应 用 数 学
MATHEMATICA APPLICATA
2019, 32(4): 879-886

Some Fixed Point Theorems in Complete ν-Generalized Metric Spaces

LIU Xianpeng(刘先鹏), JI Peisheng(纪培胜)
(School of Mathematics and Statistics, Qingdao University, Qingdao 266071, China)

Abstract: The contractive mapping of C-class function in the setting of metric space is generalized in this paper. The iterative method in complete ν-generalized metric space is used to prove the fixed point theorem of the C-class function on (ψ,ϕ)-type contractive mapping. We also prove the fixed point theorems of the generalized F-type contractive mapping and generalized θ-type contractive mapping.

Key words: Fixed point; ν-generalized metric space; (ψ,ϕ)-type contraction
CLC Number: O177.91 **AMS(2000)Subject Classification:** 47H10
Document code: A **Article ID:** 1001-9847(2019)04-0879-08

1. Introduction and Preliminaries

The Banach contraction principle introduced by Banach[1] is one of the most important results in mathematical analysis. It is the most widely applied fixed point result in many branches of mathematics and it was generalized in many different directions[2−4]. One of these generalizations was introduced by Branciari in 2000, where the triangle inequality was replaced by a so-called polygonal inequality. He introduced the concept of ν-generalized metric spaces as follows.

Definition 1.1[5−7] Let X be a nonempty set and $d : X \times X \to [0,\infty)$ be a mapping. Let $\nu \in \mathbb{N}$. Then (X, d) is called a ν-generalized metric space if the following hold:

1) $d(x, y) = 0$ if and only if $x = y$, for all $x, y \in X$;

2) $d(x, y) = d(y, x)$, for every $x, y \in X$;

3) $d(x, y) \le d(x, u_1) + d(u_1, u_2) + \cdots + d(u_\nu, y)$, for each set $\{x, u_1, \ldots, u_\nu, y\}$ of $\nu + 2$ elements of X that are all different.

Obviously, every metric space (X, d) is a 1-generalized metric space. A generalized metric space is also said to be a 2-generalized metric space. In [8], it shown that not every generalized metric space has a compatible topology. Indeed, only 3-generalized metric space has a compatible topology(see [9]).

Definition 1.2[5−7] Let (X, d) be a ν-generalized metric space and $\{x_n\}$ be a sequence in X;

1) The sequence $\{x_n\}$ is said to be a Cauchy sequence if $d(x_m, x_n) \to 0$ as $m, n \to \infty$;

---

* **Received date:** 2018-10-11
Foundation item: Supported by the National Natural Science Foundation of Shandong(ZR2016AM 05)
Corresponding author: JI Peisheng, male, Han, Shandong, professor, major in fixed point theorem.

2) The sequence $\{x_n\}$ is said to converge to $x \in X$ if $d(x_n, x) \to 0$ as $n \to \infty$.

3) The sequence $\{x_n\}$ is said to converge to x in the strong sense if $\{x_n\}$ is a Cauchy sequence and $\{x_n\}$ converges to x.

The space X is said to be complete if every Cauchy sequence in X converges.

A sequence in 2-generalized metric space may converge to more than one point and a convergent sequence may not be a Cauchy sequence(see [10]). A sequence may be convergent, but not in the strong sense.

Proposition 1.1[5] Let $\{x_n\}$ and $\{y_n\}$ be sequences in a ν-generalized metric space (X, d) that converge to x and y in the strong sense respectively. Then

$$\lim_{n\to\infty} d(x_n, y_n) = d(x, y).$$

Branciari[11] also proved a generalization of the Banach contraction principle; however, the proof is not correct. When proving theorems on ν-generalized metric spaces X, we may have to be careful, because X does not necessarily have the compatible topology(see [8-9]).

In 1997, Berinde[12] introduced the concept of comparison function.

Definition 1.3[12] A function $\psi : (0, \infty) \to (0, \infty)$ is called a comparison function if it satisfies the following:

1) ψ is monotone increasing, that is, $t_1 < t_2 \Rightarrow \psi(t_1) \le \psi(t_2)$;

2) $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t > 0$, where ψ^n stands for the nth iterate of ψ.

Clearly, if ψ is a comparison function, then $\psi(t) < t$ for each $t > 0$.

Definition 1.4[13] Let $\phi : (0, \infty) \to (0, \infty)$ be a mapping satisfying the following conditions:

(Φ_1) ϕ is non-decreasing, that is, for all $t, s \in (0, \infty)$, $t < s$, one has $\phi(t) \le \phi(s)$;

(Φ_2) for each sequence $\{t_n\} \subset (0, \infty)$, $\lim_{n\to\infty} \phi(t_n) = 0$ iff $\lim_{n\to\infty} t_n = 0$;

(Φ_3) ϕ is continuous.

We shall denote by $\varPhi$ the set of all functions satisfying the conditions (Φ_1), (Φ_2) and (Φ_3).

Lemma 1.1[13] Let $\phi : (0, \infty) \to (0, \infty)$ be a non-decreasing and continuous function with $\inf_{t\in(0,\infty)} \phi(t) = 0$ and $\{t_k\}_k$ be a sequence in $(0, \infty)$. Then the following conclusion holds.

$$\lim_{k\to\infty} \phi(t_k) = 0 \Leftrightarrow \lim_{k\to\infty} t_k = 0.$$

Lemma 1.2[7] Let (X, d) be a ν-generalized metric space and let $\{x_n\}$ be a sequence in X with distinct elements ($x_n \ne x_m$ for $n \ne m$). Suppose that $d(x_n, x_{n+1})$, $d(x_n, x_{n+2})$, $\dots$, $d(x_n, x_{n+\nu})$ tend to 0 as $n \to \infty$ and that $\{x_n\}$ is not a Cauchy sequence. Then there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $n_k > m_k > k$ and the following four sequences tend to ε as $k \to \infty$:

$$d(x_{m_k}, x_{n_k}), \quad d(x_{m_k}, x_{n_k+1}), \quad d(x_{m_k-1}, x_{n_k}), \quad d(x_{m_k-1}, x_{n_k+1}). \tag{1.1}$$

Theorem 1.1[13] Let (X, d) be a complete metric space and $T : X \to X$ be a (ψ, ϕ)-type contraction, that is, there exist $\phi \in \varPhi$ and a continuous comparison function ψ such that

$$\forall x, y \in X, d(Tx, Ty) > 0 \Rightarrow \phi(d(Tx, Ty)) \le \psi[\phi(M(x, y))],$$

where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}d(x, Ty), d(y, Tx)\}.$$

Then T has a unique fixed point $z \in X$ and the sequence $\{T^n x\}$ converges to z.

In this paper, we first prove the theorem of fixed point for (ψ, ϕ)-type contraction in the setting of complete ν-generalized metric spaces. We next prove the theorems of fixed point for generalized θ-type contraction and generalized F-type contraction.

2. Main Result

Definition 2.1 Let (X, d) be a complete ν-generalized metric space and $T : X \to X$ be a mapping. T is said to be a generalized (ψ, ϕ)-type contraction, if there exists a comparison function ψ and $\phi \in \Phi$ such that

$$\forall x, y \in X, d(Tx, Ty) > 0 \Rightarrow \phi(d(Tx, Ty)) \le \psi[\phi(M(x, y))], \tag{2.1}$$

where

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}, \frac{d(x,Tx)d(y,Ty)}{1+d(Tx,Ty)}, \frac{d(x,Tx)d(x,Ty)}{1+d(x,Ty)+d(y,Tx)}\}. \tag{2.2}$$

Theorem 2.1 Let (X, d) be a complete ν-generalized metric space and $T : X \to X$ be a generalized (ψ, ϕ)-type contraction. Then T has a unique fixed point $z \in X$ and the sequence $\{T^n x\}$ converges to z.

Proof Let x_0 be an arbitrary point in X. We construct a sequence $\{x_n\}$ such that $x_{n+1} = Tx_n = T^{n+1}x_0$. If $T^{p-1}x = T^p x$, then $T^{p-1}x$ will be a fixed point of T. So, without loss of generality, we can assume that $d(T^{n-1}x, T^n x) > 0$. From (2.1) we have

$$\phi(d(x_n, x_{n+1})) = \phi(d(Tx_{n-1}, Tx_n)) \le \psi[\phi(M(x_{n-1}, x_n))], \tag{2.3}$$

where

$$\begin{aligned} M(x_{n-1}, x_n) = & \max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_n)d(x_n, x_{n+1})}{1+d(x_{n-1}, x_n)}, \\ & \frac{d(x_{n-1}, x_n)d(x_n, x_{n+1})}{1+d(x_n, x_{n+1})}, \frac{d(x_{n-1}, x_n)d(x_{n-1}, x_{n+1})}{1+d(x_{n-1}, x_{n+1})+d(x_n, x_n)}\} \\ \le & \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}. \end{aligned} \tag{2.4}$$

Now if $\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_n, x_{n+1})$, then by (2.1) we have

$$\phi(d(x_n, x_{n+1})) \le \psi[\phi(d(x_n, x_{n+1}))].$$

This implies that $\phi(d(x_n, x_{n+1})) < \phi(d(x_n, x_{n+1}))$, this is a contradiction. Thus, from (2.4) we have

$$\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_{n-1}, x_n). \tag{2.5}$$

This together with inequality (2.3) yields that

$$\phi(d(x_n, x_{n+1})) \le \psi[\phi(d(x_{n-1}, x_n))] \le \psi^2[\phi(d(x_{n-2}, x_{n-1}))] \le \cdots \le \psi^n[\phi(d(x_0, Tx_0))]. \tag{2.6}$$

Since $\phi : (0, \infty) \to (0, \infty)$, it follows from (2.6) that

$$0 \le \lim_{n\to\infty} \phi(d(x_n, x_{n+1})) \le \lim_{n\to\infty} \psi^n[\phi(d(x_0, Tx_0))] = 0.$$

This implies that $\lim_{n\to\infty} \phi(d(x_n, x_{n+1})) = 0$. This together with (Φ_2) and Lemma 1.1 gives

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.7}$$

Now, we claim that $\lim_{n\to\infty} d(x_n, x_{n+i}) = 0$, for $i = 2, 3, 4, \cdots, \nu$.

By (2.1), we have

$$\phi(d(x_n, x_{n+i})) = \phi(d(Tx_{n-1}, Tx_{n+i-1})) \le \psi[\phi(M(x_{n-1}, x_{n+i-1}))], \tag{2.8}$$

where

$$\begin{aligned}M(x_{n-1},x_{n+i-1}) =&\max\{d(x_{n-1},x_{n+i-1}),d(x_{n-1},x_n),d(x_{n+i-1},x_{n+i}),\frac{d(x_{n-1},x_n)d(x_{n+i-1},x_{n+i})}{1+d(x_{n-1},x_{n+i-1})},\\&\frac{d(x_{n-1},x_n)d(x_{n+i-1},x_{n+i})}{1+d(x_n,x_{n+i})},\frac{d(x_{n-1},x_n)d(x_{n-1},x_{n+i})}{1+d(x_{n-1},x_{n+i})+d(x_{n+i-1},x_n)}\}\\ \le&\max\{d(x_{n-1},x_{n+i-1}),d(x_{n-1},x_n),d(x_{n+i-1},x_{n+i}),\frac{d(x_{n-1},x_n)d(x_{n+i-1},x_{n+i})}{1+d(x_{n-1},x_{n+i-1})},\\&\frac{d(x_{n-1},x_n)d(x_{n+i-1},x_{n+i})}{1+d(x_n,x_{n+i})},\frac{d(x_{n-1},x_n)d(x_{n-1},x_{n+i})}{1+d(x_{n-1},x_{n+i})}\}\\ \le&\max\{d(x_{n-1},x_{n+i-1}),d(x_{n-1},x_n),d(x_{n+i-1},x_{n+i}),\frac{d(x_{n-1},x_n)d(x_{n+i-1},x_{n+i})}{1+d(x_{n-1},x_{n+i-1})},\\&\frac{d(x_{n-1},x_n)d(x_{n+i-1},x_{n+i})}{1+d(x_n,x_{n+i})},d(x_{n-1},x_n)\}\\ =&\max\{d(x_{n-1},x_{n+i-1}),d(x_{n-1},x_n),d(x_{n+i-1},x_{n+i}),\frac{d(x_{n-1},x_n)d(x_{n+i-1},x_{n+i})}{1+d(x_{n-1},x_{n+i-1})},\\&\frac{d(x_{n-1},x_n)d(x_{n+i-1},x_{n+i})}{1+d(x_n,x_{n+i})}\}.\end{aligned}$$

From (2.4) and (2.5) we have $d(x_n,x_{n+1}) < d(x_{n-1},x_n)$. Define $a_n = d(x_n,x_{n+i})$ and $b_n = d(x_n,x_{n+1})$. Then

$$M(x_{n-1},x_{n+i-1}) \le \max\{a_{n-1},b_{n-1},b_{n+i-1},\frac{b_{n-1}b_{n+i-1}}{1+a_{n-1}},\frac{b_{n-1}b_{n+i-1}}{1+a_n}\}.$$

Since $\lim_{n\to\infty} b_n = 0$, there exists N_0 such that $b_n < 1$ for all $n > N_0$, so we have

$$\frac{b_{n-1}b_{n+i-1}}{1+a_{n-1}} \le b_{n-1}b_{n+i-1} \le b_{n-1}.$$

Similarly, we have

$$\frac{b_{n-1}b_{n+i-1}}{1+a_n} \le b_{n-1}b_{n+i-1} \le b_{n-1}.$$

Then we get $M(x_{n-1},x_{n+i-1}) \le \max\{a_{n-1},b_{n-1}\}$. Since $\psi(t) < t$ for each $t > 0$, we have

$$\phi(d(x_n,x_{n+i})) \le \psi[\phi(M(x_{n-1},x_{n+i-1}))] < \phi(M(x_{n-1},x_{n+i-1})),$$

so we have $a_n \le \max\{a_{n-1},b_{n-1}\}$.

Since $b_n < b_{n-1} \le \max\{a_{n-1},b_{n-1}\}$, we have $\max\{a_n,b_n\} \le \max\{a_{n-1},b_{n-1}\}$ for all $n\in\mathbb{N}$. Thus, the sequence $\{\max\{a_n,b_n\}\}_{n\in\mathbb{N}}$ is monotone non-increasing, so it converges to some l. Assuming that $l > 0$, since $\lim_{n\to\infty} b_n = 0$, we have

$$\limsup_{n\to\infty} a_n = \limsup_{n\to\infty}\max\{a_n,b_n\} = \lim_{n\to\infty}\max\{a_n,b_n\} = l.$$

Taking limit supremum in (2.8) by the property of ψ, we have

$$\begin{aligned}\phi(l) =&\phi(\limsup_{n\to\infty} a_n) = \limsup_{n\to\infty}\phi(a_n) \le \limsup_{n\to\infty}\psi[\phi(\max\{a_n,b_n\})]\\ =&\psi[\phi(\limsup_{n\to\infty}\max\{a_n,b_n\})] = \psi[\phi(l)],\end{aligned}$$

which is a contradiction. So $l = 0$, that is,

$$\lim_{n\to\infty} d(x_n,x_{n+i}) = 0, \quad \text{for} \quad i = 2,3,4,\cdots,\nu. \tag{2.9}$$

We will prove that $x_n \ne x_{n+j}$, for all $n\ge 0, j\ge 1$. Namely, if $x_n = x_{n+j}$ for some $n\ge 0$ and $j\ge 1$, we have $x_{n+1} = Tx_n = Tx_{n+j} = x_{n+j+1}$, which implies that

$$d(x_n,x_{n+1}) = d(x_{n+j},x_{n+j+1}) < d(x_{n+j-1},x_{n+j}) < \cdots < d(x_n,x_{n+1})$$

is a contraction. Thus, we obtain that $x_n \neq x_m$ for all distinct $n, m \in \mathbb{N}$.

In order to prove that $\{x_n\}$ is a Cauchy sequence, suppose that it is not. Then, by Lemma 1.2, using (2.7) and (2.9), we conclude that there exists $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $n_k > m_k > k$ and the sequences (1.1) tend to ε as $k \to \infty$. Using (2.1) with $x = x_{m_k-1}$ and $y = x_{n_k}$, one obtains

$$\phi(d(x_{m_k}, x_{n_k+1})) = \phi(d(Tx_{m_k-1}, Tx_{n_k})) \leq \psi[\phi(M(x_{m_k-1}, x_{n_k}))], \tag{2.10}$$

where

$$\begin{aligned} M(x_{m_k-1}, x_{n_k}) =& \max\{d(x_{m_k-1}, x_{n_k}), d(x_{m_k-1}, x_{m_k}), d(x_{n_k}, x_{n_k+1}), \frac{d(x_{m_k-1}, x_{m_k})d(x_{n_k}, x_{n_k+1})}{1 + d(x_{m_k-1}, x_{n_k})}, \\ & \frac{d(x_{m_k-1}, x_{m_k})d(x_{n_k}, x_{n_k+1})}{1 + d(x_{m_k}, x_{n_k+1})}, \frac{d(x_{m_k-1}, x_{m_k})d(x_{m_k-1}, x_{n_k+1})}{1 + d(x_{m_k-1}, x_{n_k+1}) + d(x_{n_k}, x_{m_k})}\} \\ \leq& \max\{d(x_{m_k-1}, x_{n_k}), d(x_{m_k-1}, x_{m_k}), d(x_{n_k}, x_{n_k+1}), \frac{d(x_{m_k-1}, x_{m_k})d(x_{n_k}, x_{n_k+1})}{1 + d(x_{m_k-1}, x_{n_k})}, \\ & \frac{d(x_{m_k-1}, x_{m_k})d(x_{n_k}, x_{n_k+1})}{1 + d(x_{m_k}, x_{n_k+1})}, \frac{d(x_{m_k-1}, x_{m_k})d(x_{m_k-1}, x_{n_k+1})}{1 + d(x_{m_k-1}, x_{n_k+1})}\} \\ \leq& \max\{d(x_{m_k-1}, x_{n_k}), d(x_{m_k-1}, x_{m_k}), d(x_{n_k}, x_{n_k+1}), \frac{d(x_{m_k-1}, x_{m_k})d(x_{n_k}, x_{n_k+1})}{1 + d(x_{m_k-1}, x_{n_k})}, \\ & \frac{d(x_{m_k-1}, x_{m_k})d(x_{n_k}, x_{n_k+1})}{1 + d(x_{m_k}, x_{n_k+1})}, d(x_{m_k-1}, x_{m_k})\} \\ =& \max\{d(x_{m_k-1}, x_{n_k}), d(x_{m_k-1}, x_{m_k}), d(x_{n_k}, x_{n_k+1}), \frac{d(x_{m_k-1}, x_{m_k})d(x_{n_k}, x_{n_k+1})}{1 + d(x_{m_k-1}, x_{n_k})}, \\ & \frac{d(x_{m_k-1}, x_{m_k})d(x_{n_k}, x_{n_k+1})}{1 + d(x_{m_k}, x_{n_k+1})}\}. \end{aligned}$$

Taking the upper limit as $k \to \infty$ in (2.10), we get $\phi(\varepsilon) \leq \psi[\phi(\varepsilon)]$, which is a contradiction with $\varepsilon > 0$. Thus, $\{x_n\}$ is a Cauchy sequence. By completeness of (X, d), there exists $z \in X$ such that $\lim\limits_{n\to\infty} d(x_n, z) = 0$.

Finally, we show that $Tz = z$. Arguing by contradiction, we assume that $Tz \neq z$. we have

$$\phi(d(x_{n+1}, Tz)) = \phi(d(Tx_n, Tz)) \leq \psi[\phi(M(x_n, z))], \tag{2.11}$$

where

$$\begin{aligned} M(x_n, z) =& \max\{d(x_n, z), d(x_n, x_{n+1}), d(z, Tz), \frac{d(x_n, x_{n+1})d(z, Tz)}{1 + d(x_n, z)}, \\ & \frac{d(x_n, x_{n+1})d(z, Tz)}{1 + d(x_{n+1}, Tz)}, \frac{d(x_n, x_{n+1})d(x_n, Tz)}{1 + d(x_n, Tz) + d(z, Tx_n)}\}. \end{aligned}$$

On taking limit as $n \to \infty$ in above inequation and using (2.11), we obtain

$$\phi(d(z, Tz)) \leq \psi[\phi(d(z, Tz))] < \phi(d(z, Tz)).$$

This is a contradiction. Hence $Tz = z$. This is that z is a fixed point of T.

Now we prove that z is the unique fixed point of T in X. In fact, if $z, u \in X$ are two distinct fixed points of T, that is, $Tz = z \neq u = Tu$, then it follows the assumption that

$$\phi(d(z, u)) = \phi(d(Tz, Tu)) \leq \psi[\phi(M(z, u))], \tag{2.12}$$

where

$$M(z, u) = \max\{d(z, u), d(z, Tz), d(u, Tu), \frac{d(z, Tz)d(u, Tu)}{1 + d(z, u)},$$

$$\frac{d(z,Tz)d(u,Tu)}{1+d(Tz,Tu)}, \frac{d(z,Tz)d(z,Tu)}{1+d(z,Tu)+d(u,Tz)}\}.$$

This together with (2.12) shows that

$$\phi(d(z,u)) = \phi(d(Tz,Tu)) \leq \psi[\phi(d(z,u))] < \phi(d(z,u)),$$

which is a contraction. Hence, we have $z = u$, thus z is the uniqueness fixed point of T.

In 2014, Jleli and Samet[14] introduced the following notion of θ-contraction.

Let $\theta : (0,\infty) \to (1,\infty)$ be a function satisfying the following conditions:

(Θ_1) θ is non-decreasing;

(Θ_2) For each sequence $\{t_n\} \subset (0,\infty)$, $\lim_{n\to\infty} \theta(t_n) = 1 \Leftrightarrow \lim_{n\to\infty} t_n = 0$;

(Θ_3) There exist $r \in (0,1)$ and $l \in (0,\infty]$ such that $\lim_{r\to 0^+} \frac{\theta(t)-1}{t^r} = l$.

In the sequel we denote by $\varTheta$ the set of all functions satisfying the conditions (Θ_1)-(Θ_3).

Definition 2.2[14] A mapping T is said to be a θ-contraction if there exists $\theta \in \varTheta$ and $k \in (0,1)$ such that $\forall x,y \in X, d(Tx.Ty) \neq 0 \Rightarrow \theta(d(Tx.Ty)) \leq [\theta(d(x,y))]^k$.

In this paper, we use the following condition instead of condition (Θ_3).

$(\Theta_3^{'})$ $\varTheta$ is continuous on $(0,\infty)$.

We denote by $\varOmega$ the set of all functions satisfying the conditions (Θ_1), (Θ_2), and $(\Theta_3^{'})$.

Theorem 2.2[13] Let (X,d) be a complete metric space and $T : X \to X$ be a θ-type contraction, that is, there exist $\theta \in \varOmega$ and $k \in (0,1)$ such that

$$\forall x,y \in X, d(Tx,Ty) > 0 \Rightarrow \theta(d(Tx,Ty)) \leq [\theta(M(x,y))]^k,$$

where $M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}d(x,Ty), d(y,Tx)\}$. Then T has a unique fixed point $z \in X$ and the sequence $\{T^n x\}$ converges to z.

In 2012, Wardowski[15] introduced the following notion of F-contraction.

Definition 2.3[15] Let $F : (0,\infty) \to \mathbb{R}$ be a mapping satisfying:

(F_1) F is non-decreasing, that is, for all $t,s \in \mathbb{R}, t < s$, one has $F(t) \leq F(s)$;

(F_2) for each sequence $\{t_n\} \subset (0,\infty)$, $\lim_{n\to\infty} F(t_n) = -\infty$ if and only if $\lim_{n\to\infty} t_n = 0$;

(F_3) there exist $r \in (0,1)$ such that $\lim_{t\to 0^+} t^r F(t) = 0$.

In the sequel we denote by F the set of all functions satisfying the conditions (F_1)-(F_3).

Definition 2.4[15] A mapping T is said to be a F-contraction if there exists $\tau > 0$ such that

$$\forall x,y \in X, d(Tx.Ty) \neq 0 \Rightarrow \tau + F(d(Tx.Ty)) \leq F(d(x,y)).$$

In this paper, we use the following condition instead of condition (F_3).

$(\mathrm{F}_3^{'})$ F is continuous on $(0,\infty)$.

We denote by $\varGamma$ the set of all functions satisfying the conditions $(\mathrm{F}_1), (\mathrm{F}_2)$, and $(\mathrm{F}_3^{'})$.

Theorem 2.3[16] Let (X,d) be a complete metric space and $T : X \to X$ be a F-type contraction, that is, there exist $F \in \varGamma$ and $\tau > 0$ such that

$$\forall x,y \in X, d(Tx,Ty) > 0 \Rightarrow \tau + F(d(Tx,Ty)) \leq F(M(x,y)),$$

where

$$\begin{aligned} M(x,y) = &\max\{d(x,y), d(T^2x,Tx), \frac{d(x,Ty)+d(y,Tx)}{2}, \frac{d(T^2x,x)+d(T^2x,Ty)}{2}, \\ &d(T^2x,y), d(T^2x,Ty)+d(x,Tx), d(Tx,y)+d(y,Ty)\}. \end{aligned}$$

Then T has a unique fixed point $z \in X$ and the sequence $\{T^n x\}$ converges to z.

Theorem 2.4 Let (X,d) be a complete ν-generalized metric space and $T:X\to X$ be a θ-type contraction, that is, there exist $\theta\in\Omega$ and $k\in(0,1)$ such that

$$\forall x,y\in X, d(Tx,Ty)>0\Rightarrow \theta(d(Tx,Ty))\le[\theta(M(x,y))]^k,$$

where $M(x,y)$ is given by (2.2). Then T has a unique fixed point $z\in X$ and the sequence $\{T^nx\}$ converges to z.

Proof Denote by $\psi(t)=kt$ and $\phi:=\ln\theta:(0,\infty)\to(0,\infty)$. It is easy to check that $\phi\in\Phi$. Hence , we have $\ln\theta(d(Tx,Ty))\le k\ln\theta(M(x,y))$.

The conclusion of Theorem 2.4 can be obtained from Theorem 2.1.

Definition 2.5 Let (X,d) be a complete ν-generalized metric space and $T:X\to X$ be a mapping. T is said to be a generalized JS-contraction whenever there are a function $\theta\in\Omega$ and positive real numbers k_1,k_2,k_3,k_4,k_5,k_6 with $0<k_1+k_2+k_3+k_4+k_5+k_6<1$ such that

$$\begin{aligned}\theta(d(Tx,Ty))\le&[\theta(d(x,y))]^{k_1}[\theta(d(x,Tx))]^{k_2}[\theta(d(y,Ty))]^{k_3}[\theta(\frac{d(x,Tx)d(y,Ty)}{1+d(x,y)})]^{k_4}\\&\times[\theta(\frac{d(x,Tx)d(y,Ty)}{1+d(Tx,Ty)})]^{k_5}[\theta(\frac{d(x,Tx)d(x,Ty)}{1+d(x,Ty)+d(y,Tx)})]^{k_6},\end{aligned}$$

for all $x,y\in X$ with $Tx\ne Ty$.

Corollary 2.1 Let (X,d) be a complete ν-generalized metric space and $T:X\to X$ be a generalized JS-contraction. Then T has a unique fixed point.

Proof

$$\begin{aligned}\theta(d(Tx,Ty))\le&[\theta(d(x,y))]^{k_1}[\theta(d(x,Tx))]^{k_2}[\theta(d(y,Ty))]^{k_3}[\theta(\frac{d(x,Tx)d(y,Ty)}{1+d(x,y)})]^{k_4}\\&\times[\theta(\frac{d(x,Tx)d(y,Ty)}{1+d(Tx,Ty)})]^{k_5}[\theta(\frac{d(x,Tx)d(x,Ty)}{1+d(x,Ty)+d(y,Tx)})]^{k_6}\\\le&[\theta(M(x,y))]^{k_1+k_2+k_3+k_4+k_5+k_6},\end{aligned}$$

where $M(x,y)$ is given by (2.2). Supposing $k=k_1+k_2+k_3+k_4+k_5+k_6$, from Theorem 2.4 , we obtain T has a unique fixed point.

Theorem 2.5 Let (X,d) be a complete ν-generalized metric space and $T:X\to X$ be a F-type contraction, that is, there exist $F\in\Gamma$ and $\tau>0$ such that

$$\forall x,y\in X, d(Tx,Ty)>0\Rightarrow \tau+F(d(Tx,Ty))\le F(M(x,y)),$$

where $M(x,y)$ is given by (2.2). Then T has a unique fixed point $z\in X$ and the sequence $\{T^nx\}$ converges to z.

Proof Denote $\psi(t)=\mathrm{e}^{-\tau}t$ and $\phi:=\mathrm{e}^F:(0,\infty)\to(0,\infty)$. It is easy to check that $\phi\in\Phi$. Hence , we have $\mathrm{e}^{F(d(Tx,Ty))}\le\mathrm{e}^{-\tau}\mathrm{e}^{F(M(x,y))}$.

The conclusion of Theorem 2.5 can be obtained from Theorem 2.1.

Example 1 Let $X=\{0,1,2,3,4\}$ and $d:X\times X\to[0,\infty)$ be defined by:

$$d(x,x)=0\ \text{ for }\ x\in X,\qquad d(1,2)=d(2,1)=5.$$

$$d(2,3)=d(1,3)=d(3,1)=d(3,2)=1,\quad\text{and}\quad d(x,y)=|x-y|,\ \text{otherwise}.$$

Note that (X,d) is a complete 3-generalized metric space which is not a generalized metric space since $5=d(1,2)>d(1,3)+d(3,4)+d(4,2)=4$.

Let $T:X\to X$ be defined by

$$T=\begin{cases}2, & x\in\{0,1,2,3\},\\0, & x=4.\end{cases}$$

Define $\psi:(0,\infty)\to(0,\infty)$ by $\psi(t)=\frac{1}{2}t$, and $\phi:(0,\infty)\to(0,\infty)$ by $\phi(t)=\frac{1}{2}t$. Clearly, $\psi\in\Psi$ and $\phi\in\Phi$, T is a generalized (ψ,ϕ)-type contraction mapping. Now if $\{x_n\}$ is a sequence in X such that $x_n\to x$ as $n\to\infty$, for $x\in\{0,1,2,3\}$ and $y=4$, we have

$$\psi(d(Tx,T4))\leq\psi[\phi(M(x,4))].$$

Thus, all conditions of Theorem 2.1 are satisfied. Moreover, $x=2$ is a fixed point of T.

References:

[1] BANACH S. Sur les opérationes dans les ensembles abstraits et leur application aux équations intégrales[J]. Fundam. Math., 1922, 3(1): 51-57.
[2] YIN Jiandong. Fixed point theorems of in cone metric space[J]. Math. Appl., 2010, 23(1): 171-178.
[3] HAN Yan, XU Shaoyuan. Fixed point theorems of ψ-contraction for a family of mappings in cone metric spaces[J]. Math. Appl., 2015.
[4] 麻振华, 蒋立宁, 辛巧玲. 算子值距离空间上的不动点定理[J]. 北京理工大学学报, 2014, 34(10): 1078-1080.
[5] ABTAHI M, KADELBURG Z, RADENOVIĆ S. Fixed points of Ćirić-Matkowski-type contractions in ν-generalized metric spaces[J]. Rev. Real Acad. Cienc. Exac. Fis. Naturales. Serie A. Mat., 2017, 111(1): 57-64.
[6] ALAMRI B, SUZUKI T, KHAN L A. Caristi's fixed point theorem and Subrahmanyam's fixed point theorem in ν-generalized metric spaces[J]. J. Funct. Spaces, 2015.
[7] SUZUKI T, ALAMRI B, KHAN L A. Some notes on fixed points in ν-generalized metric spaces[J]. Bull. Kyushu Inst. Tech. Pure Appl. Math., 2015, 62: 15-23.
[8] SUZUKI T. Generalized metric spaces do not have the compatible topology[J/OL]. Abstract and Applied Analysis, 2014:5[2014-08-04].http://dx.doi.org/10.1155/2014/458099.
[9] SUZUKI T, ALAMRI B, KIKKAWA M. Only 3-generalized metric spaces have a compatible symmetric topology[J/OL]. Open Mathematics, 2015[2015-09-04]. http://doi.org/10.1515/math-2015-0048.
[10] SARMA I R, RAO J M, RAO S S. Contractions over generalized metric spaces[J]. J. Nonlinear Sci. Appl., 2009, 2(3): 180-182.
[11] BRANCIARI A. A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces[J]. Publicationes Mathematicae, 2000, 57(1): 31-37.
[12] BERINDE V. Generalized contractions and applications[J]. Editura Cub Press 22, Baia Mare, 1997.
[13] LIU X D, CHANG S S, XIAO Y. Some fixed point theorems concerning (ψ,ϕ)-type contraction in complete metric spaces[J]. J. Nonlinear Sci. Appl., 2016, 9(6): 4127-4136.
[14] JLELI M, SAMET B. A new generalization of the Banach contraction principle[J]. J. Inequal. Appl., 2014, 2014(1): 38.
[15] WARDOWSKI D. Fixed points of a new type of contractive mappings in complete metric spaces[J]. Fixed Point Theory Appl., 2012, 2012(1): 94.
[16] PIRI H, KUMAM P. Wardowski type fixed point theorems in complete metric spaces[J]. Fixed Point Theory Appl., 2016, 2016(1): 1-12.

完备的ν-广义度量空间上的不动点定理

刘先鹏, 纪培胜
(青岛大学数学与统计学院, 山东 青岛 266071)

摘要: 本文对度量空间中C类函数的压缩映射进行推广. 在完备的ν-广义度量空间上, 利用构造迭代序列的方法, 证明了关于(ψ,ϕ)-类型压缩映射的不动点定理. 并且证明了广义的F类型压缩和广义θ类型压缩映射.

关键词: 不动点; ν-广义度量空间; (ψ,ϕ)-类型压缩