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Asymptotic Dynamics of Non-Autonomous Modified Swift-Hohenberg Equations with Multiplicative Noise on Unbounded Domains

Mohamed Yagoub Abaker(巴吉)[1,2], LIU Tingting(刘亭亭)[1], MA Qiaozhen(马巧珍)[1]
(1.College of Mathematics and Statistics, Northwest Normal University, Lanzhou 730070, china; 2. Department of Mathematics, College of Education, Rumbek University of Science and Technology, Rumbek, South Sudan)

Abstract: We investigate the dynamical behavior of the stochastic non-autonomous modified Swift-Hohenberg equation with time-dependent forcing term and multiplicative noise on $\mathbb{R}^2$. In order to overcome the difficulty that Sobolev embedding are not compact in the unbounded domain, we first define a continuous cocycle associated with the problem in $L^2(\mathbb{R}^2)$, and make some uniform estimates on the tails of solutions for large space variables. With the aid of uniform estimates of solution , we verify the pullback asymptotic compactness of the random dynamical system, and further obtain the existence of random attractors.

Key words: Random attractor; Swift-Hohenberg equation; Non-autonomous random dynamical system; Continuous cocycle

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1. Introduction

We consider the following modified Swift-Hohenberg equation on $\mathbb{R}^2$ perturbed by a wiener-type multiplicative noise.

$$\mathrm{d}u + (\Delta^2 u + 2\Delta u + au + b|\nabla u|^2 + u^3)\mathrm{d}t = f(x,t)\mathrm{d}t + \sigma u \circ \mathrm{d}W(t),\ \ t > \tau,\ \tau \in \mathbb{R} \tag{1.1}$$

$$u(x,\tau) = u_\tau(x),\ \ x \in \mathbb{R}^2, \tag{1.2}$$

where $u = u(x,t)$ is a real function on $\mathbb{R}^2$, Δ is the laplacian operator with respect to the variable $x \in \mathbb{R}^2$, a and b are arbitrary constants. The function $f(x,t) \in L^2_{\mathrm{loc}}(\mathbb{R}; L^2(\mathbb{R}^2))$ is an

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Corresponding author: MA Qiaozhen, female, Hui, Gansu, professor, major in infinity dimensional dynamical systems.

external force. $W(t)$ is an independent two sided real-valued wiener processes on probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$, $\mathcal{F}$ is the Borel σ-algebra induced by the compact-open topology of Ω, $\mathbb{P}$ is the corresponding wiener measure on $\mathcal{F}$, and $\circ$ denotes the Stratonovich sense in the stochastic term. In case of $b, f = 0$, and we omit the noise term, then (1.1) is the usual Swift-Hohenberg equation.

The Swift-Hohenberg equation[18] is a partial differential equation, and it takes the form

$$\frac{\partial u}{\partial t} = ru - (1 + \nabla^2)^2 u + N(u), \tag{1.3}$$

where $u(x, t)$ is a scalar function defined on the line or the plane, r is a real bifurcation parameter and $N(u)$ is smooth nonlinearity. Swift and Hohenberg proposed this model for the convective instability in the Rayleigh-Bénard convection. This equation plays a central role in studying of a pattern formation. It's an important equation in different branches of physics, spatially in thermal dynamics[12,15−16]. Doelman and Stanstede[6] proposed the following modified Swift-Hohenberg equation for a pattern formation system

$$u_t + \Delta^2 u + 2\Delta u + au + b|\nabla u|^2 + u^3 = 0, \tag{1.4}$$

where a and b are arbitrary constants. In case of $b = 0$, (1.4) becomes the usual Swift-Hohenberg equation, and the additional term $b|\nabla u|^2$ aries in the study of various pattern formation phenomena involving some kind of phase turbulence of phase transition that breaks the symmetry $u \to -u$. Both the $b|\nabla u|^2$ and u^3 terms are nonlinearities, and b is the parameter controlling the strength of the quadratic nonlinearity. A detailed analysis of a pattern formation systems reveals that pattern formation is governed by order parameters, whose spatiotemporal behaviour is determined by nonlinear partial differential equation. As the effect of thermal fluctuation on the onset of the convective motion in the Bénard system is considered in [18], the stochastic local Swift-Hohenberg equation with additive noise is proposed by Swift and Hohenberg as

$$u_t = N(u) - (1 - \partial_{xx})^2 u - u^3 + \sigma\xi. \tag{1.5}$$

SONG et al.[17] used an iteration procedure, regularity estimates for the linear semigroups and a classical existence theorem of global attractor, to prove that a modified Swift-Hohenberg equation $u_t = -\Delta^2 u + g(u)$ possesses a global attractor in Sobolev space H^k for all $k \geq 0$ which attracts any bounded subsets of H^k in the H^k-norm.

Furthermore, a local stochastic Swift-Hohenberg equation driven by multiplicative noise when the effects of small possible noise from μ is consider by Bölmker[2]

$$u_t = \mu(u) - (1 - \partial_{xx})^2 u - u^3 + \sigma u \circ \xi, \tag{1.6}$$

where $\sigma > 0$ and $\xi = \frac{dW}{dt}$ is the generalized derivative of a real-value Brownian motion. For more physics background, see [1-7] and the references therein.

Recently, non-autonomous Swift-Hohenberg equation was studied by many researchers (see [9,11,13-14,20-21]). WANG and DU[21] obtained the existence of pullback attractor for modified Swift-Hohenberg equation on unbounded domain with forcing term and additive noise. GUO[10] investigated the dynamical behaviour of modified Swift-Hohenberg equation with multiplicative noise but without any external force. In this paper we will study the existence of random attractors for the non-autonomous stochastic equation with the time-dependent external force and multiplicative noise.

We try to organize this paper as follows. In Section 2, we give some basic definitions concerning the random attractors for dynamical system which are important to get our main result. In Section 3, we transform the stochastic equation into deterministic one with random parameter and come into being a continuous cocycle. In Section 4, we show the uniform estimates of the solution. Finally, in Section 5, we prove the existence of random attractors.

2. Preliminaries and Abstract Results

In this section, we recall some basic concepts related to RDS and a random attractor for RDS in [4-5, 8, 19], which are important for getting our main results. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and (X, d) be a polish space with the Borel σ-algebra $B(X)$. The distance between $x \in X$ and $B \subseteq X$ is denoted by $d(x, B)$. If $B \subseteq X$ and $C \subseteq X$, the Hausdorff semi-distance from B to C is denoted by $d(B, C) = \sup_{x\in B} d(x, C)$.

Definition 2.1 $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ is called a metric dynamical system if $\theta : \mathbb{R}\times\Omega \longrightarrow \Omega$ is $(\mathcal{B}(\mathbb{R}) \times \mathcal{F}, \mathcal{F})$-measurable, θ_0 is the identity on Ω, $\theta_{s+t} = \theta_t \circ \theta_s$ for all s, t $\in \mathbb{R}$ and θ_0P = P for all t$\in \mathbb{R}$.

Definition 2.2 A mapping $\Phi(t, \tau\ , \omega, x) : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X$ is called continuous cocycle on X over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$, if for all $\tau \in \mathbb{R}\ , \omega \in \Omega$ and $t, s \in \mathbb{R}^+$, the following conditions are satisfied :

i) $\Phi(t, \tau, \omega, x) : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X$ is a $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F}, \mathcal{B}(\mathbb{R}))$ measurable mapping;

ii) $\Phi(0, \tau, \omega, x)$ is identity on X;

iii) $\Phi(t + s, \tau, \omega, x) = \Phi(t, \tau + s, \theta_s\omega, , x) \circ \Phi(s, \tau, \omega, , x)$;

iv) $\Phi(t, \tau, \omega, x) : X \to X$ is continuous.

Definition 2.3 Let 2^X be the collection of all subsets of X, a set valued mapping $(\tau, \omega) \mapsto \mathcal{D}(t\ \omega) : \mathbb{R} \times \Omega \mapsto 2^X$ is called measurable with respect to $\mathcal{F}$ in Ω if $\mathcal{D}(t\ \omega)$ is a (usually closed) nonempty subset of X and the mapping $\omega \in \Omega \mapsto d(X, B(\tau, \omega))$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$ -measurable for every fixed $x \in X$ and $\tau \in \mathbb{R}$. Let $B = \{B(t, \omega) \in \mathcal{D}(t, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is called a random set.

Definition 2.4 A random bounded set $B = \{B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ of X is called tempered with respect to $\{\theta(t)\}_{t\in\Omega}$, if for $\mathbb{P}$-a.e $\omega \in \Omega$,

$$\lim_{t\ \mapsto\infty} \mathrm{e}^{-\beta t}\ d(B(\theta_{-t}\omega)) = 0\ ,\ \forall\, \beta\ > 0\ ,$$

where

$$d(B) = \sup_{x\in B} \|x\|_X.$$

Definition 2.5 Let $\mathcal{D}$ be a collection of random subset of X and $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$. Then K is called an absorbing set of $\Phi \in \mathcal{D}$ if for all $\tau \in \mathbb{R}, \omega \in \Omega$ and $B \in \mathcal{D}$, there exists, $T = T(\tau, \omega, B) > 0$ such that

$$\Phi(t\ , \tau, \theta_{-t}\omega, B(\tau, \theta_{-t}\omega)) \subseteq K(\tau, \omega)\ ,\ \forall\ t\ \geq T.$$

Definition 2.6 Let $\mathcal{D}$ be a collection of random subset of X. Then Φ is said to be $\mathcal{D}$-pullback asymptotically compact in X if for $\mathbb{P}$-a.e $\omega \in \Omega$, $\{\Phi(t_n\ , \theta_{-t_n}\omega\ , x_n)\}_{n=1}^{\infty}$ has a convergent subsequence in X when $t_n \to \infty$ and $x_n \in B(\theta_{-t_n}\omega)$ with $\{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$.

Definition 2.7 Let $\mathcal{D}$ be a collection of random subset of X and $\mathcal{A} = \{\mathcal{A}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$. Then $\mathcal{A}$ is called a $\mathcal{D}$-random attractor (or $\mathcal{D}$-pullback attractor) for Φ, if the following conditions are satisfied: for all $t \in \mathbb{R}^+, \tau \in \mathbb{R}$ and $\omega \in \Omega$,

i) $\mathcal{A}(\tau,\omega)$ is compact, and $\omega \mapsto d(x,\mathcal{A}(\omega))$ is measurable for every $x \in X$;

ii) $\mathcal{A}(\tau,\omega)$ is invariant, that is

$$\varPhi(t,\tau,\omega,\mathcal{A}(\tau,\omega)) = \mathcal{A}(\tau+t,\theta_t\omega), \forall\, t \geq \tau;$$

iii) $\mathcal{A}(\tau,\omega)$ attracts every set in $\mathcal{D}$, that is for every $B = \{B(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \varOmega\} \in \mathcal{D}$,

$$\lim_{t\to\infty} d_X(\varPhi(t,\tau,\theta_{-t}\omega, B(\tau,\theta_{-t}\omega)), \mathcal{A}(\tau,\omega)) = 0,$$

where d_X is the Hausdorff semi-distance given by

$$d_X(Y,Z) = \sup_{y\in Y}\inf_{z\in Z} \|y-z\|_X$$

for any $Y \in X$ and $Z \in X$.

Remark 2.1 Let $(\varOmega,\mathcal{F},\mathbb{P})$ be a probability space with wiener measure $\mathbb{P}$ and the wiener shift $(\theta_t)_{t\in\mathbb{R}}$ be defined by

$$\theta_s\omega(t) = \omega(t+s) - \omega(s),\ t,s \in \mathbb{R}.$$

Then $(\varOmega,\mathcal{F},\mathbb{P},(\theta_t)_{t\in\mathbb{R}})$ is an ergodic metric dynamical system.

Lemma 2.1[5] Let $\mathcal{D}$ be a neighborhood-closed collection of (τ,ω)- parameterized families of nonempty subsets of X and $\varPhi$ be a continuous cocycle on X over $\mathbb{R}$ and $(\varOmega,\mathcal{F},\mathbb{P},(\theta_t)_{t\in\mathbb{R}})$. Then $\varPhi$ has a pullback $\mathcal{D}$-attractor $\mathcal{A}$ in $\mathcal{D}$ if and only if $\varPhi$ is pullback $\mathcal{D}$-asymptotically compact in X and $\varPhi$ has a closed, $\mathcal{F}$-measurable pullback $\mathcal{D}$-absorbing set $K \in \mathcal{D}$, and the unique pullback $\mathcal{D}$-attractor $\mathcal{A} = \mathcal{A}(\tau,\omega)$ is given by

$$\mathcal{A}(\tau,\omega) = \bigcap_{r\geq 0}\overline{\bigcup_{t\geq r}\varPhi(t,\tau-t,\theta_{-t}\omega,K(\tau-t,\theta_{-t}\omega))}\ \tau\in\mathbb{R}\ ,\omega\in\varOmega.$$

3. Cocycles of the Swift-Hohenberg Equation on $\mathbb{R}^2$

Consider the probability space $(\varOmega,\mathcal{F},\mathbb{P})$, where

$$\varOmega = \{\omega \in C(\mathbb{R},\mathbb{R}) : \omega(0) = 0\}.$$

Let $\mathcal{F}$ be Borel σ-algebra induced by the compact-open topology of $\varOmega$ and $\mathbb{P}$ be the corresponding wiener measure on $(\varOmega,\mathcal{F})$. We define a group $\{\theta_t\}_{t\in\mathbb{R}}$ acting on $(\varOmega,\mathcal{F},\mathbb{P})$ and the time shift by

$$\theta_t\omega(\cdot) = \omega(\cdot+t) - \omega(t),\ t\in\mathbb{R},\ \omega\in\varOmega.$$

Then $(\varOmega,\mathcal{F},\mathbb{P},\{\theta_t\}_{t\in\mathbb{R}})$ is a parametric dynamical system.

The original equation (1.1) can be written as follows

$$u(t) = u_\tau - \int_\tau^t (\Delta^2 u + 2\Delta u + au + b|\nabla u|^2 + u^3 - f(x,t))\mathrm{d}s + \sigma\int_\tau^t u(s)\circ \mathrm{d}W(s). \tag{3.1}$$

To study the dynamical behavior of problem (1.1)-(1.2), we need to convert the stochastic equation with a random multiplicative term into non-autonomous deterministic one with a random parameter. Now, we introduce an Ornstein-Uhlenbeck process $z(\theta_t\omega)$ driven by the Brownian motion, which satisfies the following Itô equation

$$\mathrm{d}z + z\mathrm{d}t = \mathrm{d}W(t),\ z(-\infty) = 0. \tag{3.2}$$

Then, the solution of (3.2) is unique and given by

$$z(\theta_t\omega) = \int_{-\infty}^0 \mathrm{e}^s(\theta_t\omega)(s)\mathrm{d}s,\ s\in\mathbb{R},\ t\in\mathbb{R},\ \omega\in\varOmega.$$

By [1,7], the random variable $|z(\theta_t\omega)|$ is tempered and there is an invariant set $\tilde{\varOmega} \subseteq \varOmega$ of full $\mathbb{P}$ measure such that $z(\theta_t\omega)$=$z(t,\omega)$ is continuous in t for every $\omega \in \tilde{\varOmega}$. For convenience we shall write $\tilde{\varOmega}$ as $\varOmega$. We put in the consideration the following properties:

$$\lim_{t\to\pm\infty}\frac{|z(\theta_t\omega)|}{|t|} = 0, \quad \text{and} \quad \lim_{t\to\pm\infty}\frac{\int_0^t z(\theta_s\omega)\mathrm{d}s}{t} = 0. \tag{3.3}$$

If u is the solution of equation (1.1), we let the variable

$$v = \mathrm{e}^{-z(\theta_t\omega)}u(t), \quad \mathrm{d}v = \mathrm{e}^{-z(\theta_t\omega)}\mathrm{d}u(t) - u(t)\mathrm{e}^{-z(\theta_t\omega)}\mathrm{d}z(\theta_t\omega). \tag{3.4}$$

Then

$$\mathrm{d}v = -(\Delta^2 v + 2\Delta v + av + \mathrm{e}^{2z(\theta_t\omega)}v^3 + b\mathrm{e}^{z(\theta_t\omega)}|\nabla v|^2 - \mathrm{e}^{-z(\theta_t\omega)}f(x,t))\mathrm{d}t + z(\theta_t\omega)v(t)\mathrm{d}t.$$

Thus, the equation (1.1) transforms into the following system:

$$\frac{\mathrm{d}v}{\mathrm{d}t} + (a - z(\theta_t\omega))v + 2\Delta v + \Delta^2 v + \mathrm{e}^{2z(\theta_t\omega)}v^3 + b\mathrm{e}^{z(\theta_t\omega)}|\nabla v|^2 - \mathrm{e}^{-z(\theta_t\omega)}f(x,t) = 0, \tag{3.5}$$

$$v(x,\tau) = \mathrm{e}^{-z(\theta_\tau\omega)}u_\tau(x). \tag{3.6}$$

For $t > \tau$, $\tau \in \mathbb{R}$ and $x \in \mathbb{R}^2$, the equation (3.5) is a deterministic equation with random parameters.

By Fatou-Galerkin methods and some a priori estimates, we can show that if f and v_τ are given satisfying $f \in L^2_{\mathrm{loc}}(\mathbb{R}, L^2(\mathbb{R}^2)), v_\tau \in L^2(\mathbb{R}^2)$, then we can obtain a unique solution, that is for $\mathbb{P}$-a.e $\omega \in \Omega, \tau \in \mathbb{R}$ and $v(\tau,\tau,\omega,v_\tau) = v_\tau$, $v_\tau \in L^2(\mathbb{R}^2)$, and for every $T > 0$, $v(\cdot,\tau,\omega,v_\tau) \in C([\tau,\infty), L^2(\mathbb{R}^2)) \cap L^2((\tau,\tau+T); H^1(\mathbb{R}^2))$. For every $t \geq \tau$, let $u(t,\tau,\omega,u_\tau) = v(t,\tau,\omega,v_\tau)\mathrm{e}^{z(\theta_t\omega)}$ with $u_\tau = v_\tau \mathrm{e}^{z(\theta_\tau\omega)}$. Then we obtain that u is continuous and $(\mathcal{F}, \mathfrak{B}(L^2(\mathbb{R}^2))$-measurable in $\omega \in \Omega$.

Define a cocycle $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$, and let

$$\Phi(t,\tau,\omega,u_\tau) = u(t+\tau,\tau,\theta_{-\tau}\omega,u_\tau) = v(t+\tau,\tau,\theta_{-\tau}\omega,v_\tau)\mathrm{e}^{z(\theta_{t+\tau}\omega)}, \tag{3.7}$$

where $v_\tau = u_\tau \mathrm{e}^{-z(\theta_{-\tau}\omega)}$. Then we can check that Φ is a continuous random cocycle associated with equation (1.1) on $L^2(\mathbb{R}^2)$ over $(\mathbb{R}, \{\theta_t\}_{t\in\mathbb{R}})$ and $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$, where $\{\theta_t\}_{t\in\mathbb{R}}$ is a family of shift operator on $\mathbb{R}$ such that $\theta_t(\tau) = t + \tau$.

Provided that $\mathfrak{D}$ is a collection of tempered random subsets of $L^2(\mathbb{R}^2)$, we will prove the existence of an absorbing set in $L^2(\mathbb{R}^2)$. Let B be a bounded nonempty subset of $L^2(\mathbb{R}^2)$, and

$$\|B\| = \sup_{\varphi\in B} \|\varphi\|_{L^2(\mathbb{R}^2)}.$$

Let $\mathcal{D} = \{\mathcal{D}(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ be a family of subset of B, which satisfies

$$\lim_{t\to+\infty} \mathrm{e}^{-\lambda t}\|\mathcal{D}(\tau - t, \theta_{-t}\omega)\|^2 = 0, \tag{3.8}$$

where λ is a positive constant.

For the external force $f(x,t)$, we assume that

$$\int_{-\infty}^0 \mathrm{e}^{\beta s}\|f(\cdot,\tau+s)\|^2_{L^2(\mathbb{R}^2)}\mathrm{d}s \ < +\infty, \ \forall \tau \in \mathbb{R}. \tag{3.9}$$

From (3.9), we deduce that

$$\lim_{k\to\infty} \int_{-\infty}^0 \mathrm{e}^{\beta s} \int_{|x|\geq k} |f(x,\tau+s)|^2 \mathrm{d}x\mathrm{d}s = 0. \tag{3.10}$$

Lemma 3.1[14] (Gagliardo-Nirenberg inequality) Let Ω be an open bounded domain of Lipschitz class in $\mathbb{R}^n$. Assume that $1 \leq p, q \leq \infty, r \geq 1, 0 \leq \theta \leq 1$, and let

$$k - \frac{n}{p} \leq \theta(m - \frac{n}{p}) + (1-\theta)\frac{n}{r}.$$

Then the following inequality holds

$$\|u\|_{k,p} \leq c(\Omega)\|u\|_r^{1-\theta}\|u\|^\theta_{m,q}.$$

Here, c is an arbitrary positive constant, which may change it's value from line to line or even in the same line.

For notation we have $L^2(\mathbb{R}^2)$ is the Hilbert space with usual inner products and norms, $(\cdot,\cdot), \|\cdot\|$, where $(u,v)=\int_{\mathbb{R}^2} u(x)v(x)\mathrm{d}x$. Also $H^\sigma(\mathbb{R}^2)$ is the Sobolev space $\{u\in L^2(\mathbb{R}^2),\ D^k u\in L^2(\mathbb{R}^2),\ k\le\sigma\}$ with norm $\|.\|_{H^\sigma}=\|.\|_\sigma$.

4. Uniform a Priori Estimates of Solution

In this part, we show uniform a priori estimates of a solution for the stochastic local modified Swift-Hohenberg equation.

Lemma 4.1 Under the assumptions in (3.9), for every $\tau\in\mathbb{R}$, $\omega\in\Omega$, $\mathcal{D}=\{\mathcal{D}(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathfrak{D}$, there exists, $T_{1\mathcal{D}}=T(\tau,\omega,\mathcal{D})>0$ such that for all $t\ge T_{1\mathcal{D}}$, the solution of problem (3.5) satisfies

$$\|v(\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t}(\theta_{-\tau}\omega)\|^2\le\rho_1(\tau,\omega), \tag{4.1}$$

where

$$\begin{aligned}\rho_1(\tau,\omega)=&M(\theta_{-\tau}\omega)\int_{-\infty}^0 \mathrm{e}^{2\beta s+2\int_0^s z(\theta_l\omega)\mathrm{d}l}\mathrm{d}s\\&+\int_{-\infty}^0 \mathrm{e}^{2\beta s+2|z(\theta_s\omega)|+2\int_0^s z(\theta_\tau\omega)\mathrm{d}\tau}\|f(s+\tau)\|^2\mathrm{d}s,\end{aligned}$$

$M(\omega)$ is a tempered random variable.

Proof Taking the inner product of equation (3.5) with v in $L^2(\mathbb{R}^2)$, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2+2(a-z(\theta_t\omega))\|v\|^2+4(\Delta v,v)+2\|\Delta v\|^2\\=&-2\mathrm{e}^{-2z(\theta_t\omega)}\|v\|_{L^4}^4-2\mathrm{e}^{-z(\theta_t\omega)}(f(x,t),v)-2b\mathrm{e}^{z(\theta_t\omega)}\int_{\mathbb{R}^2}|\nabla v|^2\cdot v\mathrm{d}x.\end{aligned} \tag{4.2}$$

Applying the Hölder inequality and the ϵ-Young inequality, we have

$$4(\Delta v,v)\le\frac{1}{2}\|\Delta v\|^2+8\|v\|^2, \tag{4.3}$$

$$|2\mathrm{e}^{-z(\theta_t\omega)}(f(x,t),v)|\le\|v\|^2+\mathrm{e}^{-2z(\theta_t\omega)}\|f(\cdot,t)\|^2. \tag{4.4}$$

Now, we deal with the last term on the right hand side of (4.2). By the Hölder inequality, the Gagliardo-Nirenberg inequality with $k=1,n=2,p=r=4,m=q=2,0<\theta<\frac{1}{2}$ and the Young inequality, we can obtain that

$$\begin{aligned}|-2b\mathrm{e}^{z(\theta_t\omega)}\int_{\mathbb{R}^2}|\nabla v|^2\cdot v\mathrm{d}x|\le&2|b|\mathrm{e}^{z(\theta_t\omega)}\|\nabla v\|_{L^4}^2\|v\|\\\le&\frac{1}{2}\|\Delta v\|^2+C\mathrm{e}^{2z(\theta_t\omega)}\|v\|_{L^4}^{\frac{3-2\theta}{1-\theta}}.\end{aligned} \tag{4.5}$$

Substituting (4.3)-(4.5) into (4.2), we obtain that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2+2(a-z(\theta_t\omega))\|v\|^2+\|\Delta v\|^2\\\le&9\|v\|^2-2\mathrm{e}^{-2z(\theta_t\omega)}\|v\|_{L^4}^4+C\mathrm{e}^{2z(\theta_t\omega)}\|v\|_{L^4}^{\frac{3-2\theta}{1-\theta}}+\mathrm{e}^{-2z(\theta_t\omega)}\|f(\cdot,t)\|^2.\end{aligned} \tag{4.6}$$

Namely for some $\beta>0$, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2+2(\beta-z(\theta_t\omega))\|v\|^2+\|\Delta v\|^2\\\le&(9+2(\beta-a))\|v\|^2-2\mathrm{e}^{-2z(\theta_t\omega)}\|v\|_{L^4}^4+C\mathrm{e}^{2z(\theta_t\omega)}\|v\|_{L^4}^{\frac{3-2\theta}{1-\theta}}+\mathrm{e}^{-2z(\theta_t\omega)}\|f(\cdot,t)\|^2.\end{aligned} \tag{4.7}$$

Since $3<\frac{3-2\theta}{1-\theta}<4(0<\theta<\frac{1}{2})$, and the properties of $z(\theta_t\omega)$, there exists a random variable $M(\omega)>0$ such that

$$(9+2(\beta-a))\|v\|^2-2\mathrm{e}^{-2z(\theta_t\omega)}\|v\|_{L^4}^4+C\mathrm{e}^{2z(\theta_t\omega)}\|v\|_{L^4}^{\frac{3-2\theta}{1-\theta}}\le M(\omega). \tag{4.8}$$

Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2+2(\beta-z(\theta_t\omega))\|v\|^2+\|\Delta v\|^2\leq M(\omega)+\mathrm{e}^{-2z(\theta_t\omega)}\|f(\cdot,t)\|^2. \tag{4.9}$$

By the Gronwall inequality we have

$$\begin{aligned}&\|v(\tau,\tau-t,\omega,v_{\tau-t}(\theta_{-\tau}\omega)\|^2+\int_{\tau-t}^{\tau}\mathrm{e}^{-2\beta(\tau-s)+2\int_\tau^s z(\theta_l\omega)\mathrm{d}l}\|\Delta v(s)\|^2\mathrm{d}s\\ \leq&\mathrm{e}^{-2\beta t+2\int_{\tau-t}^{\tau}z(\theta_s\omega)\mathrm{d}s}\|v(\tau-t)\|^2+M(\omega)\int_{\tau-t}^{\tau}\mathrm{e}^{-2\beta(\tau-s)+2\int_\tau^s z(\theta_l\omega)\mathrm{d}l}\mathrm{d}s\\ &+\int_{\tau-t}^{\tau}\mathrm{e}^{-2z(\theta_s\omega)+2\beta(s-\tau)+2\int_\tau^s z(\theta_l\omega)\mathrm{d}l}\cdot\|f(s)\|^2\mathrm{d}s.\end{aligned} \tag{4.10}$$

Replacing ω with $\theta_{-\tau}\omega$ in (4.10) we get

$$\begin{aligned}&\|v(\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t}\|^2+\int_{-t}^{0}\mathrm{e}^{-2\beta(\tau-s)+2\int_0^s z(\theta_l\omega)\mathrm{d}l}\|\Delta v(s)\|^2\mathrm{d}s\\ \leq&\mathrm{e}^{-2\beta t+2\int_{-t}^{0}z(\theta_s\omega)\mathrm{d}s}\|v(\tau-t)\|^2+M(\theta_{-\tau}\omega)\int_{-\infty}^{0}\mathrm{e}^{2\beta s+2\int_0^s z(\theta_l\omega)\mathrm{d}l}\mathrm{d}s\\ &+\int_{-\infty}^{0}\mathrm{e}^{2\beta s+2|z(\theta_s\omega)|+2\int_0^s z(\theta_\tau\omega)\mathrm{d}\tau}\cdot\|f(s+\tau)\|^2\mathrm{d}s.\end{aligned} \tag{4.11}$$

Since $|z(\theta_t\omega)|$ is stationary and ergodic, from (3.3) we can get there exists $T_{1\mathcal{D}}>0$ such that for all $t\geq T_{1\mathcal{D}}$,

$$\frac{\int_{-t}^0 2z(\theta_\tau\omega)\mathrm{d}\tau}{t}\leq\frac{\beta}{2}.$$

It follows that

$$\mathrm{e}^{-2\beta t+2\int_{-t}^{0}z(\theta_s\omega)\mathrm{d}s}\|v(\tau-t)\|^2\leq\mathrm{e}^{-2\beta t}\|v(\tau-t)\|^2\to0,\ \ t\to+\infty, \tag{4.12}$$

$$\int_{-\infty}^{0}\mathrm{e}^{2\beta s+2\int_0^s z(\theta_l\omega)\mathrm{d}l}\mathrm{d}s\leq\int_{-\infty}^{0}\mathrm{e}^{4\beta s}\mathrm{d}s\leq\frac{1}{4\beta}, \tag{4.13}$$

$$\int_{-\infty}^{0}\mathrm{e}^{2\beta(s+\frac{|z(\theta_s\omega)|}{s}+\frac{\int_0^s z(\theta_\tau\omega)\mathrm{d}\tau}{s})}\|f(s+\tau)\|^2\mathrm{d}s\leq\int_{-\infty}^{0}\mathrm{e}^{4\beta s}\|f(s+\tau)\|^2\mathrm{d}s<+\infty. \tag{4.14}$$

So there exists a random variable $\rho_1(\tau,\omega)$, for $\mathbb{P}$-a.e. $\omega\in\Omega$ and $t>T_{1\mathcal{D}}$,

$$\begin{aligned}\rho_1(\tau,\omega)=&M(\theta_{-\tau}\omega)\int_{-\infty}^{0}\mathrm{e}^{2\beta s+2\int_0^s z(\theta_l\omega)\mathrm{d}l}\mathrm{d}s+\int_{-\infty}^{0}\mathrm{e}^{2\beta s+2|z(\theta_s\omega)|+2\int_0^s z(\theta_\tau\omega)\mathrm{d}\tau}\|f(s+\tau)\|^2\mathrm{d}s\\ &<+\infty,\end{aligned} \tag{4.15}$$

then we can get $\|v(\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t}\|^2\leq C(1+\rho_1(\tau,\omega))$.

Lemma 4.2 Under the assumptions in (3.9), for every $\tau\in\mathbb{R}$, $\omega\in\Omega$, $\mathcal{D}=\{\mathcal{D}(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathfrak{D}$, there exists $T_{2\mathcal{D}}=T(\tau,\omega,\mathcal{D})>0$ such that for all $t\geq T_{2\mathcal{D}}$, and there holds

$$\int_{\tau}^{\tau+1}\|\Delta v(s-\tau,\theta_{-\tau-1}\omega,v_{\tau-t}(\theta_{-\tau-1}\omega))\|^2\mathrm{d}s\leq\rho(\tau,\omega),\ \tau>T_{2\mathcal{D}}, \tag{4.16}$$

where

$$\begin{aligned}\rho(\tau,\omega)=&C(1+\mathrm{e}^{2\beta+2\max_{-1\leq\tau\leq0}|z(\theta_s\omega)|}[\int_{-\infty}^{0}\mathrm{e}^{-2z(\theta_s\omega)+2\beta s+\int_s^0 2z(\theta_\tau\omega)\mathrm{d}\tau}\|f(s+\tau+1)\|^2\mathrm{d}s\\ &+M(\theta_{-\tau-1}\omega)\int_{-\infty}^{0}\mathrm{e}^{2\beta s+\int_s^0 2z(\theta_\tau\omega)\mathrm{d}\tau}\mathrm{d}s]),\end{aligned}$$

Proof Multiplying equation (4.9) by $e^{2\beta t-\int_0^t 2z(\theta_\tau\omega)d\tau}$, we get

$$\begin{aligned}&\frac{d}{dt}[e^{2\beta t-\int_0^t 2z(\theta_\tau\omega)d\tau}\|v\|^2]+e^{2\beta t-\int_0^t 2z(\theta_\tau\omega)d\tau}\|\Delta v\|^2\\ \le&M(\omega)e^{2\beta t-2\int_0^t z(\theta_\tau\omega)d\tau}+e^{-2z(\theta_\tau\omega)+2\beta t-2\int_0^t z(\theta_l\omega)dl}\|f(t)\|^2.\end{aligned}\tag{4.17}$$

Let $\tilde{T}\le t\le t+1$. Integrating from $\tilde{T}$ to τ, we have

$$\begin{aligned}&e^{2\beta t-\int_0^t 2z(\theta_\tau\omega)d\tau}\|v(\tau)\|^2+\int_{\tilde{T}}^{\tau}e^{2\beta s-\int_0^s 2z(\theta_\tau\omega)d\tau}\|\Delta v(s)\|^2ds\\ \le&e^{2\beta t-\int_0^t 2z(\theta_\tau\omega)d\tau}\|v(\tilde{T})\|^2+M(\omega)\int_{\tilde{T}}^{\tau}e^{2\beta s-2\int_0^s 2z(\theta_\tau\omega)d\tau}ds\\ &+\int_{\tilde{T}}^{\tau}e^{-2z(\theta_s\omega)+2\beta s-2\int_0^s z(\theta_l\omega)dl}\|f(s)\|^2ds.\end{aligned}\tag{4.18}$$

Multiplying (4.18) by $e^{-2\beta\tau+\int_0^\tau 2z(\theta_\tau\omega)d\tau}$ and omitting the first term yield that

$$\begin{aligned}&\int_{\tilde{T}}^{\tau}e^{2\beta(s-\tau)+\int_s^\tau 2z(\theta_l\omega)dl}\|\Delta v(s,s-\tau,\omega,v_{\tau-t})\|^2ds\\ \le&e^{2\beta(\tilde{T}-\tau)-\int_{\tilde{T}}^{\tau}2z(\theta_l\omega)dl}\|v(\tilde{T},\omega,v_{\tau-t}(\omega))\|^2+M(\omega)\int_{\tilde{T}}^{\tau}e^{2\beta(s-\tau)+2\int_s^\tau z(\theta_\tau\omega)d\tau}ds\\ &+\int_{\tilde{T}}^{\tau}e^{-2z(\theta_s\omega)+2\beta(s-\tau)-\int_s^\tau 2z(\theta_l\omega)dl}\|f(s)\|^2ds.\end{aligned}\tag{4.19}$$

By substituting τ for $\tilde{T}$ in (4.11), we get

$$\begin{aligned}\|v(\tilde{T},\tilde{T}-t,\omega,v_{\tau-t}(\omega))\|^2\le&e^{-2\beta\tilde{T}+\int_{\tilde{T}-t}^{\tilde{T}}2z(\theta_\tau\omega)d\tau}\|v_\tau(\omega)\|^2+M(\omega)\int_{\tilde{T}-t}^{\tilde{T}}e^{-2\beta(\tilde{T}-s)+2\int_{\tilde{T}}^{s}z(\theta_l\omega)dl}ds\\ &+\int_{\tilde{T}-t}^{\tilde{T}}e^{-2z(\theta_s\omega)-2\beta(\tilde{T}-s)-\int_s^{\tilde{T}}2z(\theta_\tau\omega)d\tau}\|f(s)\|^2ds.\end{aligned}\tag{4.20}$$

Substituting (4.20) into (4.19), we have

$$\begin{aligned}&\int_{\tilde{T}}^{\tau}e^{2\beta(s-\tau)+\int_s^\tau 2z(\theta_l\omega)dl}\|\Delta v(s,s-\tau,\omega,v_{\tau-t}(\omega))\|^2ds\\ \le&e^{-2\beta\tau+\int_{\tilde{T}-t}^{\tau}2z(\theta_\tau\omega)d\tau}\|v_\tau(\omega)\|^2+M(\omega)\int_{\tilde{T}-t}^{\tau}e^{-2\beta(s-\tau)+2\int_s^\tau z(\theta_\tau\omega)d\tau}ds\\ &+\int_{\tilde{T}-t}^{\tau}e^{-2z(\theta_s\omega)-2\beta(\tau-s)-\int_s^\tau 2z(\theta_\tau\omega)d\tau}\|f(s)\|^2ds.\end{aligned}\tag{4.21}$$

Replacing ω by $\theta_{-\tau}\omega$ in (4.21), we obtain

$$\begin{aligned}&\int_{\tilde{T}}^{\tau}e^{2\beta(s-\tau)+\int_s^\tau 2z(\theta_{l-\tau}\omega)dl}\|\Delta v(s,s-\tau,\theta_{-\tau}\omega,v_{\tau-t}(\theta_{-\tau}\omega))\|^2ds\\ \le&e^{-2\beta\tau+\int_{\tilde{T}-t}^{\tau}2z(\theta_{l-\tau}\omega)dl}\|v_{\tau-t}(\theta_{-\tau}\omega)\|^2+M(\theta_{-\tau}\omega)\int_{\tilde{T}-t}^{\tau}e^{-2\beta(s-\tau)+2\int_s^\tau z(\theta_{l-\tau}\omega)dl}ds\\ &+\int_{\tilde{T}-t}^{\tau}e^{-2z(\theta_{s-\tau}\omega)-2\beta(\tau-s)-\int_s^\tau 2z(\theta_{l-\tau}\omega)dl}\|f(s)\|^2ds.\end{aligned}\tag{4.22}$$

To get the desired result, we substitute $\tilde{T}$ for τ and τ for $\tau+1$ in (4.22) as follows:

$$\begin{aligned}&\int_{\tau}^{\tau+1}e^{2\beta(s-\tau-1)+\int_s^{\tau+1}2z(\theta_{l-\tau-1}\omega)dl}\|\Delta v(s,s-\tau,\theta_{-\tau-1}\omega,v_{\tau-t}(\theta_{-\tau-1}\omega))\|^2ds\\ \le&e^{-2\beta(\tau+1)+\int_{\tau-t}^{\tau+1}2z(\theta_{l-\tau-1}\omega)dl}\|v_{\tau-t}(\theta_{-\tau-1}\omega)\|^2\\ &+\int_{\tau-t}^{\tau+1}e^{-2z(\theta_{s-\tau-1}\omega)-2\beta(\tau+1-s)+\int_s^{\tau+1}2z(\theta_{l-\tau-1}\omega)dl}\|f(s)\|^2ds\end{aligned}$$

$$+ M(\theta_{-\tau-1}\omega)\int_{\tau-t}^{\tau+1} \mathrm{e}^{-2\beta(\tau+1-s)+\int_s^{\tau+1} 2z(\theta_{l-\tau-1}\omega)\mathrm{d}l}\mathrm{d}s. \tag{4.23}$$

Then we have

$$\int_{\tau}^{\tau+1} \mathrm{e}^{2\beta(s-\tau-1)+\int_{s-\tau-1}^{0} 2z(\theta_l\omega)\mathrm{d}l}\|\Delta v(s, s-\tau, \theta_{-\tau-1}\omega, v_{\tau-t}(\theta_{-\tau-1}\omega))\|^2\mathrm{d}s$$

$$\le \mathrm{e}^{-2(\tau+1)(\beta-\frac{\int_{-\tau-1}^{0} 2z(\theta_l\omega)\mathrm{d}l}{2(\tau+1)})}\|v_\tau(\theta_{-\tau-1}\omega)\|^2 + \int_{-\infty}^{0} \mathrm{e}^{-z(\theta_s\omega)+2\beta s+\int_s^0 2z(\theta_l\omega)\mathrm{d}l}\|f(s+\tau+1)\|^2\mathrm{d}s$$

$$+ M(\theta_{-\tau-1}\omega)\int_{-\infty}^{0} \mathrm{e}^{2\beta s+\int_s^0 2z(\theta_l\omega)\mathrm{d}l}\mathrm{d}s. \tag{4.24}$$

Since $t \ge T_{1\mathcal{D}}$, according to the properties of $z(\theta_t\omega)$, when $-1 \le s-\tau-1 \le 0$, we infer that

$$\int_{\tau}^{\tau+1} \mathrm{e}^{2\beta(s-\tau-1)+\int_{s-\tau-1}^{0} 2z(\theta_l\omega)\mathrm{d}l}\|\Delta v(s, s-\tau, \theta_{-\tau-1}\omega, v_0(\theta_{-t-1}\omega))\|^2\mathrm{d}s$$

$$\ge \mathrm{e}^{-2\beta-2\max_{-1\le\tau\le 0}|z(\theta_l\omega)|}\int_{\tau}^{\tau+1} \|\Delta v(s, s-\tau, \theta_{-\tau-1}\omega, v_0(\theta_{-\tau-1}\omega))\|^2\mathrm{d}s. \tag{4.25}$$

Then, from (4.24) and (4.25), we prove that there exists a random variable $\rho(\tau,\omega)$ and $T_{2\mathcal{D}} \ge 0$ such that, for $\mathbb{P}$-a.e. $\omega \in \Omega$ and all $t \ge T_{2\mathcal{D}}$,

$$\int_{\tau}^{\tau+1} \|\Delta v(s, s-\tau, \theta_{-\tau-1}\omega, v_0(\theta_{-\tau-1}\omega))\|^2\mathrm{d}s \le \rho(\tau,\omega),$$

which completes the proof.

Then, similar to the proof of Lemma 4.1 in [10], we can also get the result.

Lemma 4.3 Under the assumptions in (3.9), for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, $\mathcal{D} = \{\mathcal{D}(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathfrak{D}$, there exists $T_{3\mathcal{D}} = T(\tau,\omega,\mathcal{D}) > 0$ such that for all $t \ge T_{3\mathcal{D}}$, and there holds

$$\|\Delta v(\tau, \tau-t, \theta_{-\tau}\omega, v_{\tau-t}\|^2 \le \rho_3(\tau,\omega), \tag{4.26}$$

where

$$\rho_3(\tau,\omega) = C(1+r(\omega)\rho(\tau,\omega)) + \mathrm{e}^{\frac{22}{3}\beta}\cdot \mathrm{e}^{\frac{16}{3}\max_{-1\le\tau\le 0}|z(\theta_\tau\omega)|}\cdot (\int_{-\infty}^{0} \mathrm{e}^{-2z(\theta_s\omega)+2\beta s+\int_s^0 2|z(\theta_r\omega)|dr}\mathrm{d}s)^{\frac{11}{3}})$$

$$+ \mathrm{e}^{\max_{-1\le\tau\le 0} 2|z(\theta_r\omega)|}\int_{-\infty}^{0} \mathrm{e}^{s+1}\|f(s+\tau+1)\|^2\mathrm{d}s.$$

Lemma 4.4 If $f \in L^2_{\mathrm{loc}}(\mathbb{R}, L^2(\mathbb{R}^2)), v_\tau \in L^2(\mathbb{R}^2)$. Then, for any $t > 0$, $\mathbb{P}$-a.e. $\omega \in \Omega$, $\tau \in \mathbb{R}$ and $\mathcal{D} = \{\mathcal{D}(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \mathbb{R}\} \in \mathfrak{D}$, there exists $T_{\mathcal{D}} = T(\tau,\omega,\mathcal{D},\eta) > 1$ and $K = K(\tau,\omega,\eta) \ge 1$ such that for any $t \ge T_{\mathcal{D}}$, the solution v of equation (3.5) with ω replaced by $\theta_{-\tau}\omega$ satisfies

$$\int_{|x|\ge k} |v(\tau, \tau-t, \theta_{-\tau}\omega, v_{\tau-t})(x)|^2\mathrm{d}x \le \eta,$$

where $v_{\tau-t} \in \mathcal{D}(\tau-t, \theta_{-\tau}\omega)$.

Proof Let θ be a smooth function defined on $\mathbb{R}^+$, such that $0 \le \theta(s) \le 1$ for all $s \in \mathbb{R}^+$, and

$$\theta(s) = \begin{cases} 0 & \text{for } 0 \le s \le 1, \\ 1 & \text{for } s \ge 2. \end{cases} \tag{4.27}$$

Then there exists a positive constant C such that $|\theta'(s)| \le C$ for all $s \in \mathbb{R}^+$. For convenience, we write $\theta_k = \theta(\frac{|x|^2}{k^2})$.

Taking the inner product of (3.5) with $\theta_k v$ in $L^2(\mathbb{R}^2)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\theta_k|v|^2\mathrm{d}x+2(a-z(\theta_t\omega))\int_{\mathbb{R}^2}\theta_k|v|^2\mathrm{d}x+4\int_{\mathbb{R}^2}\Delta v\theta_k v\mathrm{d}x+2\int_{\mathbb{R}^2}\Delta^2 v\theta_k v\mathrm{d}x$$
$$+2\mathrm{e}^{2z(\theta_t\omega)}\int_{\mathbb{R}^2}v^3\theta_k v\mathrm{d}x+2b\mathrm{e}^{z(\theta_t\omega)}\int_{\mathbb{R}^2}|\nabla v|^2\theta_k v\mathrm{d}x-2\int_{\mathbb{R}^2}\mathrm{e}^{-z(\theta_t\omega)}f(x,t)\theta_k v\mathrm{d}x=0. \quad (4.28)$$

For the terms of (4.28), by the Hölder inequality and the ϵ-Young inequality, we obtain

$$|4\int_{\mathbb{R}^2}\Delta v\theta_k v\mathrm{d}x|\le\int_{\mathbb{R}^2}\theta_k|\Delta v|^2\mathrm{d}x+4\int_{\mathbb{R}^2}\theta_k|v|^2\mathrm{d}x, \quad (4.29)$$

$$|2\int_{\mathbb{R}^2}\Delta^2 v\theta_k v\mathrm{d}x|\le\int_{\mathbb{R}^2}\theta_k|v|^2\mathrm{d}x+\int_{\mathbb{R}^2}\theta_k|\Delta^2 v|^2\mathrm{d}x, \quad (4.30)$$

$$|2\mathrm{e}^{2z(\theta_t\omega)}\int_{\mathbb{R}^2}v^3\theta_k v\mathrm{d}x|\le\int_{\mathbb{R}^2}\theta_k|v|^2\mathrm{d}x+\mathrm{e}^{4z(\theta_t\omega)}\int_{\mathbb{R}^2}\theta_k v^6\mathrm{d}x, \quad (4.31)$$

$$|2b\mathrm{e}^{z(\theta_t\omega)}\int_{\mathbb{R}^2}|\nabla v|^2\theta_k v\mathrm{d}x|\le\int_{\mathbb{R}^2}\theta_k|v|^2\mathrm{d}x+|b|\mathrm{e}^{2z(\theta_t\omega)}\int_{\mathbb{R}^2}\theta_k|\nabla v|^4\mathrm{d}x, \quad (4.32)$$

$$|-2\int_{\mathbb{R}^2}\mathrm{e}^{-z(\theta_t\omega)}f(x,t)\theta_k v\mathrm{d}x|\le\int_{\mathbb{R}^2}\theta_k|v|^2\mathrm{d}x+\mathrm{e}^{-2z(\theta_t\omega)}\int_{\mathbb{R}^2}\theta_k|f(x,t)|^2\mathrm{d}x. \quad (4.33)$$

Then, from (4.29)-(4.33) it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\theta_k|v|^2\mathrm{d}x+2(\beta-z(\theta_t\omega))\int_{\mathbb{R}^2}\theta_k|v|^2\mathrm{d}x+\int_{\mathbb{R}^2}\theta_k|\Delta v|^2\mathrm{d}x$$
$$\le 2|\beta-\alpha+4|\int_{\mathbb{R}^2}|v|^2\mathrm{d}x+b^2\mathrm{e}^{2z(\theta_t\omega)}\|\nabla v\|^4_{L^4(\mathbb{R}^2)}+\mathrm{e}^{4z(\theta_t\omega)}\|v\|^6_{L^6(\mathbb{R}^2)}+\mathrm{e}^{-2z(\theta_t\omega)}\int_{\mathbb{R}^2}\theta_k|f(x,t)|^2\mathrm{d}x. \quad (4.34)$$

By the Gagliardo-Nirenberg inequality, we know

$$b^2\mathrm{e}^{2z(\theta_t\omega)}\|\nabla v\|^4_{L^4(\mathbb{R}^2)}\le C\mathrm{e}^{2z(\theta_t\omega)}\|v\|^2\|\Delta v\|\le\|\Delta v\|^2+C\mathrm{e}^{4z(\theta_t\omega)}\|v\|^6, \quad (4.35)$$

and

$$\mathrm{e}^{4z(\theta_t\omega)}\|v\|^6_{L^6(\mathbb{R}^2)}\le C\mathrm{e}^{4z(\theta_t\omega)}\|v\|^5\|\Delta v\|\le\|\Delta v\|^2+C\mathrm{e}^{8z(\theta_t\omega)}\|v\|^{10}. \quad (4.36)$$

Combining with (4.35)-(4.36) we finally get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\theta_k|v|^2\mathrm{d}x+2(\beta-z(\theta_t\omega))\int_{\mathbb{R}^2}\theta_k|v|^2\mathrm{d}x$$
$$\le 2|\beta-\alpha+4|\|v\|^2+2\|\Delta v\|^2+C\mathrm{e}^{4z(\theta_t\omega)}\|v\|^6+C\mathrm{e}^{8z(\theta_t\omega)}\|v\|^{10}+\mathrm{e}^{-2z(\theta_t\omega)}\int_{\mathbb{R}^2}\theta_k|f(x,t)|^2\mathrm{d}x. \quad (4.37)$$

By applying the Gronwall inequality to (4.37) on $[\tau-t,\tau]$

$$\int_{\mathbb{R}^2}\theta_k|v(\tau,\tau-t,\omega,v_{\tau-t})|^2\mathrm{d}x$$
$$\le\mathrm{e}^{-2\beta t+2\int_{\tau-t}^{\tau}z(\theta_l\omega)\mathrm{d}l}\|v_{\tau-t}(\theta_{-\tau}\omega)\|^2+2|\beta-\alpha+4|\int_{\tau-t}^{\tau}\mathrm{e}^{-2\beta(\tau-s)+2\int_s^{\tau}z(\theta_l\omega)\mathrm{d}l}\|v(s)\|^2\mathrm{d}s$$
$$+2\int_{\tau-t}^{\tau}\mathrm{e}^{-2\beta(\tau-s)+2\int_s^{\tau}z(\theta_l\omega)\mathrm{d}l}\|\Delta v(s)\|^2\mathrm{d}s+C\int_{\tau-t}^{\tau}\mathrm{e}^{-2\beta(\tau-s)+2\int_s^{\tau}z(\theta_l\omega)\mathrm{d}l+4z(\theta_s\omega)}\|v(s)\|^6\mathrm{d}s$$
$$+C\int_{\tau-t}^{\tau}\mathrm{e}^{-2\beta(\tau-s)+2\int_s^{\tau}z(\theta_l\omega)\mathrm{d}l+8z(\theta_s\omega)}\|v(s)\|^{10}\mathrm{d}s$$
$$+\int_{\tau-t}^{\tau}\mathrm{e}^{-2\beta(\tau-s)+2\int_s^{\tau}z(\theta_l\omega)\mathrm{d}l-2z(\theta_s\omega)}\int_{\mathbb{R}^2}\theta_k|f(x,s)|^2\mathrm{d}x\mathrm{d}s. \quad (4.38)$$

Replacing ω by $\theta_{-\tau}\omega$ in (4.38), we obtain

$$\begin{aligned}&\int_{\mathbb{R}^2}\theta_k|v(\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2\mathrm{d}x\\ \le&\mathrm{e}^{-2\beta t+2\int_{-t}^0 z(\theta_l\omega)\mathrm{d}l}\|v_{(\tau-t)}\|^2+2|\beta-\alpha+4|\int_{-t}^0\mathrm{e}^{2\beta s+2\int_s^0 z(\theta_l\omega)\mathrm{d}l}\|v(s+\tau)\|^2\mathrm{d}s\\ &+2\int_{-t}^0\mathrm{e}^{2\beta s+2\int_s^0 z(\theta_l\omega)\mathrm{d}l}\|\Delta v(s+\tau)\|^2\mathrm{d}s+C\int_{-t}^0\mathrm{e}^{2\beta s+2\int_s^0 z(\theta_l\omega)\mathrm{d}l+4z(\theta_s\omega)}\|v(s+\tau)\|^6\mathrm{d}s\\ &+C\int_{-t}^0\mathrm{e}^{2\beta s+2\int_s^0 z(\theta_l\omega)\mathrm{d}l+8z(\theta_s\omega)}\|v(s+\tau)\|^{10}\mathrm{d}s\\ &+\int_{-t}^0\mathrm{e}^{2\beta s+2\int_s^0 z(\theta_l\omega)\mathrm{d}l-2z(\theta_s\omega)}\int_{\mathbb{R}^2}\theta_k|f(x,s)|^2\mathrm{d}x\mathrm{d}s. \end{aligned}\tag{4.39}$$

For any initial data $v_{\tau-t}\in\mathcal{D}(\tau-t,\theta_{-t}\omega)$, we have

$$\mathrm{e}^{-2\beta t+2\int_{-t}^0 z(\theta_l\omega)\mathrm{d}l}\|v_{(\tau-t)}\|^2\le\mathrm{e}^{-\beta t}\|\mathcal{D}(\tau-t,\theta_{-t}\omega)\|^2\to0,\ \ t\to+\infty.$$

So for an arbitrarily given $\eta>0$, there exists $T_3=T_3(\tau,\omega,\mathcal{D},\eta)$, such that for all $t\ge T_3$,

$$\mathrm{e}^{-2\beta t+2\int_{-t}^0 z(\theta_l\omega)\mathrm{d}l}\|v_{(\tau-t)}\|^2\le\eta.\tag{4.40}$$

By Lemma 4.1 and Lemma 4.3, similar to (4.12) and (4.14) for all $t\ge T_3$, we obtain

$$C\int_{-t}^0\mathrm{e}^{2\beta s+2\int_s^0 z(\theta_l\omega)\mathrm{d}l+4z(\theta_s\omega)}\|v(s+\tau)\|^6\mathrm{d}s\le\eta\rho_1^3(\tau,\omega),\tag{4.41}$$

$$C\int_{-t}^0\mathrm{e}^{2\beta s+2\int_s^0 z(\theta_l\omega)\mathrm{d}l+8z(\theta_s\omega)}\|v(s+\tau)\|^{10}\mathrm{d}s\le\eta\rho_1^5(\tau,\omega),\tag{4.42}$$

$$2\int_{-t}^0\mathrm{e}^{2\beta s+2\int_s^0 z(\theta_l\omega)\mathrm{d}l}\|\Delta v(s+\tau)\|^2\mathrm{d}s\le\eta\rho_2(\tau,\omega).\tag{4.43}$$

Finally, from the condition of (3.10), there exists $k_1=k_1(\tau,\omega)>1$, such that for $k>k_1$, we have

$$\int_{-t}^0\mathrm{e}^{2\beta s+2\int_s^0 z(\theta_l\omega)\mathrm{d}l-2z(\theta_s\omega)}\int_{\mathbb{R}^2}\theta_k|f(x,s+\tau)|^2\mathrm{d}x\mathrm{d}s\le\int_{-t}^0\mathrm{e}^{C\beta s}\int_{|x|\ge k}|f(x,s+\tau)|^2\mathrm{d}x\mathrm{d}s\le\eta.\tag{4.44}$$

Then it holds that for all $k>k_1,t\ge T_3$,

$$\int_{\mathbb{R}^2}\theta_k|v(\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t}(\theta_{-\tau}\omega))|^2\mathrm{d}x\le\eta(1+\rho_1^3(\tau,\omega)+\rho_1^5(\tau,\omega)+\rho_2(\tau,\omega)).\tag{4.45}$$

From (4.45), there exists $k_2=k_2(\tau,\eta)\ge k_1$ such that for all $k\ge k_2,t\ge T_2$,

$$\int_{|x|\ge\sqrt{2}k}|v(\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2\mathrm{d}x\le2\eta(1+\rho_1^3(\tau,\omega)+\rho_1^5(\tau,\omega)+\rho_2(\tau,\omega)).$$

Then the proof is completed.

5. Existence of Random Attractor

Theorem 5.1 Suppose that (3.9)-(3.10) holds. Let $\mathcal{D}$ be defined in (3.8). Then the continuous cocycle Φ associated with the problem (1.1) possess a unique $\mathcal{D}$-random attractor $\mathcal{A}=\{\mathcal{A}(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}$ in the initial space $L^2(\mathbb{R}^2)$.

Proof We have $\tau\in\mathbb{R},\omega\in\Omega$ and $\mathcal{D}\in\mathfrak{D}$. Define

$$\tilde{D}_T(\tau,\omega)=:\cup_{t\ge T}\Phi(t,\tau-t,\theta_{-t}\omega),\mathcal{D}(\tau-t,\theta_{-t}\omega)).$$

Let $\eta>0$. From (4.1) there exists $T_1=T_1(\tau,\omega,\mathcal{D},\eta)$ and a ball $B_{L^2(\mathbb{R}^2)}(0,C(\tau,\omega,\eta))$ centred at zero with radius less than or equal to $C(\tau,\omega,\eta)$ such that

$$\tilde{D}_{T1}(\tau,\omega)\subset B_{L^2(\mathbb{R}^2)}(0,C(\tau,\omega,\eta)).$$

From the compact of Sobolev embedding in the bounded domain, for every $\eta > 0$ there exists a finite $\eta\backslash 4 - net$ in $L^2(Q_K)$ covering $\tilde{D}_{1T}(\tau,\omega)\backslash_{Q_K}$. Therefore

$$k_{L^2}(\tilde{D}_{T1}(\tau,\omega)\backslash_{Q_K}) \leq \frac{\eta}{2},$$

where $k_{L^2}(\cdot)$ is non-compact measure in $L^2(Q_K)$. By Lemma 4.3, there exists $T_2 = T_2(\tau,\omega,\mathcal{D},\eta)$ and $K(\tau,\omega,\mathcal{D},\eta)$ such that

$$\tilde{D}_{T2}(\tau,\omega)\backslash_{Q_K^c} \subset B_{L^2(\mathbb{R}^2)}(0,\frac{\eta}{4}).$$

$T = \max\{T_1, T_2\}$. By additive property of non-compact measure, we have

$$\begin{aligned} k_{L^2}(\tilde{D}_T(\tau,\omega)) &\leq k_{L^2}(\tilde{D}_{T1}(\tau,\omega)\backslash_{Q_K}) + k_{L^2}(\tilde{D}_{T2}(\tau,\omega)\backslash_{Q_K^c}) \\ &\leq \frac{\eta}{2} + k(B_{L^2(\mathbb{R}^2)}(0,\frac{\eta}{4})) \leq \frac{\eta}{2} + \frac{\eta}{2} = \eta. \end{aligned} \tag{5.1}$$

Therefore, by the arbitrariness of η, Φ is omega-limit compact in $L^2(\mathbb{R}^2)$.

By the inequality (4.26) in Lemma 4.3, we deduce that for $\{t > \tau : \tau \in \mathbb{R} \text{ and } \omega \in \Omega\}$ there exists $K = \{K(\tau,\omega) = \{u : \|\Delta u\|^2 \leq \eta\}\} \in \mathfrak{D}$.

Hence, the continuous cocycle Φ has a closed random absorbing set $\{A(\omega)\}_{\omega\in\Omega}$ in $\mathcal{D}$. By Lemma 4.1 and Lemma 4.3, the continuous cocycle Φ is $\mathcal{D}$-random asymptotically compact in $\mathbb{R}^2$. Then the existence of a unique $\mathcal{D}$- random attractor for Φ follows from Lemma 2.1 immediately.

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具有乘积噪声的非自治Swift-Hohenberg方程在无界区域上的渐近性

巴吉[1,2], 刘亭亭[1], 马巧珍[1]

(1. 西北师范大学数学与统计学院, 甘肃 兰州 730070; 2. 伦贝克科技大学教育学院数学部, 伦贝克 南苏丹)

摘要： 本文研究$\mathbb{R}^2$上带有时间依赖外力项与乘性噪声的随机非自治修正Swift-Hohenberg方程的动力行为. 为了克服无界域上Sobolev嵌入不紧的困难, 我们先定义了问题在$L^2(\mathbb{R}^2)$ 上的连续共圈, 并且建立了当空间变量足够大时, 解尾部的一致估计. 通过解的一致估计, 我们证明了随机动力系统的拉回渐近紧性, 进一步得到了随机吸引子的存在性.

关键词： 随机吸引子; Swift-Hohenberg方程; 非自治随机动力系统; 连续余圈