应 用 数 学
MATHEMATICA APPLICATA
2019, 32(4): 832-837

Limit Cycles by Perturbing a Piecewise Near-Hamiltonian System with 4 Switching Lines

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Abstract: By using the first order Melnikov function method for piecewise near-Hamiltonian systems, we study limit cycle bifurcations by perturbing a compound global center with 4 regions. When the perturbed terms are polynomials with degree n, we give the number of limit cycles bifurcated from the center.

Key words: Piecewise smooth dynamical system; Limit cycle; Bifurcation; Melnikov function
CLC Number: O175.1 **AMS(2000)Subject Classification:** 34C05; 34C07; 37G15
Document code: A **Article ID:** 1001-9847(2019)04-0832-06

1. Introduction And Main Results

For many applications involving biology, medicine, friction and devices with switching components, piecewise smooth (PWS) dynamical systems are more precise to model these problems. As a consequence, the study of PWS systems has become very active in the last decades. One of the most important problems about PWS dynamical systems is how to determine the number of limit cycles. The problem is much more difficult than that in smooth systems. So far, there have been at least three common ways to consider multiple limit cycles for a given PWS system. The first method is to study small amplitude limit cycles that bifurcate from Hopf or center bifurcation[1−2,7,9]. The second is to study limit cycles which bifurcate from a periodic annulus[3−5]. The third method is to study the multiple limit cycles through homoclinic bifurcation[8,13,16].

Most of works assumed that discontinuity sets of PWS planar systems consist of only one switching manifold, particular a straight line.[7−10,13,17] However, as pointed out by Akhmet and Arugaslan[6], due to exterior effects, discontinuities may occur on multiple lines, curves or surfaces. Recently, some work has been done on PWS dynamical systems with multiple switching lines.[12,14−15] In [12], HU and DU studied bifurcations of periodic orbits in discontinuous planar systems, whose discontinuities occur on finitely many rays starting at the origin. More precisely, they considered the following system

$$\dot{x} = g_k(x) + \varepsilon f_k(x), \quad x \in D_k, \quad k = 1, 2, \cdots m, \tag{1.1}$$

* **Received date:** 2018-09-26
Foundation item: Supported by the National Natural Science Foundation of China(11671013)
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where D_k is the open region between the switching rays l_k and l_{k+1} for $k = 1, 2, \cdots m$, and $l_{m+1} = l_1$, $g_k, f_k \in C^2(D_k \cup l_k \cup l_{k+1}, \mathbb{R}^2)$, $|\varepsilon| \le \varepsilon_0$ for some $\varepsilon_0 > 0$. For $\varepsilon = 0$, the unperturbed system is given by

$$\dot{x} = g_k(x), \quad x \in D_k, \quad k = 1, 2, \cdots m. \tag{1.2}$$

By establishing the first order Melnikov function for system (1.1), the authors considered limit cycle bifurcations for several piecewise systems with low degree perturbations.

Further, in [11] WANG and HAN considered the piecewise near-Hamiltonian system with 4 regions

$$(\dot{x}, \dot{y}) = (-\omega_k y + \varepsilon p_k(x, y), \omega_k x + \varepsilon q_k(x, y)), (x, y) \in D_k, \tag{1.3}$$

$k = 1, 2, 3, 4$. Here, D_1, D_2, D_3, D_4 denote the first, second, third and fourth quadrants, respectively. The unperturbed systems of (1.3) has a global center at the origin. They proved that systems (1.3) have n limit cycles around the origin under nth-degree perturbations.

In this paper, we study limit cycle bifurcations for a class of perturbed planar discontinuous system with 4 switching rays as follows

$$(\dot{x}, \dot{y}) = \begin{cases} (-1 + \varepsilon p_1(x, y), 1 + \varepsilon q_1(x, y)), & x > 0, y > 0, \\ (-1 + \varepsilon p_2(x, y), -1 + \varepsilon q_2(x, y)), & x < 0, y > 0, \\ (1 + \varepsilon p_3(x, y), -1 + \varepsilon q_3(x, y)), & x < 0, y < 0, \\ (1 + \varepsilon p_4(x, y), 1 + \varepsilon q_4(x, y)), & x > 0, y < 0, \end{cases} \tag{1.4}$$

where

$$p_k(x, y) = \sum_{i+j=0}^{n} a_{kij} x^i y^j, q_k(x, y) = \sum_{i+j=0}^{n} b_{kij} x^i y^j, \tag{1.5}$$

for $k = 1, 2, 3, 4$.

It is obvious that the unperturbed system of (1.4) is different from that of system (1.3), although it also has a global center at the origin. For system (1.4), we have the main result below.

Theorem 1.1 Let (1.5) hold. Then for $\varepsilon > 0$ small system (1.4) has at most n limit cycles on the plane if the first order Melnikov function of system (1.4) does not vanish identically. Moreover, the upper bound can be achieved.

2. Fundamental Lemmas and the Proof of the Main Result

First, we assume that the space $\mathbb{R}^2$ is splited into 4 disjoint regions by four rays $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4$, and let $\Sigma_5 = \Sigma_1$, where

$$\begin{aligned} \Sigma_1 &= \{(x, y) | y = 0, x \ge 0\} = \{(x, y) | c_1^{\mathrm{T}} (x, y)^{\mathrm{T}} = 0, x \ge 0\}, c_1^{\mathrm{T}} = (0, 1), \\ \Sigma_2 &= \{(x, y) | x = 0, y \ge 0\} = \{(x, y) | c_2^{\mathrm{T}} (x, y)^{\mathrm{T}} = 0, y \ge 0\}, c_2^{\mathrm{T}} = (-1, 0), \\ \Sigma_3 &= \{(x, y) | y = 0, x \le 0\} = \{(x, y) | c_3^{\mathrm{T}} (x, y)^{\mathrm{T}} = 0, x \le 0\}, c_3^{\mathrm{T}} = (0, -1), \\ \Sigma_4 &= \{(x, y) | x = 0, y \le 0\} = \{(x, y) | c_4^{\mathrm{T}} (x, y)^{\mathrm{T}} = 0, y \le 0\}, c_4^{\mathrm{T}} = (1, 0). \end{aligned}$$

The open region between Σ_k and Σ_{k+1} was denoted by D_k for $k = 1, 2, 3, 4$. Then,

$$\begin{aligned} D_1 &= \{(x, y) | x > 0, y > 0\}, \qquad D_2 = \{(x, y) | x < 0, y > 0\}, \\ D_3 &= \{(x, y) | x < 0, y < 0\}, \qquad D_4 = \{(x, y) | x > 0, y < 0\}. \end{aligned}$$

It is easy to see that $D_k \cap (\Sigma_k \cup \Sigma_{k+1})$ is empty for $k = 1, 2, 3, 4$.

In order to give the first order Melnikov function of system (1.4), we consider the following planar piecewise system defined on $D_1 \cup D_2 \cup D_3 \cup D_4$

$$\begin{cases} x' = H_{ky} + \varepsilon p_k(x,y), \\ y' = -H_{kx} + \varepsilon q_k(x,y), \qquad (x,y) \in D_k, k = 1,2,3,4, \end{cases} \tag{2.1}$$

where $H_{ky} = \frac{\partial H_k(x,y)}{\partial y}, H_{kx} = \frac{\partial H_k(x,y)}{\partial x}$, and $H_k, p_k, q_k \in C^\infty(D_k \cup \Sigma_k \cup \Sigma_{k+1}, \mathbb{R}^2), |\varepsilon| \in \mathbb{R}$ is small. For $\varepsilon = 0$, the unperturbed system of (2.1) is

$$\begin{cases} x' = H_{ky}, \\ y' = -H_{kx}, \quad (x,y) \in D_k, k = 1,2,3,4. \end{cases} \tag{2.2}$$

For the system (2.2), we give the following two hypotheses first.

(H1) There exist four points $A_1 = (a_1(h), 0), A_2 = (0, a_2(h)), A_3 = (a_3(h), 0), A_4 = (0, a_4(h))$, $A_k \in \Sigma_k$ for $k = 1,2,3,4$, and an interval $J = (0, \alpha)$ such that for $h \in J$, $H_1(A_1) = H_1(A_2) = h, H_2(A_2) = H_2(A_3), H_3(A_3) = H_3(A_4), H_4(A_4) = H_4(A_1)$, where $a_1(h) > 0 > a_3(h), a_2(h) > 0 > a_4(h)$.

(H2) System (2.2) has a family of periodic orbits $L_h = L_h^1 \cup L_h^2 \cup L_h^3 \cup L_h^4$, $h \in J$, surrounding the origin counterclockwise. For $k = 1,2,3,4$, let A_k be the intersection of L_h with Σ_k, and L_h^k be the intersection of L_h with D_k. We have

$$L_h^1 = \{(x,y) \in D_1 : H_1(x,y) = h\},$$
$$L_h^i = \{(x,y) \in D_i : H_i(x,y) = H_i(A_i)\},$$

where $i = 2,3,4$.

By (H1) and (H2), we can see that the orbit of system (2.1) starting from A_1 at Σ_1 crosses $\Sigma_2, \Sigma_3, \Sigma_4, \Sigma_1$ in turn. For $|\varepsilon|$ small, let $A_{i\varepsilon}$ denote its first intersection point with Σ_i, $i = 2,3,4$, and let B_ε denote the second intersection point with Σ_1 when it returns to Σ_1 for the first time. And set $A_{2\varepsilon} = (0, a_{2\varepsilon}(h)), A_{3\varepsilon} = (a_{3\varepsilon}(h), 0), A_{4\varepsilon} = (0, a_{4\varepsilon}(h)), B_\varepsilon = (a_{1\varepsilon}(h), 0)$. Similar to Theorem 2.2 in [12] we have

$$H_4(B_\varepsilon) - H_4(A_1) = \varepsilon F(h, \varepsilon). \tag{2.3}$$

It is clear that the function F in (2.3) is smooth because $A_{i\varepsilon}, i = 2,3,4$ and B_ε are smooth in (ε, h). The function $F(h,\varepsilon)$ is called a bifurcation function of system (2.1). So the system (2.1) has a periodic orbit near L_{h_0} for $h_0 \in J$ if and only if $B_\varepsilon = A_1$ for (h, ε) near $(h_0, 0)$. It follows that an isolated zero of F in h corresponds to a limit cycle of (2.1). Let $M(h) = F(h, 0)$. Then $M(h)$ is called the first order Melnikov function of the system (2.1).

From [11], we have the following lemma for the expression of $M(h)$.

Lemma 2.1 Suppose that (H1) and (H2) hold. Then the first order Melnikov function $M(h)$ of system(2.1) has the following form

$$\begin{aligned} M(h) =& \frac{H_{4y}(A_4)H_{3x}(A_3)H_{2y}(A_2)}{H_{3y}(A_4)H_{2x}(A_3)H_{1y}(A_2)} M_1(h) \\ &+ \frac{H_{4y}(A_4)H_{3x}(A_3)}{H_{3y}(A_4)H_{2x}(A_3)} M_2(h) + \frac{H_{4y}(A_4)}{H_{3y}(A_4)} M_3(h) + M_4(h), \quad h \in J, \end{aligned} \tag{2.4}$$

where $M_k(h) = \int_{\widehat{A_k A_{k+1}}} q_k \mathrm{d}x - p_k \mathrm{d}y, \widehat{A_k A_{k+1}} = \overline{L_h^k}$ (the closure of open arc L_h^k), $k = 1,2,3,4$, and $A_5 = A_1$.

For system (1.4), it is clear that

$$H_1(x,y) = -y - x = h, H_2(x,y) = -y + x, H_3(x,y) = y + x, H_4(x,y) = y - x.$$

By Lemma 2.1, for system (1.4) we have

$$\begin{aligned} M(h) &= \frac{1\times 1\times(-1)}{1\times 1\times(-1)}M_1(h) + \frac{1\times 1}{1\times 1}M_2(h) + \frac{1}{1}M_3(h) + M_4(h) \\ &= M_1(h) + M_2(h) + M_3(h) + M_4(h), \quad h\in(-\infty,0). \end{aligned} \tag{2.5}$$

Lemma 2.2 Suppose (1.5) holds, the function $M_1(h)$ in (2.5) has the following expansion

$$M_1(h) = h\sum_{l=0}^{n} b_{1l}h^l, \quad h\in(-\infty,0),$$

where $b_{1l} = \sum\limits_{i+j=l}\sum\limits_{s=0}^{j}(b_{1ij}+a_{1ij})\frac{C_j^s(-1)^{l+s}}{i+s+1}$, and $b_{10}, b_{11}, b_{12}\cdots b_{1n}$ can be taken as free parameters.

Proof By Lemma 2.1 and (1.5), we get for $h\in(-\infty,0)$

$$\begin{aligned} M_1(h) &= \int_{\widehat{A_1A_2}} q_1\mathrm{d}x - p_1\mathrm{d}y = \sum_{i+j=0}^{n}\int_{\widehat{A_1A_2}} b_{1ij}x^iy^j\mathrm{d}x - a_{1ij}x^iy^j\mathrm{d}y \\ &= \sum_{i+j=0}^{n}\int_{-h}^{0}\big[b_{1ij}x^i(-h-x)^j + a_{1ij}x^i(-h-x)^j\big]\mathrm{d}x \\ &= \sum_{i+j=0}^{n}(b_{1ij}+a_{1ij})\int_{-h}^{0}x^i(-h-x)^j\mathrm{d}x = \sum_{i+j=0}^{n}(b_{1ij}+a_{1ij})M_{ij}^1, \end{aligned}$$

where $M_{ij}^1 = \int_{-h}^{0}x^i(-h-x)^j\mathrm{d}x$. Then,

$$M_{ij}^1 = (-1)^j\sum_{s=0}^{j}\int_{-h}^{0}x^i(C_j^sh^{j-s}x^s)\mathrm{d}x = \sum_{s=0}^{j}\frac{C_j^s(-1)^{i+j+s}}{i+s+1}h^{i+j+1} = \alpha_{1ij}h^{i+j+1}$$

with $\alpha_{1ij} = \sum\limits_{s=0}^{j}\frac{C_j^s(-1)^{i+j+s}}{i+s+1}$. Further, letting $\tilde{a}_{1ij} = \alpha_{1ij}(a_{1ij}+b_{1ij})$, we obtain

$$M_1(h) = \sum_{i+j=0}^{n}\tilde{a}_{1ij}h^{i+j+1} = h\sum_{l=0}^{n}(\sum_{i+j=l}\tilde{a}_{1ij})h^l = h\sum_{l=0}^{n}b_{1l}h^l, \quad h\in(-\infty,0),$$

where $b_{1l} = \sum\limits_{i+j=l}\tilde{a}_{1ij}$ for $0\le l\le n$. It is clear that

$$\begin{aligned} b_{10} &= \tilde{a}_{100} = \alpha_{100}(a_{100}+b_{100}), \\ b_{11} &= \tilde{a}_{110} + \tilde{a}_{101} = \alpha_{110}(a_{110}+b_{110}) + \alpha_{101}(a_{101}+b_{101}), \\ &\vdots \\ b_{1n} &= \tilde{a}_{1n0} + \tilde{a}_{1(n-1)1} + \cdots + \tilde{a}_{10n} \\ &= \alpha_{1n0}(a_{1n0}+b_{10n}) + \cdots + \alpha_{10n}(a_{10n}+b_{10n}). \end{aligned} \tag{2.6}$$

Let

$$A_1 = \frac{\partial(b_{10}, b_{11}, \cdots, b_{1n})}{\partial(a_{100}, b_{100}, \cdots, b_{1n0}, \cdots, b_{10n})}.$$

By (2.6), we have

$$A_1 = \begin{pmatrix} \alpha_{100} & \alpha_{100} & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \alpha_{110} & \alpha_{101} & \alpha_{110} & \alpha_{101} & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \alpha_{1n0} & \cdots & \alpha_{10n} & \alpha_{1n0} & \cdots & \alpha_{10n} \end{pmatrix}.$$

Note that $\alpha_{1i0} = \frac{(-1)^i}{i+1} \neq 0, 0 \leq i \leq n$. We see that $\text{Rank}(A_1) = n+1$, which means that $b_{10}, b_{11}, b_{12} \cdots b_{1n}$ can be taken as free parameters. The proof is completed.

Remark 2.1 We can use the similar method as Lemma 2.2 to get

$$M_k(h) = h\sum_{l=0}^{n} b_{kl}h^l, \quad h \in (-\infty, 0), \quad k = 2, 3, 4,$$

where

$$b_{2l} = \sum_{i+j=l}\sum_{s=0}^{j}(b_{2ij} - a_{2ij})\frac{C_j^s(-1)^{j-s}}{i+s+1},$$

$$b_{3l} = \sum_{i+j=l}\sum_{s=0}^{j}(b_{3ij} + a_{3ij})\frac{C_j^s(-1)^{s+1}}{i+s+1},$$

$$b_{4l} = \sum_{i+j=l}\sum_{s=0}^{j}(b_{4ij} - a_{4ij})\frac{C_j^s(-1)^{i+s+1}}{i+s+1}.$$

Moreover, for each $k \in \{2, 3, 4\}$, $b_{k0}, b_{k1}, b_{k2} \cdots b_{kn}$ can be also taken as free parameters.

By Lemma 2.2 and Remark 2.1, it is easy to obtain the lemma below.

Lemma 2.3 Let (1.5) hold. Then, for the system (1.4) we have

$$M(h) = M_1(h) + M_2(h) + M_3(h) + M_4(h) = h\sum_{l=0}^{n} b_l h^l, \qquad h \in (-\infty, 0), \tag{2.7}$$

where $b_l = b_{1l} + b_{2l} + b_{3l} + b_{4l}$. Obviously, $b_0, b_1, \cdots, b_n$ can be taken as free parameters.

Proof of Theorem 1.1 By Lemma 2.3, we know

$$M(h) = h\sum_{l=0}^{n} b_l h^l = h[b_0 + b_1 h + \cdots + b_n h^n], \qquad h \in (-\infty, 0),$$

where $b_0, b_1, \cdots, b_n$ can be taken as free parameters. It is clear that $M(h)$ has at most n zeros in the interval $(-\infty, 0)$ if $M(h) \not\equiv 0$. This means that for $\varepsilon > 0$ small the system (1.4) has at most n limit cycles on the plane. On the other hand, we show that n negative simple zeros of $M(h)$ can appear near $h = 0$. Since $b_0, b_1, \cdots, b_n$ can be taken as free parameters, we first take $b_0 = b_1 = \cdots = b_{n-1} = 0$ and $(-1)^n b_n > 0$ such that $M(h) > 0$ for $h < 0$. Then, we only change b_{n-1} with $b_n b_{n-1} > 0$ and $|b_{n-1}| \ll |b_n|$. This means $M(h) = b_{n-1}h^{n-1}(1 + O(h)) < 0$ for $-h > 0$ small. Hence, $M(h)$ has a negative simple zero near $h = 0$, denoted by h_1. Using the same method, we change $b_{n-2}, b_{n-3}, \cdots, b_0$ in turn such that

$$b_{l-1}b_l > 0, l = n-1, n-2, \cdots, 1, \quad \text{and} \quad 0 < |b_0| \ll |b_1| \ll \cdots \ll |b_{n-1}| \ll 1.$$

Then, other $n-1$ negative simple zeros $h_2, h_3, \cdots, h_n$ are found with $0 < |h_n| \ll |h_{n-1}| \ll \cdots \ll |h_1|$. Therefore, n limit cycles can appear near the origin for system (1.4). This completes the proof.

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一类具有四条分界射线的近哈密顿系统的极限环分支

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摘要: 应用分片光滑近哈密顿系统一阶Melnikov函数方法, 研究一类由四个角形区域合成的全局中心的极限环扰动分支. 当扰动项为n次多项式时, 给出由中心分支出来的极限环的个数的上下界.

关键词: 分片光滑动力系统; 极限环; 分支; Melnikov函数