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A New Exceptional Family of Elements for Complementarity Problems

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Abstract: In this paper, we present a generalized exceptional family of elements for complementarity problems, which is a generalization for the concepts of exceptional family of elements and d-orientation sequence for a continuous function. We show that if there exists no generalized exceptional family for a continuous function, then the corresponding complementarity problem has a solution. It is also shown that a continuous function does not possess the generalized exceptional family under Karamardian type condition, Isac and Gowda type condition or p-order generalized coercive type condition. Applying the new concept to the P_*-mapping CP, a new existence result is established.
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1. Introduction

A complementarity problem (CP for short) is to find a $z \in \mathbb{R}^n$ such that

$$z \geq 0, \quad f(z) \geq 0, \quad z^{\mathrm{T}} f(z) = 0, \tag{1.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function. It is known that CP(1.1) has many wide applications in economics, engineering, operation research etc.[1] CP(1.1) has received increasing attention recently. The research of the existence conditions of the solution for CP(1.1) has played an important role in both theory and practical applications. Among these researches, the concept of exceptional family is a very powerful tool to study the existence properties of the solution to CP(1.1) and variational inequality(VI for short) problems. Smith[2] firstly presented the concept of exceptional family of elements for a continuous function. Subsequently, a more general notion of exceptional family of elements was introduced by Isac[3]. Based on this notion, some results on existence of solutions to nonlinear complementarity problems were established in [3-4]. Exceptional family of elements is also extended by ZHAO et al.[5−8] to study the existence conditions of solutions to variational inequality problems. Recently, this notion is extended to study the existence theorems of solutions to semidefinite complementarity problems and copositive cone complementarity problem[9−14]. In [15], ZHAO

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presented the d-orientation sequence concept for a continuous function which is different from other notions of exceptional family of elements, and he proved some similar properties as that of the exceptional family of elements.

In this paper, motivated by the previous studies, we introduce a new concept of exceptional family of elements for a continuous function, which unifies the two concepts of d-orientation sequence and exceptional family of elements for a continuous function. We prove that if there exists no generalized exceptional family for a continuous function, then CP(1.1) has a solution. Under the Karamardian type condition, Isac and Gowda tpye condition or p-order generalized coercive type condition, the property that a continuous function does not possess the generalized exceptional family is established, respectively. By applying the new concept to the P_*-mapping CP, a new existence result is presented.

2. Generalized Exceptional Family of Elements

Throughout this paper, let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function and let $d \in \mathbb{R}^n$ be a given vector with positive orthant, i.e., $d > 0$. f_i represents the i-th component of a vector-valued function f and similar notations are used for vectors. In this section, we first recall the concepts about the exceptional family of elements and d-orientation sequence for f.

Definition 2.1[3] Let $\{z^r\} \subset \mathbb{R}^n_+$ be a set of points, $\{z^r\}$ is defined as an exceptional family of elements for f with respect to $\mathbb{R}^n_+$ if $\|z^r\| \to \infty$ as $r \to \infty$, and for each z^r there exists a scalar $\mu^r > 0$ such that

$$\begin{aligned} &f_i(z^r) = -\mu^r z_i^r, \quad \text{if } z_i^r > 0,\\ &f_i(z^r) \geq 0, \qquad\quad\ \text{if } z_i^r = 0. \end{aligned}$$

Definition 2.2[15] Given $d > 0$, let $\{z^r\} \subset \mathbb{R}^n_+$ be a set of points, $\{z^r\}$ is defined as a d-orientation sequence of the function f if $\|z^r\| \to \infty$ as $r \to \infty$, and for each z^r there exists a scalar $\mu^r > 0$ satisfying

$$\begin{aligned} &f_i(z^r) = -\mu^r d_i, \quad \text{if } z_i^r > 0,\\ &f_i(z^r) \geq -\mu^r d_i, \quad \text{if } z_i^r = 0. \end{aligned}$$

Remark 2.1 Definition 2.2 is quite different from Definition 2.1. In the Definition 2.2, for each $z^r > 0, f_i(z^r) = -\mu^r d_i$ for some scalar μ^r, that is to say, all the vectors $f(z^r)$ with $z^r > 0$ have the same direction $-d$.

In the sequel, we shall present a new concept of a generalized exceptional family of elements as follows.

Definition 2.3 Given $d > 0$, let $\{z^r\} \subset \mathbb{R}^n_+$ be a set of points, $\{z^r\}$ is called a generalized exceptional family of elements for the function f if $\|z^r\| \to \infty$ as $r \to \infty$, and for each z^r there exists a positive number μ^r and a number $\theta \in [0, 1]$ such that

$$\begin{aligned} &f_i(z^r) = -\mu^r((1-\theta)z_i^r + \theta d_i), \quad \text{if } z_i^r > 0,\\ &f_i(z^r) \geq -\mu^r \theta d_i, \qquad\qquad\qquad\ \text{if } z_i^r = 0. \end{aligned}$$

Remark 2.2 In the above definition, if $\theta = 0$, it reduces to the exceptional family of elements for the function[3]. If $\theta = 1$, it becomes to the d-orientation sequence for the function[15]. Therefore it is a generalization of the concepts of exceptional family of elements and d-orientation sequence for the function f.

We all know that $\mathrm{VI}(K, f)$ is to find a solution z^* satisfying

$$(z - z^*)^{\mathrm{T}} f(z^*) \geq 0, \quad \forall z \in K,$$

where $K \subseteq \mathbb{R}^n$ is closed and convex. In particular, when the set $K = \mathbb{R}^n_+ = \{z \in \mathbb{R}^n : z \geq 0\}$, $\mathrm{VI}(K, f)$ reduces to CP(1.1).

For a given positive vector $d \in \mathbb{R}^n$, let

$$K_r = \mathbb{R}^n_+ \cap \{z \in \mathbb{R}^n : z^{\mathrm{T}}(\frac{1}{2}(1-\theta)z + \theta d) \leq r\},$$

where $r > 0$ and $\theta \in [0, 1]$. It is obvious that K_r is bounded and convex, which implies that $\mathrm{VI}(K, f)$ has at least one solution[1].

In order to get a general existence theorem of CP(1.1), it is necessary to give the following lemma which is similar to the one in [15].

Lemma 2.1 Given $d > 0$, let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuous, then CP(1.1) has a solution if and only if there exists a scalar $r > 0$ and $\theta \in [0, 1]$ such that $\mathrm{VI}(K_r, f)$ has a solution z^r with $(z^r)^{\mathrm{T}}(\frac{1}{2}(1-\theta)z^r + \theta d) < r$.

Proof If z^* is a solution to CP(1.1), then

$$(z - z^*)^{\mathrm{T}} f(z^*) \geq 0, \quad \forall z \in \mathbb{R}^n_+.$$

Let $r > (z^*)^{\mathrm{T}}(\frac{1}{2}(1-\theta)z^* + \theta d)$. It is clear that

$$(z - z^*)^{\mathrm{T}} f(z^*) \geq 0, \quad \forall z \in K_r.$$

We can conclude from the definition of $\mathrm{VI}(K_r, f)$[15] that z^* is a solution to $\mathrm{VI}(K_r, f)$.

Next suppose that there exists $r > 0$ and $\theta \in [0, 1]$ such that $\mathrm{VI}(K_r, f)$ has a solution z^r with $(z^r)^{\mathrm{T}}(\frac{1}{2}(1-\theta)z^r + \theta d) < r$, i.e.,

$$(z - z^r)^{\mathrm{T}} f(z^r) \geq 0, \quad \forall z^r \in K_r. \tag{2.1}$$

To prove that z^r is a solution to CP(1.1), it is necessary to prove that

$$(z - z^r)^{\mathrm{T}} f(z^r) \geq 0, \quad \forall z^r \in \mathbb{R}^n_+ \backslash K_r. \tag{2.2}$$

In fact, denote

$$p(\lambda) = \lambda z + (1-\lambda) z^r \in \mathbb{R}^n_+, \quad \forall z \in \mathbb{R}^n_+, \quad \forall \lambda \in [0, 1].$$

Taking into account that $(z^r)^{\mathrm{T}}(\frac{1}{2}(1-\theta)z^r + \theta d) < r$, there exists a sufficiently small positive number λ^* such that $(p(\lambda^*))^{\mathrm{T}}(\frac{1}{2}(1-\theta)p(\lambda^*) + \theta d) < r$, i.e., $p(\lambda^*) \in K_r$. From (2.1), we have

$$\begin{aligned} 0 \leq (p(\lambda^*) - z^r)^{\mathrm{T}} f(z^r) &= (\lambda^* z + (1-\lambda^*) z^r - z^r)^{\mathrm{T}} f(z^r) \\ &= \lambda^* (z - z^r)^{\mathrm{T}} f(z^r), \end{aligned}$$

which shows that (2.2) holds. So z^r is a solution to CP(1.1).

By utilizing the above lemma, we can obtain the following main result.

Theorem 2.1 If $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous, then there exists either the generalized exceptional family of elements for the function f or a solution to CP(1.1).

Proof Suppose that there does not exist any solution for CP(1.1). For a given vector $d > 0$, we shall prove that there exists the generalized exceptional family of elements for the function f. In fact, it follows from Lemma 2.1 that there exists no solution z^r of $\mathrm{VI}(K_r, f)$ such that $(z^r)^{\mathrm{T}}(\frac{1}{2}(1-\theta)z^r + \theta d) < r$ for each $r > 0$. Since K_r is a bounded set, the solution set of $\mathrm{VI}(K_r, f)$ is always nonempty. Hence, for any $r > 0$, the solution z^r of $\mathrm{VI}(K_r, f)$ must satisfy $(z^r)^{\mathrm{T}}(\frac{1}{2}(1-\theta)z^r + \theta d) = r$. The next objective is to show that such $\{z^r\}$ is the generalized exceptional family of elements for the function f.

Taking into account z^r being a solution of VI(K_r, f), we can get

$$z^r = P_{K_r}(z^r - f(z^r)),$$

i.e., z^r is the unique solution to the following convex programming where Slater's constrained qualification is satisfied.

$$\begin{aligned} &\text{Min} \quad \tfrac{1}{2}\|y - [z^r - f(z^r)]\|^2 \\ &\text{s.t.} \quad\ \ y \geq 0, \ \ y^{\mathrm{T}}(\tfrac{1}{2}(1-\theta)y + \theta d) \leq r, \ \ y \in \mathbb{R}^n. \end{aligned}$$

Therefore the Karush Kuhn Tucker conditions must be satisfied at z^r, i.e., there exists a vector $\lambda^r \in \mathbb{R}^n_+$ and a nonnegative scalar μ^r such that

$$[z^r - (z^r - f(z^r))] - \lambda^r + \mu^r(\theta d + (1-\theta)z^r) = 0, \tag{2.3}$$

$$(\lambda^r)^{\mathrm{T}}(z^r) = 0, \tag{2.4}$$

$$\mu^r((z^r)^{\mathrm{T}}(\tfrac{1}{2}(1-\theta)z^r + \theta d) - r) = 0, \tag{2.5}$$

$$z^r \geq 0, \ (z^r)^{\mathrm{T}}(\tfrac{1}{2}(1-\theta)z^r + \theta d) \leq r. \tag{2.6}$$

Take into account z^r being a solution of VI(K_r, f) and $(z^r)^{\mathrm{T}}(\frac{1}{2}(1-\theta)z^r + \theta d) = r$. It is obvious that (2.5) and (2.6) hold. Thus, from the above conditions, we can conclude the following result

$$f(z^r) = \lambda^r - \mu^r((1-\theta)z^r + \theta d), \ \ (\lambda^r)^{\mathrm{T}}(z^r) = 0,$$

which implies that

$$\begin{aligned} f_i(z^r) &= -\mu^r((1-\theta)z_i^r + \theta d_i), \ \ \text{if} \ \ z_i^r > 0, \\ f_i(z^r) &\geq -\mu^r\theta d_i, \qquad\qquad\qquad \text{if} \ \ z_i^r = 0. \end{aligned}$$

The next object is to show that $\mu^r > 0$. In fact, if $\mu^r = 0$, then the above two relations reduce to

$$f(z^r) \geq 0, \ z^r \geq 0, \ (z^r)^{\mathrm{T}} f(z^r) = 0,$$

which implies that z^r is a solution to CP(1.1). This contradicts the assumption of the proof. Taking into account the fact that $d > 0$ and $\{z^r\} \subset \mathbb{R}^n_+$, we have $(z^r)^{\mathrm{T}}(\frac{1}{2}(1-\theta)z^r + \theta d) = r$ and $\|z^r\| \to +\infty$ as $r \to +\infty$. By Definition 2.3, we get that $\{z^r\}$ is the generalized exceptional family of elements for the function f. The proof is complete.

The following result is a direct consequence of Theorem 2.1.

Corollary 2.1 If there exists no generalized exceptional family of elements for the function f, then CP (1.1) has a solution.

3. Existence Conditions of Solution to CP

In this section, we shall show that Karamardian type condition, Isac and Gowda type condition or p-order generalized coercivity type condition is the sufficient condition for the existence of solutions to CP(1.1), respectively. Finally applying the new concept to the P_*-mapping CP, we also present a new existence result.

Firstly, we shall give an existence theorem related to the Karamardian type condition for CP(1.1).

Theorem 3.1 Let f be a function satisfying the Karamardian type condition, i.e., there exists a compact convex set $D \subset \mathbb{R}^n_+$ such that for every $z \in \mathbb{R}^n_+ \setminus D$, there exists $y \in D$ satisfying $(z-y)^{\mathrm{T}} f(z) \geq 0$. Then there exists no generalized exceptional family of elements for the function f and consequently CP(1.1) has a solution.

Proof Suppose that there exists the generalized exceptional family of elements $\{z^r\}$ for f. We shall prove that f does not satisfy the Karamardian type condition on $\mathbb{R}^n_+$, that is to say, we shall show that for every compact convex set $D \subset \mathbb{R}^n_+$, there exists a positive scalar r such that $z^r \in \mathbb{R}^n_+ \setminus D$ and $(z^r - y)^{\mathrm{T}} f(z^r) < 0$ for each $y \in D$.

If $z_i^r > 0$, from the first condition of Definition 2.3, for a given vector $d > 0$, we can get

$$(z_i^r - y_i) f_i(z^r) = (z_i^r - y_i)(-\mu^r((1-\theta)z_i^r + \theta d)).$$

If $z_i^r = 0$, from the second condition of Definition 2.3, for a given vector $d > 0$, one has

$$(z_i^r - y_i) f_i(z^r) = (0 - y_i) f_i(z^r) \le (0 - y_i)(-\mu^r \theta d).$$

Therefore

$$(z^r - y)^{\mathrm{T}} f(z^r) \le (z^r - y)^{\mathrm{T}}(-\mu^r((1-\theta)z^r + \theta d)) = -\mu^r[\theta (z^r)^{\mathrm{T}} d - \theta y^{\mathrm{T}} d + (1-\theta)\|z^r\|^2 - (1-\theta)(z^r)^{\mathrm{T}} y].$$

Take into account D being compact. There exists some positive scalar c such that $y^{\mathrm{T}} d \le c$ for every $y \in D$. Since $\|z^r\| \to +\infty$, we have

$$(z^r - y)^{\mathrm{T}} f(z^r) \le -\mu^r[\theta (z^r)^{\mathrm{T}} d - \theta y^{\mathrm{T}} d + (1-\theta)\|z^r\|^2 - (1-\theta)(z^r)^{\mathrm{T}} y] < 0, \text{ as } r \to +\infty,$$

which implies that the Karamardian type condition on $\mathbb{R}^n_+$ does not hold. This contradicts the assumption of the theorem.

In the following, we shall provide the Isac and Gowda type sufficient condition for the existence of a solution to CP(1.1).

Definition 3.1[16] We say that $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is monotone decreasing with respect to $\mathbb{R}^n_+$ if there exists a positive scalar t_0 such that for every $z \in \mathbb{R}^n_+$ and every s, t with $s \ge t \ge t_0$, we obtain

$$z^{\mathrm{T}}(\phi(tz) - \phi(sz)) \ge 0.$$

We present the following lemma in order to show the nonexistence of the exceptional families of elements for the monotone decreasing function.

Lemma 3.1[17] ϕ is monotone decreasing with respect to $\mathbb{R}^n_+$ if and only if for every $\alpha \ge 1$ and every $z \in \mathbb{R}^n_+$, we get

$$z^{\mathrm{T}}(\phi(z) - \phi(\alpha z)) \ge 0.$$

Theorem 3.2 Let f be a function satisfying the Isac and Gowda type condition, i.e., there exists a scalar $p \ge 1$ such that $\phi(z) = \|z\|^{p-1} z - f(z)$ is monotone decreasing with respect to $\mathbb{R}^n_+$. Then there does not exist the generalized exceptional family of elements for the function f. Hence CP (1.1) has a solution.

Proof Suppose that there exists the generalized exceptional family of elements $\{z^r\}$ for the function f. Since $\phi(z) = \|z\|^{p-1} z - f(z)$ is monotone decreasing with respect to $\mathbb{R}^n_+$, we can obtain from Lemma 3.1 that for every $\alpha \ge 1$ and every $z \in \mathbb{R}^n_+$,

$$z^{\mathrm{T}}(\phi(z) - \phi(\alpha z)) \ge 0. \tag{3.1}$$

In view of the fact $\|z^r\| \to +\infty$ as $r \to +\infty$, we can conclude that there exists a scalar $r_0 > 0$ such that for all $r \ge r_0$, $\|z^r\| \ge 1$ holds. Letting $\alpha = \|z^r\|$ and $z = \frac{z^r}{\|z^r\|}$ in (3.1), one gets

$$\left(\frac{z^r}{\|z^r\|}\right)^{\mathrm{T}}\left[\phi\left(\frac{z^r}{\|z^r\|}\right) - \phi(z^r)\right] \ge 0,$$

i.e.,

$$(z^r)^{\mathrm{T}}\left[\phi\left(\tfrac{z^r}{\|z^r\|}\right) - \|z^r\|^{p-1} z^r + f(z^r)\right] \ge 0. \tag{3.2}$$

Noting the fact that for a given vector $d > 0$, $\{z^r\}$ is the generalized exceptional family of elements for the function f, we obtain from Definition 2.3

$$(z^r)^{\mathrm{T}} f(z^r) = -\mu^r \sum_{i\in I_+} z_i^r((1-\theta)z_i^r + \theta d_i) = -\mu^r (z^r)^{\mathrm{T}}((1-\theta)z^r + \theta d),$$

where $I_+ = \{i : z_i^r > 0\}$. Combining the above equation and (3.2), one can get

$$(z^r)^{\mathrm{T}}[\phi(\frac{z^r}{\|z^r\|}) - \|z^r\|^{p+1} - \mu^r(z^r)^{\mathrm{T}}((1-\theta)z^r + \theta d)] \geq 0.$$

Thus

$$(z^r)^{\mathrm{T}}[\phi(\frac{z^r}{\|z^r\|})] \geq \|z^r\|^{p+1}.$$

Because f is a continuous function and $\phi(z)$ is bounded on $B(0,1) = \{z : \|z\| \leq 1\}$, there exists some positive scalar M such that

$$M\|z^r\| \geq (z^r)^{\mathrm{T}}[\phi(\frac{z^r}{\|z^r\|})] \geq \|z^r\|^{p+1}.$$

This shows that

$$M \geq \|z^r\|^p \to +\infty \text{ as } r \to +\infty.$$

This is a contradiction.

The sufficient condition related to p-order generalized coercivity is presented as follows for the existence theorem of a solution to CP(1.1).

Theorem 3.3 Let f be a p-order generalized coercive function, i.e., there exists $\widehat{z} \in \mathbb{R}_+^n$ and $p \in (-\infty, 1]$ such that for each sequence $\{z^\alpha\} \subset \mathbb{R}_+^n$ with $\|z^\alpha\| \to +\infty$,

$$\limsup_{z^\alpha \in \mathbb{R}_+^n, \|z^\alpha\| \to +\infty} \frac{f(z^\alpha)^{\mathrm{T}}(z^\alpha - \widehat{z})}{\|z^\alpha\|^p} > 0.$$

Then there does not exist the generalized exceptional family of elements for the function f and consequently CP(1.1) has a solution.

Proof Suppose that $\{z^r\}$ is the generalized exceptional family of elements for the function f. From Definition 2.3, for a given vector $d > 0$, we have that for any $p \in (-\infty, 1]$,

$$\limsup_{\|z^r\| \to +\infty} \frac{f(z^r)^{\mathrm{T}}(z^r - \widehat{z})}{\|z^r\|^p} \leq \limsup_{\|z^r\| \to +\infty} \frac{-\mu^r[\theta(z^r)^{\mathrm{T}} d - \theta \widehat{z}^{\mathrm{T}} d + (1-\theta)\|z^r\|^2 - (1-\theta)(z^r)^{\mathrm{T}}\widehat{z}]}{\|z^r\|^p} \leq 0,$$

which shows that f can not be a p-order generalized coercive function. This is a contradiction.

The above results imply that these conditions provide some new existence results of CP(1.1). However, these are not necessary in general. A natural question is whether there exists a necessary and sufficient condition for the existence result of solution to CP(1.1). The following result will give a positive answer.

Theorem 3.4 Let f be a pseudo-monotone mapping, i.e., for each different point $z, y \in \mathbb{R}^n$,

$$f(z)^{\mathrm{T}}(y - z) \geq 0 \text{ implies } f(y)^{\mathrm{T}}(y - z) \geq 0.$$

Then CP(1.1) has a solution if and only if there exists no generalized exceptional family of elements for the function f.

Proof In view of Corollary 2.1, we only need to prove that if CP(1.1) has a solution, then there exists no generalized exceptional family of elements. Let z^* be a solution for CP(1.1), i.e.,

$$z^* \geq 0, f(z^*) \geq 0, (z^*)^{\mathrm{T}} f(z^*) = 0,$$

which is equivalent to

$$(z-z^*)^{\mathrm{T}} f(z^*) \geq 0, \text{ for all } z \in \mathbb{R}^n_+.$$

Taking into account f being pseudo-monotone, one can get

$$(z-z^*)^{\mathrm{T}} f(z) \geq 0, \text{ for all } z \in \mathbb{R}^n_+. \tag{3.3}$$

Assuming that there exists the generalized exceptional family of elements $\{z^r\}$ for the function f, we have the following inequality from Definition 2.3, for a given vector $d > 0$,

$$(z^r - z^*)^{\mathrm{T}} f(z^r) \leq -\mu^r [\theta (z^r)^{\mathrm{T}} d - \theta (z^*)^{\mathrm{T}} d + (1-\theta)\|z^r\|^2 - (1-\theta)(z^r)^{\mathrm{T}} z^*].$$

Since $\|z^r\| \to +\infty$ as $r \to +\infty$, the above inequality implies that for sufficiently large r,

$$(z^r - z^*)^{\mathrm{T}} f(z^r) < 0,$$

which is in contradiction with (3.3).

The linear P_* mapping was first defined by Kojima et al.[18] Since then, CP(1.1) with P_* mapping has been applied extensively to interior point algorithm. In the following, by using the new concept of generalized exceptional family of elements, we shall prove that CP(1.1) with P_* mapping has a solution under the strictly feasible condition.

Definition 3.2[18] For a mapping $f : \mathbb{R}^n \to \mathbb{R}^n$, f is defined as a P_* mapping if there exists a constant $\gamma \geq 0$ such that for any distinct $z, y \in \mathbb{R}^n$,

$$(1+\gamma) \sum_{j \in I_+(z,y,f)} (z_j - y_j)(f_j(z) - f_j(y)) + \min_{1 \leq j \leq n} (z_j - y_j)(f_j(z) - f_j(y)) \geq 0, \tag{3.4}$$

holds, where $I_+(z,y,f) = \{j : (z_j - y_j)(f_j(z) - f_j(y)) \geq 0\}$.

Remark 3.1 Denote

$$j_0 = \{j | \min_{1 \leq j \leq n} (z_j - y_j)(f_j(z) - f_j(y))\},$$

then (3.4) can be converted into

$$(z-y)^{\mathrm{T}}(f(z)-f(y)) \geq -\gamma \sum_{j \in I_+(z,y,f)} (z_j - y_j)(f_j(z) - f_j(y)) + \sum_{j \in I_-(z,y,f) \setminus \{j_0\}} (z_j - y_j)(f_j(z) - f_j(y)),$$

where $I_-(z,y,f) = \{j : (z_j - y_j)(f_j(z) - f_j(y)) < 0\}$.

It is clear that a monotone mapping is a P_* mapping. Now, we can obtain the following result.

Theorem 3.5 Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a given P_* mapping. If there exists $u \in \mathbb{R}^n_+$ such that $f(u) > 0$, then there does not exist the generalized exceptional family of elements for the function f. Hence the corresponding CP has a solution.

Proof Suppose that $\{z^r\}$ is the generalized exceptional family of elements for a function f. From Definition 2.3, for a given vector $d > 0$, we have that

$$(f_i(z^r) - f_i(u))(z_i^r - u_i) = -(\mu^r(1-\theta) z_i^r + \mu^r \theta d_i + f_i(u))(z_i^r - u_i), \text{ if } z_i^r > 0;$$

$$(f_i(z^r) - f_i(u))(z_i^r - u_i) \leq -(\mu^r(1-\theta) z_i^r + \mu^r \theta d_i + f_i(u))(z_i^r - u_i), \text{ if } z_i^r = 0,$$

i.e., we have

$$(f_i(z^r) - f_i(u))(z_i^r - u_i) \leq -(\mu^r(1-\theta) z_i^r + \mu^r \theta d_i + f_i(u))(z_i^r - u_i), i = 1, 2, \cdots, n. \tag{3.5}$$

In view of the fact that $\|z^r\| \to +\infty$, it is obvious that there exists at least one component index i_0 such that $z_{i_0} \to +\infty$ as $r \to +\infty$. Therefore, we can obtain that

$$\begin{aligned}(f_{i_0}(z^r) - f_{i_0}(u))(z_{i_0}^r - u_{i_0}) &\leq -(\mu^r(1-\theta) z_{i_0}^r + \mu^r \theta d_{i_0} + f_{i_0}(u))(z_{i_0}^r - u_{i_0}) \\ &\leq - f_{i_0}(u)(z_{i_0}^r - u_{i_0}) \to -\infty, \end{aligned} \tag{3.6}$$

which shows that $I_-(z,y,f)$ is not an empty-set. Thus, $I_+(z,y,f)$ is not an empty-set from Definition 3.2. This implies that there exists a sequence $\{z^{r_j}\} \subset \{z^r\}$ such that for some fixed index p, q and all the sequence $\{z^{r_j}\}$

$$(f_p(z^{r_j}) - f_p(u))(z_p^{r_j} - u_p) = \min_{1\le i\le n}(z_i^{r_j} - u_i)(f_i(z^{r_j}) - f_i(u)) \tag{3.7}$$

and

$$(f_q(z^{r_j}) - f_q(u))(z_q^{r_j} - u_q) = \max_{1\le i\le n}(z_i^{r_j} - u_i)(f_i(z^{r_j}) - f_i(u)) > 0 \tag{3.8}$$

hold. In view of the fact that f is a P_* mapping, by (3.7) (3.8) and Definition 3.2, we can get

$$\begin{aligned}(f_{i_0}(z^r) - f_{i_0}(u))(z_{i_0}^r - u_{i_0}) \ge& (f_p(z^{r_j}) - f_p(u))(z_p^{r_j} - u_p)\\ \ge& -(1+\gamma)\sum_{I_+(z,u,f)}(z_i^{r_j} - u_i)(f_i(z^{r_j}) - f_i(u))\\ \ge& -(1+\gamma)(n-1)\max_{1\le i\le n}(z_i^{r_j} - u_i)(f_i(z^{r_j}) - f_i(u))\\ =& -(1+\gamma)(n-1)(f_q(z^{r_j}) - f_q(u))(z_q^{r_j} - u_q).\end{aligned} \tag{3.9}$$

Suppose that $z_i^{r_j} > u_i$, then

$$(z_i^{r_j} - u_i)(f_i(z^{r_j}) - f_i(u)) = -(\mu^{r_j}(1-\theta)z_i^{r_j} + \mu^{r_j}\theta d_i + f_i(u))(z_i^{r_j} - u_i) < 0.$$

This is a contradiction with (3.8). Therefore, we conclude that $0 \le z_q^{r_j} \le u_q$.

If $z_q^{r_j} = 0$, we can get that

$$(f_q(z^{r_j}) - f_q(u))(z_q^{r_j} - u_q) \le (\mu^{r_j}(1-\theta)z_q^{r_j} + \mu^{r_j}\theta d_q + f_q(u))u_q.$$

If $0 < z_q^{r_j} \le u_q$, we have

$$\begin{aligned}(f_q(z^{r_j}) - f_q(u))(z_q^{r_j} - u_q) =& (\mu^{r_j}(1-\theta)z_q^{r_j} + \mu^{r_j}\theta d_q + f_q(u))(u_q - z_q^{r_j})\\ \le& (\mu^{r_j}(1-\theta)z_q^{r_j} + \mu^{r_j}\theta d_q + f_q(u))u_q.\end{aligned}$$

Thus, (3.9) can be rewritten as follows,

$$(f_{i_0}(z^r) - f_{i_0}(u))(z_{i_0}^r - u_{i_0}) \ge -(1+\gamma)(n-1)(\mu^{r_j}(1-\theta)z_q^{r_j} + \mu^{r_j}\theta d_q + f_q(u))u_q,$$

i.e.,

$$-(\mu^{r_j}(1-\theta)z_{i_0}^{r_j} + \mu^{r_j}\theta d_{i_0} + f_{i_0}(u))(z_{i_0}^{r_j} - u_{i_0}) \ge -(1+\gamma)(n-1)(\mu^{r_j}(1-\theta)z_q^{r_j} + \mu^{r_j}\theta d_q + f_q(u))u_q.$$

By $\dfrac{1}{(z_{i_0}^{r_j} - u_{i_0})}$ multiplying both sides, we can get

$$-(\mu^{r_j}[(1-\theta)z_{i_0}^{r_j} + \mu^{r_j}\theta d_{i_0} - \frac{(1+\gamma)(n-1)d_q}{z_{i_0}^{r_j} - u_{i_0}}] \ge f_{i_0}(u) - \frac{(1+\gamma)(n-1)f_q(u))u_q}{z_{i_0}^{r_j} - u_{i_0}}.$$

In view of the fact that $z_{i_0}^{r_j} \to +\infty$, thus, for sufficiently large r_j, the left side of the above formular is negative, which is a contradiction to the fact that the right side of the above inequality tends to $f_{i_0}(u) > 0$.

4. Conclusions

We presented a new concept of exceptional family of elements, which is a generalization for the concept of d-orientation sequence and exceptional family of elements for a continuous function. We have proved that if there exists no generalized exceptional family for a continuous function, then CP(1.1) has a solution. It is also shown that a continuous function does not possess the generalized exceptional family under the Karamardian type condition, the Isac and Gowda type condition or p-order generalized coercive type condition, respectively. By applying the new concept to the P_*-mapping CP, we have obtained a new existence result.

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互补问题的一个新例外族

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摘要: 本文对互补问题提出一个新的广义例外族, 它是例外族和d方向序列概念的推广; 证明如果连续函数不存在广义例外族, 那么该互补问题一定有解; 也分别证明在Karamardian型条件, Isac and Gowda型条件或P-阶广义强制型条件下, 连续函数不存在广义例外族; 应用该广义例外族概念到P_* 映射互补问题, 得到了一个新的存在性定理.

关键词: 互补问题; 例外族; P_*映射