}\|+\|x_{n+1}-x_n\|$$
$$\le (t_n\|P_{F(T)}f(x_n)-u\|+\beta_n\|P_{F(T)}f(x_n)-x_n\|+\gamma_n\epsilon_n)+\|x_{n+1}-x_n\|,$$

which implies that

$$(1-\beta_n)\|P_{F(T)}f(x_n)-x_n\| \le t_n\|P_{F(T)}f(x_n)-u\|+\gamma_n\epsilon_n+\|x_{n+1}-x_n\|.$$

Since $\lim\limits_{n\to\infty} t_n=0$, $\lim\limits_{n\to\infty}\epsilon_n=0$ and $\lim\limits_{n\to\infty}\beta_n=\eta\in(0,1)$, it follows from (3.5) that

$$\lim_{n\to\infty}\|P_{F(T)}f(x_n)-x_n\|=0. \tag{3.7}$$

Since

$$\|x_n-x^*\| \le \|x_n-P_{F(T)}f(x_n)\|+\|P_{F(T)}f(x_n)-P_{F(T)}f(x^*)\|$$
$$\le \|x_n-P_{F(T)}f(x_n)\|+\alpha\|x_n-x^*\|,$$

we have that $(1-\alpha)\|x_n-x^*\| \le \|x_n-P_{F(T)}f(x_n)\| \to 0$, as $n\to\infty$. This shows that $x_n\to x^*$, as $n\to\infty$.

Remark 3.2 Theorem 3.1 improves Theorem 3.2 in [9] (i.e. Theorem 1.2 in this paper) in the following sense:

The condition (A2) is removed. Algorithm 3.1 converges strongly to the unique solution of the variational inequality (1.4) under only the condition (A1).

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Hilbert空间中关于不动点问题粘滞逼近法的一个修正

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摘要: 本文对Hilbert空间中关于非扩张映射不动点问题的粘滞逼近法提出一个修正, 该修正后的算法取消了粘滞逼近法中的一个控制条件, 并建立了该算法的一个强收敛定理.

关键词: 非扩张映射; 度量投影; 不动点; 压缩; 粘滞逼近法