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An Iterative Criteria for Nongeneralized $\mathcal{H}$-Tensors

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Abstract: A necessary and sufficient condition for identifying nongeneralized $\mathcal{H}$-tensors is given, theoretically, it thoroughly solves the problem of identifying irreducible nongeneralized $\mathcal{H}$-tensors. Moreover, an iterative algorithm for judging irreducible nongeneralized $\mathcal{H}$-tensors is given. Numerical examples are also given to illustrate the advantages of our derived results.

Key words: Generalized $\mathcal{H}$-tensor; Nongeneralized $\mathcal{H}$-tensor; $\mathcal{H}$-tensor; Diagonally dominant tensor; Diagonally minor tensor; Irreducible tensor

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1. Introduction

A complex order m dimension n tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ consists of n^m complex entries:

$$a_{i_1 i_2 \cdots i_m} \in \mathbb{C},$$

where $i_j = 1, \ldots, n$ for $j = 1, \ldots, m$.[1] It is obvious that a matrix is an order 2 tensor.

By introducing the definition of $\mathcal{H}$-tensors[2−3], LI et al.[3] provided a practical sufficient condition for identifying the positive definiteness of an even-order symmetric tensor. They pointed out that $\mathcal{H}$-tensor is a special kind of tensors and an even order symmetric $\mathcal{H}$-tensor with positive diagonal entries is positive definite. Later, with the help of generalized diagonally dominant tensor, various sufficient conditions for $\mathcal{H}$-tensors are established[4−9]. However, if the tensor is a nongeneralized $\mathcal{H}$-tensor or $\mathcal{H}$-tensor, these conditions are hard to verify. In order to identify nongeneralized $\mathcal{H}$-tensor, in this paper, we establish a necessary and sufficient condition for a nongeneralized $\mathcal{H}$-tensor which depends only on the elements of the tensor. The obtained results extend the corresponding conclusions for nongeneralized diagonally dominant matrices.

Now some notation and definitions are given, which will be used in the sequel. Vectors are written as italic lowercase letters such as $x, y, \ldots$, matrices correspond to italic capitals

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such as $A, B, \ldots$, and tensors are written as calligraphic capitals such as $\mathcal{A}, \mathcal{B}, \ldots$ For two vectors $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ and $y = (y_1, y_2, \ldots, y_n)^{\mathrm{T}}$, it is denoted, by $x \circ y$, the hadamard product of x and y, i.e., $x \circ y = (x_1y_1, x_2y_2, \ldots, x_ny_n)^{\mathrm{T}}$.

Let I denote a nonempty subset of N, and let $|I|$ be the number of the elements of I. Given an order m dimension n complex tensor $\mathcal{A} = (a_{i_1\cdots i_m})$, we denote

$$R_i(\mathcal{A}) = \sum_{\substack{i_2,\ldots,i_m \in N \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| = \sum_{i_2,\ldots,i_m \in \mathbb{N}} |a_{ii_2\cdots i_m}| - |a_{ii\cdots i}|,$$

$$N_1(\mathcal{A}) = \{i \in \mathbb{N} : |a_{ii\cdots i}| > R_i(\mathcal{A})\},$$
$$N_2(\mathcal{A}) = \{i \in \mathbb{N} : |a_{ii\cdots i}| = R_i(\mathcal{A})\},$$
$$N_3(\mathcal{A}) = \{i \in \mathbb{N} : |a_{ii\cdots i}| < R_i(\mathcal{A})\}.$$

Definition 1.1[10] Let $\mathcal{A} = (a_{i_1i_2\cdots i_m})$ be a complex tensor of order m dimension n. If for all $i \in \mathbb{N}$, $|a_{ii\cdots i}| \geq R_i(\mathcal{A})$, then $\mathcal{A}$ is called a diagonally dominant tensor. Otherwise, we say $\mathcal{A}$ is a diagonally minor tensor.

Definition 1.2 Let $\mathcal{A} = (a_{i_1i_2\cdots i_m})$ be a complex tensor of order m dimension n. If there is an entrywise positive vector $x = (x_1, x_2, \ldots x_n)^{\mathrm{T}} \in \mathbb{R}^n$ such that for all $i \in \mathbb{N}$,

$$|a_{ii\cdots i}|x_i^{m-1} \geq \sum_{\substack{i_2,\ldots,i_m \in N \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|x_{i_2}\cdots x_{i_m}, \tag{1.1}$$

then $\mathcal{A}$ is called a generalized $\mathcal{H}$-tensor. Otherwise, we say $\mathcal{A}$ is a nongeneralized $\mathcal{H}$-tensor. $\mathcal{A}$ is called an $\mathcal{H}$-tensor[2−3] if the strict inequality holds in (1.1) for all $i \in \mathbb{N}$.

Definition 1.3[11] A complex tensor $\mathcal{A} = (a_{i_1i_2\cdots i_m})$ of order m dimension n is called reducible, if there exists a nonempty proper index subset $I \subset \mathbb{N}$ such that

$$a_{i_1i_2\cdots i_m} = 0, \text{ for all } i_1 \in I, \text{ for all } i_2, \ldots, i_m \notin I.$$

Otherwise, we say $\mathcal{A}$ is irreducible.

2. Main Results

In this section, we provide an iterative algorithm for tensors which are nongenerialized $\mathcal{H}$-tensors and give its theoretical analysis.

Algorithm T INPUT: an irreducible tensor $\mathcal{A} = (a_{i_1i_2\cdots i_m})$ and any $\epsilon > 0$.

OUTPUT: an entrywise positive vector $x = x^{(1)} \circ x^{(2)} \circ \cdots \circ x^{(p)}$ if $\mathcal{A}$ is a nongenerialized $\mathcal{H}$-tensor.

1) If $N_3(\mathcal{A}) = N$ or $a_{ii\cdots i} = 0$ for some $i \in N_3(\mathcal{A})$, then $\mathcal{A}$ is a nongenerialized $\mathcal{H}$-tensor, stop; otherwise

2) Set $\mathcal{A}^{(0)} = \mathcal{A}$, $x^{(0)} = (1, 1, \ldots, 1)^{\mathrm{T}}$, $p = 1$;

3) Compute $\mathcal{A}^{(p)} = (a^{(p)}_{i_1i_2\cdots i_m})$, where $a^{(p)}_{ii_2\cdots i_m} = a^{(p-1)}_{ii_2\cdots i_m} x^{(p-1)}_{i_2} \cdots x^{(p-1)}_{i_m}$ for any $i \in \mathbb{N}$;

4) If $N_3(\mathcal{A}^{(p)}) = N$, then $\mathcal{A}$ is a nongeneralized $\mathcal{H}$-tensor, stop; otherwise

5) Set $d^{(p)} = (d^{(p)}_1, d^{(p)}_2, \ldots, d^{(p)}_n)^{\mathrm{T}}$, where

$$d_i^{(p)} = \begin{cases} \left(1 + \frac{R_i(\mathcal{A}^{(p)}) - |a^{(p)}_{i\cdots i}|}{|a^{(p)}_{ii\cdots i}| + \epsilon}\right)^{\frac{1}{m-1}}, & \text{if} \quad i \in N_3(\mathcal{A}^{(p)}), \\ 1, & \text{if} \quad i \in N_1(\mathcal{A}^{(p)}) \cup N_2(\mathcal{A}^{(p)}); \end{cases}$$

6) Set $x^{(p)} = d^{(p)}$, $p = p + 1$; go to Step 3).

The theoretical basis for the functionality of Algorithm T as criteria for nongenerialized $\mathcal{H}$-tensors is provided by the following theorems.

Theorem 2.1 Let $\mathcal{A}=(a_{i_1i_2\cdots i_m})$ be an irreducible tensor of order m dimension n, then $\mathcal{A}$ is a nongenerialized $\mathcal{H}$-tensor if and only if Algorithm T terminates after a finite number of iterations by producing a diagonally minor tensor.

Before the proof of Theorem 2.1, we give the following lemmas.

Lemma 2.1 Algorithm T either terminates or produces an infinite sequence of distinct tensors $\{\mathcal{A}^{(p)}=(a^{(p)}_{i_1i_2\cdots i_m})\}$ such that $\lim\limits_{p\to+\infty}|a^{(p)}_{i_1i_2\cdots i_m}|$ exists ($or+\infty$) for all $i_1,i_2,\ldots,i_m\in\mathbb{N}$.

Proof Suppose that Algorithm T does not terminate, that is, it produces an infinite sequence of tensors. This means $|N_1(\mathcal{A})\cup N_2(\mathcal{A})|\geq 1$ and $a_{i\cdots i}\neq 0$ for all $i\in N_3(\mathcal{A})$. Then from the definition of $d_i^{(p)}$ in Step 5), we have that for all $i\in N_3(\mathcal{A}^{(p)})$, $d_i^{(p)}>1$, and

$$d_i^{(p)}=\left(\frac{R_i(\mathcal{A}^{(p)})+\epsilon}{|a^{(p)}_{ii\cdots i}|+\epsilon}\right)^{\frac{1}{m-1}}.$$

Hence, d_i is not an increasing function of $\epsilon\in[0,+\infty)$. Moreover,

$$|a^{(p+1)}_{ii\cdots i}|=|a^{(p)}_{ii\cdots i}|(d_i^{(p)})^{m-1}\quad<\quad|a^{(p)}_{ii\cdots i}|\frac{R_i(\mathcal{A}^{(p)})}{|a^{(p)}_{ii\cdots i}|}=R_i(\mathcal{A}^{(p)})\leq R_i(\mathcal{A}^{(p+1)}),$$

which implies that $N_3(\mathcal{A})=N_3(\mathcal{A}^{(1)})\subseteq N_3(\mathcal{A}^{(2)})\cdots\subseteq N_3(\mathcal{A}^{(p)})\subseteq\cdots$. Consequently, there exists a smallest integer l, such that

$$N_3(\mathcal{A}^{(l)})=N_3(\mathcal{A}^{(l+k)}),k=1,2,\cdots.$$

Since Algrothm T terminates for input tensor if and only if it terminates for the input $\mathcal{A}^{(l)}$, we may without loss of generality assume that $l=1$. Furthermore, we may suppose that

$$N_3(\mathcal{A})=N_3(\mathcal{A}^{(1)})=\{j_1,j_2,\ldots,j_q\},$$

where $1\leq q<n$. Moreover, let $x^{(p)}=(x_1^{(p)},x_2^{(p)},\ldots,x_n^{(p)})^{\mathrm{T}}$. Then from the step 6) of Algorithm T, we have that

$$\begin{cases} x_j^{(p)}>1, & \text{if}\quad j\in\{j_1,j_2,\ldots,j_q\},\\ x_j^{(p)}=1, & \text{if}\quad i\notin\{j_1,j_2,\ldots,j_q\}.\end{cases}$$

Note that for any $i\in\mathbb{N}$, $|a^{(p+1)}_{ii_2\cdots i_m}|=|a^{(p)}_{ii_2\cdots i_m}|x^{(p)}_{i_2}\cdots x^{(p)}_{i_m}$, $p=1,2,\ldots$ Then for any $i_1,i_2,\ldots,i_m\in\mathbb{N}$, $\{|a^{(p)}_{i_1i_2\cdots i_m}|\}$ is an increasing sequence. Therefore, $\lim\limits_{p\to+\infty}|a^{(p)}_{i_1i_2\cdots i_m}|$ exists (or $+\infty$) for all $i_1,i_2,\ldots,i_m\in\mathbb{N}$, and

$$\begin{cases} |a^{(p)}_{ii\cdots i}|\geq R_i(\mathcal{A}^{(p)}), & \text{if}\quad i\in N_1(\mathcal{A})\cup N_2(\mathcal{A}),\\ |a^{(p)}_{ii\cdots i}|<R_i(\mathcal{A}^{(p)}), & \text{if}\quad i\in N_3(\mathcal{A}).\end{cases}$$

The proof is completed.

Lemma 2.2 If Algorithm T produces the infinite sequence $\{\mathcal{A}^{(p)}=(a^{(p)}_{i_1i_2\cdots i_m})\}$, and $\lim\limits_{p\to+\infty}|a^{(p)}_{i_1i_2\cdots i_m}|$ is a finite number, then for all $i\in N_3(\mathcal{A})$,

$$\lim_{p\to+\infty}\left(|a^{(p)}_{ii\cdots i}|-R_i(\mathcal{A}^{(p)})\right)=0.$$

Proof From Lemma 2.1, we have that both $\left\{|a^{(p)}_{ii\cdots i}|\right\}$ and $\left\{R_i(\mathcal{A}^{(p)})\right\}$ converge, and the limit is a finite number or $+\infty$. When the limit is a finite number, we suppose that $\lim\limits_{p\to+\infty}|a^{(p)}_{ii\cdots i}|=\alpha_{ii\cdots i}$, $\lim\limits_{p\to+\infty}R_i(\mathcal{A}^{(p)})=\beta_{ii\cdots i}$. For all $i\in N_3(\mathcal{A})$, we have

$$|a^{(p+1)}_{ii\cdots i}|=|a^{(p)}_{ii\cdots i}|(d_i^{(p)})^{m-1}=|a^{(p)}_{ii\cdots i}|\frac{R_i(\mathcal{A}^{(p)})+\epsilon}{|a^{(p)}_{ii\cdots i}|+\epsilon}.$$

Letting $p \to +\infty$, we have

$$\alpha_{ii\cdots i} = \alpha_{ii\cdots i}(\frac{\beta_{ii\cdots i}+\epsilon}{\alpha_{ii\cdots i}+\epsilon}).$$

Note that $\alpha_{ii\cdots i} > 0$ and therefore, $\alpha_{ii\cdots i} = \beta_{ii\cdots i}$. Hence, $\lim\limits_{p\to+\infty} |a^{(p)}_{ii\cdots i}| = \lim\limits_{p\to+\infty} R_i(\mathcal{A}^{(p)})$ for all $i \in N_3(\mathcal{A})$, that is, $\lim\limits_{p\to+\infty}\Big(|a^{(p)}_{ii\cdots i}| - R_i(\mathcal{A}^{(p)})\Big) = 0$. The proof is completed.

Lemma 2.3 Let $\mathcal{A} = (a_{i_1 i_2\cdots i_m})$ be a generialized $\mathcal{H}$-tensor of order m dimension n. Let $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ be an entrywise positive vector, and $\mathcal{B} = (b_{i_1 i_2\cdots i_m})$, where

$$b_{ii_2\cdots i_m} = a_{ii_2\cdots i_m} x_{i_2}\cdots x_{i_m}$$

for any $i \in \mathbb{N}$, then $\mathcal{B}$ is a generialized $\mathcal{H}$-tensor.

Proof Since $\mathcal{A}$ is a generialized $\mathcal{H}$-tensor, there is an entrywise positive vector $y = (y_1, y_2, \ldots, y_n)^{\mathrm{T}}$ such that for all $i \in \mathbb{N}$,

$$|a_{ii\cdots i}| y_i^{m-1} \ge \sum_{\substack{i_2,\ldots,i_m\in\mathbb{N}\\ \delta_{ii_2\ldots i_m}=0}} |a_{ii_2\cdots i_m}| y_{i_2}\cdots y_{i_m}.$$

Let $z = (z_1, z_2, \ldots, z_n)^{\mathrm{T}}$, where $z_i = x_i^{-1} y_i$. Then for any $i \in \mathbb{N}$, we obtain

$$\begin{aligned} |b_{ii\cdots i}| z_i^{m-1} =& |a_{ii\cdots i} x_i\cdots x_i| z_i^{m-1} = |a_{ii\cdots i}| y_i^{m-1} \\ \ge& \sum_{\substack{i_2,\ldots,i_m\in N\\ \delta_{ii_2\ldots i_m}=0}} |a_{ii_2\cdots i_m}| y_{i_2}\cdots y_{i_m} = \sum_{\substack{i_2,\ldots,i_m\in N\\ \delta_{ii_2\ldots i_m}=0}} |a_{ii_2\cdots i_m} x_{i_2}\cdots x_{i_m}| z_{i_2}\cdots z_{i_m} \\ =& \sum_{\substack{i_2,\ldots,i_m\in N\\ \delta_{ii_2\ldots i_m}=0}} |b_{ii_2\cdots i_m}| z_{i_2}\cdots z_{i_m}. \end{aligned}$$

Therefore, $\mathcal{B}$ is a generialized H-tensor. The proof is completed.

Now we give the proof of Theorem 2.1 in the following.

Proof Sufficiency: If Algorithm T terminates after k iterations, then we obtain an entrywise positive vector $x = x^{(1)} \circ x^{(2)} \circ \cdots \circ x^{(k-1)}$ such that $\mathcal{A}^{(k)}$ is a diagonally minor tensor, that is, $|a^{(k)}_{ii\cdots i}| < R_i(\mathcal{A}^{(k)}), i \in \mathbb{N}$. It follows that $\mathcal{A}^{(k)}$ is a nongenerialized $\mathcal{H}$-tensor. From Definition 1.2, $\mathcal{A}$ is a nongenerialized $\mathcal{H}$-tensor.

Necessity: Let irreducible tensor $\mathcal{A}$ be a nongenerialized $\mathcal{H}$-tensor. Suppose, on the contrary, that Algorithm T produces the infinite sequences

$$\{\mathcal{A}^{(p)}\},\ \{|a^{(p)}_{ii\cdots i}|\},\ \{R_i(\mathcal{A}^{(p)})\},\ \{N_3(\mathcal{A}^{(p)})\}.$$

As in the proof of Lemma 2.2, without loss generality, we can assume that

$$N_3(\mathcal{A}) = N_3(\mathcal{A}^{(p)}) = \{j_1, j_2, \cdots, j_q\}$$

for all $p = 1, 2, \ldots$ and $1 \le q < n$. Note that for any $i \in \mathbb{N}$, we have

$$\begin{aligned} a^{(p+1)}_{ii_2\cdots i_m} =& a^{(p)}_{ii_2\cdots i_m} x^{(p)}_{i_2}\cdots x^{(p)}_{i_m} = a^{(p-1)}_{ii_2\cdots i_m} x^{(p)}_{i_2} x^{(p-1)}_{i_2}\cdots x^{(p)}_{i_m} x^{(p-1)}_{i_m} = \cdots \\ =& a_{ii_2\cdots i_m} x^{(p)}_{i_2} x^{(p-1)}_{i_2}\cdots x^{(1)}_{i_2}\cdots x^{(p)}_{i_m} x^{(p-1)}_{i_m}\cdots x^{(1)}_{i_m} \\ =& a_{ii_2\cdots i_m} x^{<p>}_{i_2}\cdots x^{<p>}_{i_m}, \end{aligned}$$

where $x^{<p>}_i = x^{(p)}_i x^{(p-1)}_i \cdots x^{(1)}_i$ for any $i \in \mathbb{N}$. Let $x^{<p>} = (x^{<p>}_1, x^{<p>}_2, \ldots, x^{<p>}_n)^{\mathrm{T}}$. Then from Lemma 2.2, Step 5) and Step 6) of Algorithm T, we have that

$$x^{<p>}_j = \begin{cases} x^{(p)}_i x^{(p-1)}_i \cdots x^{(1)}_i < 1, & \text{if } \ j \in \{j_1, j_2, \ldots, j_q\}, \\ 1, & \text{if } \ j \notin \{j_1, j_2, \ldots, j_q\}, \end{cases}$$

that $\lim\limits_{p\to+\infty}\mathcal{A}^{(p)}$ exists (or $+\infty$), and that $\lim\limits_{p\to+\infty}x^{<p>}$ exists (or $+\infty$). Furthermore, let these limits be $\mathcal{B}=(b_{i_1i_2\cdots i_m})$ and $x=(x_1,x_2,\ldots,x_n)^{\mathrm{T}}$, respectively, where

$$\begin{cases} x_j=\lim\limits_{p\to+\infty}x_j^{<p>}, & \text{if} \quad j\in\{j_1,j_2,\ldots,j_q\},\\ x_j=1, & \text{if} \quad j\notin\{j_1,j_2,\ldots,j_q\}.\end{cases}$$

Then for any $i\in\mathbb{N}$, we get

$$b_{ii_2\cdots i_m}=a_{ii_2\cdots i_m}x_{i_2}\cdots x_{i_m}, \qquad i_2,\cdots,i_m\in\mathbb{N}.$$

1) If x_j is a positively finite number for all $j\in N_3(\mathcal{A})$ and other $x_j=1$, by Lemma 2.2 and Lemma 2.3, we have

$$\begin{cases} R_j(\mathcal{B})=|b_{jj\cdots j}|, & \text{if} \quad j\in N_3(\mathcal{A}),\\ R_j(\mathcal{B})\le|b_{jj\cdots j}|, & \text{if} \quad j\notin N_3(\mathcal{A}).\end{cases}$$

Hence, $\mathcal{B}$ is a diagonally dominant tensor, so $\mathcal{A}$ is a generalized $\mathcal{H}$-tenor, which is a contradiction.

2) If $x_j=+\infty$ for all $j\in N_3(\mathcal{A})$, since $\mathcal{A}$ is irreducible, then there exists $a_{ii_2^0\cdots i_m^0}\neq0$, $i\in N_1(\mathcal{A})\cup N_2(\mathcal{A})$, $i_2^0,\cdots,i_m^0\in N_3(\mathcal{A})$, and for all $i\in N_1(\mathcal{A})\bigcup N_2(\mathcal{A})$, we have

$$\begin{aligned}R_i(\mathcal{B})=&\sum_{\substack{i_2,\ldots,i_m\in\mathbb{N}\\ \delta_{ii_2\ldots i_m}=0}}|b_{ii_2\cdots i_m}|\ge\sum_{i_2,\ldots,i_m\in N_3(\mathcal{A})}|a_{ii_2\cdots i_m}|x_{i_2}\cdots x_{i_m}+\sum_{\substack{i_2,\ldots,i_m\in N_1(\mathcal{A})\bigcup N_2(\mathcal{A})\\ \delta_{ii_2\ldots i_m}=0}}|a_{ii_2\cdots i_m}|+\cdots\\ =&+\infty,\end{aligned}$$

and $b_{ii\cdots i}=a_{ii\cdots i}$.

This is a contradiction, because $R_i(\mathcal{B})\le|b_{ii\cdots i}|$, for all $i\in N_1(\mathcal{A})\cup N_2(\mathcal{A})$.

3) If $x_j=+\infty$ for some $j\in N_3(\mathcal{A})$, without loss generality, we can assume that $x_j=+\infty$, $j\in\widetilde{N}_3(\mathcal{A})=\{j_1,j_2,\cdots,j_l\}\subset N_3(\mathcal{A})$; x_j is a positively finite number for all $j\in N_3(\mathcal{A})\setminus\widetilde{N}_3(\mathcal{A})$. Since $\mathcal{A}$ is irreducible, there exists $a_{ii_2^0\cdots i_m^0}\neq0$,$i\in\mathbb{N}\setminus\widetilde{N}_3(\mathcal{A})$, $i_2^0,\cdots,i_m^0\in\widetilde{N}_3(\mathcal{A})$, and two cases occur.

i) For all $i\in N_1(\mathcal{A})\cup N_2(\mathcal{A})$, we have

$$\begin{aligned}R_i(\mathcal{B})=&\sum_{\substack{i_2,\ldots,i_m\in\mathbb{N}\\ \delta_{ii_2\ldots i_m}=0}}|b_{ii_2\cdots i_m}|\ge\sum_{i_2,\ldots,i_m\in\widetilde{N}_3(\mathcal{A})}|a_{ii_2\cdots i_m}|x_{i_2}\cdots x_{i_m}\\ &+\sum_{i_2,\ldots,i_m\in N_3(\mathcal{A})\setminus\widetilde{N}_3(\mathcal{A})}|a_{ii_2\cdots i_m}|x_{i_2}\cdots x_{i_m}+\sum_{\substack{i_2,\ldots,i_m\in N_1(\mathcal{A})\bigcup N_2(\mathcal{A})\\ \delta_{ii_2\ldots i_m}=0}}|a_{ii_2\cdots i_m}|+\cdots\\ =&+\infty,\end{aligned}$$

and $b_{ii\cdots i}=a_{ii\cdots i}$.

This is a contradiction, because $R_i(\mathcal{B})\le|b_{ii\cdots i}|$, for all $i\in N_1(\mathcal{A})\cup N_2(\mathcal{A})$.

ii) For all $i\in N_3(\mathcal{A})\setminus\widetilde{N}_3(\mathcal{A})$, we get

$$\begin{aligned}R_i(\mathcal{B})=&\sum_{\substack{i_2,\ldots,i_m\in\mathbb{N}\\ \delta_{ii_2\ldots i_m}=0}}|b_{ii_2\cdots i_m}|\ge\sum_{i_2,\ldots,i_m\in\widetilde{N}_3(\mathcal{A})}|a_{ii_2\cdots i_m}|x_{i_2}\cdots x_{i_m}\\ &+\sum_{\substack{i_2,\ldots,i_m\in N_3(\mathcal{A})\setminus\widetilde{N}_3(\mathcal{A})\\ \delta_{ii_2\ldots i_m}=0}}|a_{ii_2\cdots i_m}|x_{i_2}\cdots x_{i_m}+\sum_{i_2,\ldots,i_m\in N_1(\mathcal{A})\bigcup N_2(\mathcal{A})}|a_{ii_2\cdots i_m}|\\ =&+\infty.\end{aligned}$$

Furthermore,

$$\lim_{p\to+\infty} d_i^{(p)} = \lim_{p\to+\infty}\left(\frac{R_i(\mathcal{A}^{(p)})+\epsilon}{|a_{ii\cdots i}^{(p)}|+\epsilon}\right)^{\frac{1}{m-1}} = +\infty.$$

Therefore, we have $x_i = +\infty$. This is a contradiction, since x_i is a positively finite number for all $i \in N_3(\mathcal{A}) \setminus \widetilde{N}_3(\mathcal{A})$.

By the above statements, we obtain that if $\mathcal{A}$ is a nongenerialized $\mathcal{H}$-tensor, then Algorithm T terminates after finite iterations. The proof is completed.

3. Numerical Examples

In this section, we present two numerical examples to illustrate the advantages of our derived results.

Example 3.1 Let $\mathcal{A} = (a_{ijkl})$ be a irreducible tensor of order 4 dimension 2 with elements defined as follows: $a_{1111} = a$, $a_{2222} = b$, $a_{1112} = 0.2$, $a_{1121} = 0.7$, $a_{1211} = 0.2$, $a_{1122} = 0.3$, $a_{1212} = 0.5$, $a_{1222} = 1$, $a_{1221} = 0.2$, $a_{2111} = 0.5$, $a_{2121} = 0.2$, $a_{2221} = 0.4$, $a_{2212} = 0.3$, $a_{2122} = 0.3$, $a_{2211} = 0.6$, $a_{2112} = 0.2$, where $(a, b) \in \{(2, 3), (3, 2.5), (4, 1.5), (5, 1.6)\}$. By Algorithm T, we can obtain that there is an entrywise positive vector x such that $\mathcal{A}$ is a nongeneralized $\mathcal{H}$-tensor (see Tab. 1).

Tab. 1 Algorithm T terminates after t iterations and x is the desired positive vector

a	b	ϵ	t	x
2	3	0.1	2	$(1.1507, 1.0000)^{\mathrm{T}}$
3	2.5	0.5	2	$(1.0094, 1.0000)^{\mathrm{T}}$
4	1.5	1	3	$(1.0000, 1.2046)^{\mathrm{T}}$
5	1.6	1	6	$(1.0000, 1.2707)^{\mathrm{T}}$

Example 3.2 Let $\mathcal{A} = (a_{ijkl})$ be a irreducible tensor of order 4 dimension 3 with elements defined as follows:

$$\begin{aligned}\mathcal{A} &= [A(:,:,1,1), A(:,:,2,1), A(:,:,3,1)A(:,:,1,2), A(:,:,2,2),\\ &\quad A(:,:,3,2), A(:,:,1,3), A(:,:,2,3), A(:,:,3,3)],\end{aligned}$$

$$A(:,:,1,1) = \begin{pmatrix} a & 0.3 & -1.2 \\ -1.2 & 1.0 & 0.1 \\ -0.2 & 1.8 & -0.7 \end{pmatrix}, \quad A(:,:,2,1) = \begin{pmatrix} 1.0 & -0.1 & 0.2 \\ -1.4 & -0.5 & -1.2 \\ -0.4 & 1.6 & -1.3 \end{pmatrix},$$

$$A(:,:,3,1) = \begin{pmatrix} -1.6 & 0.05 & -0.01 \\ -0.08 & -1.0 & 1.8 \\ -1.2 & 0.05 & -1.3 \end{pmatrix}, \quad A(:,:,1,2) = \begin{pmatrix} 0.2 & 1.4 & -0.8 \\ -1.5 & -1.3 & 0.5 \\ 1.4 & -0.3 & 1.5 \end{pmatrix},$$

$$A(:,:,2,2) = \begin{pmatrix} -1.4 & 0.1 & 1.0 \\ -1.3 & b & -0.8 \\ 0.6 & 1.5 & -1.6 \end{pmatrix}, \quad A(:,:,3,2) = \begin{pmatrix} -1.6 & -0.6 & 0.6 \\ -1.8 & -0.9 & -0.5 \\ 0.7 & 1.4 & -0.5 \end{pmatrix},$$

$$A(:,:,1,3) = \begin{pmatrix} 0.6 & 1.3 & -1.2 \\ -0.5 & -0.9 & 0.5 \\ 0.6 & 0.7 & -1.4 \end{pmatrix}, \quad A(:,:,2,3) = \begin{pmatrix} 1.4 & -0.7 & 0.7 \\ -1.3 & -0.2 & 1.8 \\ 0.3 & -1.5 & -1.2 \end{pmatrix},$$

$$A(:,:,1,3) = \begin{pmatrix} -0.6 & -0.5 & 0.2 \\ -0.3 & 0.2 & 0.7 \\ 0.8 & 1.5 & c \end{pmatrix},$$

where $(a,b,c) \in \{(40,20,16),(42,20,15),(45,18,16),(38,20,15),(32,15,12),(20,30,15)\}$. By Algorithm T, we can obtain that there is an entrywise positive vector x such that $\mathcal{A}$ is a nongeneralized $\mathcal{H}$-tensor (see Tab. 2).

Tab. 2 Algorithm T terminates after t iterations and x is the desired positive vector

a	b	c	ϵ	t	x
40	20	16	0.1	15	$(1.0000, 1.3359, 1.5180)^{\mathrm{T}}$
42	20	15	1	11	$(1.0000, 1.3584, 1.5776)^{\mathrm{T}}$
45	18	16	0.5	12	$(1.0000, 1.4372, 1.5836)^{\mathrm{T}}$
38	20	15	0.1	7	$(1.0000, 1.3056, 1.5121)^{\mathrm{T}}$
32	15	12	0.5	3	$(1.0000, 1.3159, 1.4843)^{\mathrm{T}}$
20	30	15	0.1	3	$(1.0523, 1.0000, 1.2780)^{\mathrm{T}}$

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非广义$\mathcal{H}$-张量的判定准则

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摘要: 给出判定非广义$\mathcal{H}$-张量的充要条件, 从理论上彻底解决了不可约非广义$\mathcal{H}$-张量的判定问题, 并给出判定不可约非广义$\mathcal{H}$-张量的具体算法. 最后, 利用数值算例表明了结果的有效性.

关键词: 广义$\mathcal{H}$-张量; 非广义$\mathcal{H}$-张量; $\mathcal{H}$-张量; 对角占优张量; 对角占劣张量; 不可约张量