应 用 数 学
MATHEMATICA APPLICATA
2020, 33(1): 12-24

Solvability for Fractional p-Laplacian Differential Equation with Integral Boundary Conditions at Resonance on Infinite Interval

LIU Zongbao(刘宗宝)[1], LIU Wenbin(刘文斌)[2], ZHANG Wei(张伟)[2]
(1.Department of Fundamental Course, Wuxi Institute of Technology, Wuxi 214121, China;
2.School of Mathematics, China University of Mining and Technology, Xuzhou 221116, China)

Abstract: In this paper, we investigate the existence of solutions for a class of fractional integral boundary value problems with p-Laplacian operator at resonance on infinite interval, by using Mawhin's continuation theorem. An example is given to show the application of our main result.

Key words: Fractional boundary value problem; p-Laplacian operator; Resonance; Infinite interval; Mawhin's continuation theorem
CLC Number: O175.8 **AMS(2000)Subject Classification:** 34A08; 34B15
Document code: A **Article ID:** 1001-9847(2020)01-0012-13

1. Introduction

Fractional calculus is a generalization of the classical integer order calculus. In contrast to the integer order calculus, fractional calculus has nonlocal behavior. This characteristic lets fractional differential equation be an excellent tool in describing some complex problems. For example, in the description of memory and hereditary properties of various materials and processes.[1−2] Besides, the fractional calculus and its applications appear frequently in various fields, such as physics, chemistry, biology, control theory, economics, biophysics, signal and image processing, etc.[3−8] The height loss over time of the granular material contained in a silo can be modeled with fractional derivatives as follows:

$$^{C}D^{\alpha}_{T-}D^{\alpha}_{0+}h^{*}(t)+\beta h^{*}(t)=0,\ t\in[0,T],\ \alpha\in(0,1),$$

where D^{α}_{0+} is the Riemann-Liouville fractional derivative and $^{C}D^{\alpha}_{T-}$ is the right-sided Caputo fractional derivative, $h^{*}(t)=h_{\mathrm{bed}}-h(t)$ represents an investigated function, h_{bed} is the initial bed height, $h(t)$ is the height loss of the granular bed due to silo emptying[3].

Differential equations with integral boundary conditions have various applications in applied fields.[9−10] In the past few years, this kind of boundary value problems (short for BVPs) has drawn increasing attention of scholars.[10−17]

* **Received date:** 2018-10-11
Foundation item: Supported by the National Natural Science Foundation of China(11271364)
Biography: LIU Zongbao, male, Han, Jiangsu, associate professor, major in differential equation.

In [11], YANG, MIAO and GE considered the following integral BVPs by using extension of Mawhin's continuation theorem.

$$\begin{cases} (c(t)\phi_p(x'(t)))' + g(t)h(t,x(t),x'(t)) = 0, \quad 0 < t < \infty, \\ x(0) = \int_0^{+\infty} g(s)x(s)\mathrm{d}s, \quad \lim\limits_{t\to+\infty} c(t)\phi_p(x'(t)) = 0, \end{cases}$$

where $h : [0,+\infty) \times \mathbb{R}^2 \to \mathbb{R}$ satisfies g-Carathéodory conditions, ϕ_p is a p-Laplacian operator, defined as $\phi_p(s) = |s|^{p-2}s\ (s \neq 0),\ \phi_p(0) = 0$.

In [12], Cabada and WANG investigated the following fractional differential equations with integral boundary value conditions by using Guo-Krasnoselskii fixed point theorem.

$$\begin{cases} {}^C D^\alpha u(t) + f(t,u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u''(0) = 0,\ u(1) = \lambda\int_0^1 u(s)\mathrm{d}s, \end{cases}$$

where $2 < \alpha < 3$, $0 < \lambda < 2$, ${}^C D^\alpha$ is the Caputo fractional derivative and $f : [0,1]\times[0,+\infty) \to [0,+\infty)$ is a continuous function.

In [13], JIANG studied the following fractional differential equations with integral boundary value conditions on the half line by using Mawhin's continuation theorem.

$$\begin{cases} D^\alpha_{0+}u(t) = f(t,u(t),D^{\alpha-2}_{0+}u(t),D^{\alpha-1}_{0+}u(t)), \quad \text{a.e. } t \in [0,+\infty), \\ u(0) = 0,\ D^{\alpha-2}_{0+}u(0) = 0,\ D^{\alpha-1}_{0+}u(+\infty) = \int_0^{+\infty} h(t)D^{\alpha-1}_{0+}u(t)\mathrm{d}t, \end{cases}$$

where D^α_{0+} is the standard Riemann-Liouville fractional derivative with $2 < \alpha \le 3$ and $f : [0,+\infty) \times \mathbb{R}^3 \to \mathbb{R}$ satisfies Carathéodory conditions.

Recently, the existence of solutions to boundary value problems of fractional differential equations on infinite interval has been extensively studied.[13,16,18−23] However, to the best of our knowledge, the research has proceeded more slowly for fractional boundary value problems on infinite interval with p-Laplacian operator, especially for resonance problem. Thus, motivated by the results mentioned, in this paper, we discuss the following integral boundary value problems by using Mawhin's continuation theorem.

$$\begin{cases} D^\beta_{0+}(w(t)\phi_p(D^\alpha_{0+}x(t))) = f(t,x(t),D^{\alpha-1}_{0+}x(t),D^\alpha_{0+}x(t)), \quad \text{a.e. } t \in [0,+\infty), \\ (w(t)\phi_p(D^\alpha_{0+}x(t)))'|_{t=0} = I^{2-\alpha}_{0+}x(0) = D^\alpha_{0+}x(0) = 0, \\ D^{\alpha-1}_{0+}x(+\infty) = \int_0^{+\infty} g(t)D^{\alpha-1}_{0+}x(t)\mathrm{d}t, \end{cases} \tag{1.1}$$

where D^α_{0+} and D^β_{0+} are the standard Riemann-Liouville fractional derivative with $1 < \alpha,\beta \le 2$, ϕ_p is a p-Laplacian operator, $g(t) > 0$ and $g(t) \in L^1[0,+\infty)$, with $\int_0^{+\infty} g(t)\mathrm{d}t = 1$.

Throughout this paper, we assume that the following conditions hold:

(A_1) $w(t) > 0$ on $[0,+\infty)$ and $\phi_p^{-1}(t^{\beta-1}/w(t)) \in C[0,+\infty) \cap L^1[0,+\infty)$;

(A_2) $f : [0,+\infty)\times\mathbb{R}^3 \to \mathbb{R}$ is an L^1-Carathéodory function, that is, f satisfies Carathéodory conditions, and for each $r > 0$, there exists a nonnegative function $\varphi_r(t) \in L^1[0,+\infty)$ such that $|f(t,x,y,z)| \le \varphi_r(t)$ for all $\dfrac{|x|}{1+t^{\alpha-1}},\ |y|,\ |z| \in [0,r]$, a.e. $t \in [0,+\infty)$.

Remark The condition (A_1) implies that

$$\sup_{t\in[0,+\infty)} \phi_p^{-1}(t^{\beta-1}/w(t)) < +\infty, \quad \lim_{t\to+\infty} \phi_p^{-1}(t^{\beta-1}/w(t)) = 0.$$

The rest of this paper is built up as follows. In Section 2, we recall some definitions and lemmas. In Section 3, based on the Mawhin's continuation theorem, we establish an existence result for the problem (1.1). In Section 4, an example is given to illustrate the usefulness of our main results.

2. Preliminaries

In this section, we present some definitions and lemmas.

Let X and Z be two Banach spaces with the norms $\|\cdot\|_X$ and $\|\cdot\|_Z$, respectively. Let $L: \mathrm{dom}(L) \subset X \to Z$ be a Fredholm operator with index zero, $P: X \to X$, $Q: Z \to Z$ be two projectors such that

$$\mathrm{Im}\, P = \mathrm{Ker}L,\ \mathrm{Im}\, L = \mathrm{Ker}Q,\ X = \mathrm{Ker}L \oplus \mathrm{Ker}P,\ Y = \mathrm{Im}\, L \oplus \mathrm{Im}\, Q,$$

then, $L\,|_{\mathrm{dom}L\cap \mathrm{Ker}P}: \mathrm{dom}L \to \mathrm{Im}\, L$ is invertible. We denote the inverse by K_p. Let Ω be an open bounded subset of X and $\mathrm{dom}L \cap \bar{\Omega} \neq \varnothing$, then the map $N: X \to Y$ is called L-compact on $\bar{\Omega}$, if $QN\left(\bar{\Omega}\right)$ is bounded and $K_{P,Q}N = K_p(I-Q)N : \bar{\Omega} \to X$ is compact (see [24]).

Lemma 2.1[24] Let $L: \mathrm{dom}(L) \subset X \to Y$ be a Fredholm operator of index zero and $N: X \to Y$ is L-compact on $\bar{\Omega}$. Assume that the following conditions are satisfied:

(i) $Lu \neq \lambda Nu$ for any $u \in (\mathrm{dom}L \backslash \mathrm{Ker}L) \cap \partial\Omega,\ \lambda \in (0,1)$;

(ii) $Nu \notin \mathrm{Im}\, L$ for any $u \in \mathrm{Ker}L \cap \partial\Omega$;

(iii) $\deg(QN\,|_{\mathrm{Ker}L}, \Omega \cap \mathrm{Ker}L, 0) \neq 0$.

Then the equation $Lx = Nx$ has at least one solution in $\mathrm{dom}L \cap \bar{\Omega}$.

Proposition 2.1[25] ϕ_p has the following properties:

(i) ϕ_p is continuous, monotonically increasing and invertible. Moreover, $\phi_p^{-1} = \phi_q$ with $q > 1$ satisfying $1/p + 1/q = 1$;

(ii) For $\forall s, t \geq 0,\ \phi_p(s+t) \leq \phi_p(s) + \phi_p(t)$, if $1 < p < 2$ and $\phi_p(s+t) \leq 2^{p-1}(\phi_p(s) + \phi_p(t))$, if $p \geq 2$.

Next, we introduce the definitions of Riemann-Liouville fractional integrals and fractional derivatives on the half-axis and some lemmas, which can be found in [2, 4, 6, 19].

Definition 2.1 The fractional Riemann-Liouville integral of order $\alpha > 0$ for a function $f: (0, +\infty) \to \mathbb{R}$ is defined by

$$(I_{0+}^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\mathrm{d}s,\ t > 0,$$

provided the right-hand side is pointwise defined on $(0, +\infty)$.

Definition 2.2 The Riemann-Liouville fractional derivative of order $\alpha > 0$ for a function $f: (0, +\infty) \to \mathbb{R}$ is defined by

$$(D_{0+}^{\alpha} f)(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n (I_{0+}^{n-\alpha} f)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s)\mathrm{d}s,\ t > 0,$$

provided the right-hand side is pointwise defined on $(0, +\infty)$, where $n = [\alpha] + 1$.

Lemma 2.2 Assume that $f \in L^1[0, +\infty)$, $\gamma > \delta > 0$, then

$$D_{0+}^{\delta} I_{0+}^{\delta} f(t) = f(t),\ D_{0+}^{\delta} I_{0+}^{\gamma} f(t) = I_{0+}^{\gamma-\delta} f(t).$$

Lemma 2.3 Assume that $\alpha > 0,\ \lambda > -1,\ t > 0$, then

$$I_{0+}^{\alpha} t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1+\alpha)} t^{\alpha+\lambda},\ D_{0+}^{\alpha} t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\alpha)} t^{\lambda-\alpha}.$$

Lemma 2.4 $D_{0+}^{\alpha} x(t) = 0$ if and only if

$$x(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

where n is the smallest integer greater than or equal to $\alpha, c_i \in \mathbb{R},\ i = 1, 2, \cdots, n$.

3. Main Result

Take

$$X = \left\{ x : x, D_{0+}^{\alpha} x \in C[0, +\infty), \sup_{t\in[0,+\infty)} \frac{|x(t)|}{1+t^{\alpha-1}} < +\infty, \right.$$
$$\left. \sup_{t\in[0,+\infty)} |D_{0+}^{\alpha-1} x(t)| < +\infty, \sup_{t\in[0,+\infty)} |D_{0+}^{\alpha} x(t)| < +\infty \right\},$$
$$Z = C[0, +\infty) \cap L^1[0, +\infty),$$

and endowed with the norms

$$||x||_X = \max\left\{ \left\| \frac{x}{1+t^{\alpha-1}} \right\|_{\infty},\ ||D_{0+}^{\alpha-1} x||_{\infty},\ ||D_{0+}^{\alpha} x||_{\infty} \right\},\ ||z||_Z = \max\left\{ ||z||_{\infty},\ ||z||_1 \right\},$$

where $||x||_{\infty} = \sup_{t\in[0,+\infty)} |x(t)|,\ ||z||_1 = \int_0^{+\infty} |z(t)|\mathrm{d}t$. Clearly, $(X, \|\cdot\|_X)$ and $(Z, \|\cdot\|_Z)$ are two Banach spaces.

Lemma 3.1 The problem (1.1) is equivalent to the following BVPs:

$$\begin{cases} D_{0+}^{\alpha} x(t) = \phi_q \left(\dfrac{1}{w(t)} I_{0+}^{\beta} f(t, x(t), D_{0+}^{\alpha-1} x(t), D_{0+}^{\alpha} x(t)) \right), \quad \text{a.e. } t \in [0, +\infty), \\ I_{0+}^{2-\alpha} x(0) = 0, \quad D_{0+}^{\alpha-1} x(+\infty) = \displaystyle\int_0^{+\infty} g(t) D_{0+}^{\alpha-1} x(t)\mathrm{d}t. \end{cases} \tag{3.1}$$

Proof By Lemma 2.4, $D_{0+}^{\beta}(w(t)\phi_p(D_{0+}^{\alpha} x(t))) = f(t, x(t), D_{0+}^{\alpha-1} x(t), D_{0+}^{\alpha} x(t))$ has solution

$$w(t)\phi_p(D_{0+}^{\alpha} x(t)) = I_{0+}^{\beta} f(t, x(t), D_{0+}^{\alpha-1} x(t), D_{0+}^{\alpha} x(t)) + c_1 t^{\beta-1} + c_2 t^{\beta-2}.$$

By the boundary conditions $(w(t)\phi_p(D_{0+}^{\alpha} x(t)))'|_{t=0} = D_{0+}^{\alpha} x(0) = 0$ and Lemmas 2.2-2.3, we have $c_1 = c_2 = 0$. Then

$$D_{0+}^{\alpha} x(t) = \phi_q \left(\frac{1}{w(t)} I_{0+}^{\beta} f(t, x(t), D_{0+}^{\alpha-1} x(t), D_{0+}^{\alpha} x(t)) \right).$$

So, (1.1) can be rewritten in the form (3.1) and we can easily verify that (1.1) has a solution $x(t)$ if and only if $x(t)$ is the solution of (3.1).

Define the linear operator $L : \mathrm{dom}L \subset X \to Z$ and nonlinear operator $N : X \to Z$ as follows:

$$Lx(t) = D_{0+}^{\alpha} x(t),\ x(t) \in \mathrm{dom}L,$$
$$Nx(t) = \phi_q \left(\frac{1}{w(t)} I_{0+}^{\beta} f(t, x(t), D_{0+}^{\alpha-1} x(t), D_{0+}^{\alpha} x(t)) \right),\ x \in X,$$

where

$$\mathrm{dom}L = \left\{ x \in X : I_{0+}^{2-\alpha} x(0) = 0,\ \ D_{0+}^{\alpha-1} x(+\infty) = \int_0^{+\infty} g(t) D_{0+}^{\alpha-1} x(t)\mathrm{d}t \right\}.$$

Then the problem (3.1) is equivalent to the operator equation $Lx = Nx,\ x \in \mathrm{dom}L$.

Lemma 3.2 The operator $L : \mathrm{dom}L \subset X \to Z$ satisfies

$$\mathrm{Ker}L = \left\{ x \in \mathrm{dom}L : x(t) = ct^{\alpha-1}, \forall t \in [0, +\infty), c \in \mathbb{R} \right\}, \tag{3.2}$$

$$\mathrm{Im}L = \left\{ z \in Z : \int_0^{+\infty} z(t) \int_0^t g(s)\mathrm{d}s\mathrm{d}t = 0 \right\}. \tag{3.3}$$

Proof By Lemmas 2.3-2.4 and boundary conditions of (3.1), it is easy to get (3.2). Thus, $\dim \mathrm{Ker}L$=1 and $\mathrm{Ker}L$ is linearly homeomorphic to $\mathbb{R}$. If $z \in \mathrm{Im}L$, then there exists a function $x\in\mathrm{dom}L$ such that $z(t)=D_{0+}^{\alpha}x(t)$. Thus, by Lemma 2.4 we have

$$x(t) = ct^{\alpha-1} + \tilde{c}t^{\alpha-2} + I_{0+}^{\alpha}z(t).$$

Considering the boundary condition $I_{0+}^{2-\alpha}x(0)$=0 and Lemma 2.3, we have

$$x(t) = ct^{\alpha-1} + I_{0+}^{\alpha}z(t).$$

Then, by Lemma 2.2 and Lemma 2.3, we get

$$D_{0+}^{\alpha-1}x(t) = c\Gamma(\alpha) + \int_0^t z(s)\mathrm{d}s.$$

It follows from the conditions $D_{0+}^{\alpha-1}x(+\infty)=\int_0^{+\infty} g(t)D_{0+}^{\alpha-1}x(t)\mathrm{d}t$, $\int_0^{+\infty} g(t)\mathrm{d}t=1$ that

$$\begin{aligned} D_{0+}^{\alpha-1}x(+\infty) &= c\Gamma(\alpha) + \int_0^{+\infty} z(s)\mathrm{d}s \\ &= \int_0^{+\infty} g(t)\left[c\Gamma(\alpha) + \int_0^t z(s)\mathrm{d}s\right]\mathrm{d}t, \end{aligned}$$

that is,

$$\int_0^{+\infty} z(t)\int_0^t g(s)\mathrm{d}s\mathrm{d}t = 0. \tag{3.4}$$

On the other hand, suppose $z\in Z$ and satisfies (3.4), take $x(t) = I_{0+}^{\alpha}z(t)$, then $x \in \mathrm{dom}L$ and $Lx(t) = D_{0+}^{\alpha}x(t) = z(t)$, i.e., $z(t) \in \mathrm{Im}L$. Consequently, (3.3) is satisfied.

Lemma 3.3[22] Let X be the space of all bounded continuous vector-valued functions on $[0, +\infty)$ and $S \subset X$. Then S is relatively compact if the following conditions hold:

(a) S is bounded in X;

(b) All functions from S are equicontinuous on any compact subinterval of $[0, +\infty)$;

(c) All functions from S are equiconvergent at infinity.

Let

$$\Delta := \int_0^{+\infty} g(t)\mathrm{e}^{-t}\mathrm{d}t, \quad m = \left\|\phi_q\left(\frac{t^{\beta-1}}{w(t)}\right)\right\|_1.$$

Based on $\int_0^{+\infty} g(t)\mathrm{d}t = 1$, we derive

$$0 < \Delta = \int_0^{+\infty} g(t)\mathrm{e}^{-t}\mathrm{d}t \le \int_0^{+\infty} g(t)\mathrm{d}t=1.$$

Lemma 3.4 Define the linear operators $P : X \to X_1$ and $Q : Z \to Z_1$ by

$$Px(t) = \frac{D_{0+}^{\alpha-1}x(0)}{\Gamma(\alpha)}t^{\alpha-1}, \quad Qz(t) = \frac{\mathrm{e}^{-t}}{\Delta}\int_0^{+\infty} z(t)\int_0^t g(s)\mathrm{d}s\mathrm{d}t, \ \ t \in [0, +\infty),$$

where $X_1 := \mathrm{Ker}L,\ Z_1 := \mathrm{Im}\,Q$. Then L is a Fredholm operator with index zero.

Proof Easily check that P is a continuous projector with

$$\dim Z_1 = \dim \mathrm{Im}Q = \dim \mathrm{Im}P = \dim \mathrm{Ker}L = 1.$$

It follows from $x = (x - Px) + Px$ that $X = \mathrm{Ker}P + \mathrm{Ker}L$. For $x \in \mathrm{Ker}P \cap \mathrm{Ker}L$, that is, $x \in \mathrm{Ker}P$ and $x \in \mathrm{Ker}L$, x can be rewritten as $x(t) = ct^{\alpha-1},\ c \in \mathbb{R}$ and 0= $(Px)(t) = ct^{\alpha-1}$, thus, c=0. So, Ker$P\cap$Ker$L$= $\{0\}$. Therefore, $X=\mathrm{Ker}P\oplus\mathrm{Ker}L$. On the other hand, for any $z\in Z$, we have

$$\begin{aligned}Q^2z(t)=Q[(Qz(t))(t)]&=\frac{\mathrm{e}^{-t}}{\Delta}\int_0^{+\infty}Qz(t)\int_0^t g(s)\mathrm{d}s\mathrm{d}t\\&=Qz(t)\frac{1}{\Delta}\int_0^{+\infty}\mathrm{e}^{-t}\int_0^t g(s)\mathrm{d}s\mathrm{d}t\\&=Qz(t).\end{aligned}$$

Hence, Q is a continuous projector. Obviously, $\mathrm{Im}L=\mathrm{Ker}Q$. Set $z=(z-Qz)+Qz$, then $(z-Qz)\in\mathrm{Ker}Q=\mathrm{Im}L$, $Qz\in\mathrm{Im}Q$. So, $Z=\mathrm{Im}L+\mathrm{Im}Q$. Furthermore, from $\mathrm{Ker}Q=\mathrm{Im}L$ and Q^2z=Qz, we can obtain that $\mathrm{Im}L\cap\mathrm{Im}Q$= $\{0\}$. Thus, $Z=\mathrm{Im}L\oplus\mathrm{Im}Q$. Then, we have $\dim\mathrm{Ker}L=\dim\mathrm{Im}Q=\mathrm{co}\dim\mathrm{Im}L=1$, it follows that L is a Fredholm operator with index zero.

Lemma 3.5 Define the linear operators $K_p:\mathrm{Im}L\to\mathrm{dom}L\cap\mathrm{Ker}P$ by

$$K_pz=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}z(s)\mathrm{d}s,$$

then K_p is the inverse of the operator $L\mid_{\mathrm{dom}L\cap\mathrm{Ker}P}$.

Proof Obviously, $LK_pz=D_{0+}^{\alpha}I_{0+}^{\alpha}z=z$, for $z\in\mathrm{Im}L$. Besides, for $x\in\mathrm{dom}L\cap\mathrm{Ker}P$, we have $D_{0+}^{\alpha-1}x(0)=0$ and $I_{0+}^{2-\alpha}x(0)=0$, by Lemma 2.3, we get

$$K_pLx=x(t)+c_1t^{\alpha-1}+c_2t^{\alpha-1}=x(t).$$

Thus we arrive at the conclusion that K_p= $(L\mid_{\mathrm{dom}L\cap\mathrm{Ker}P})^{-1}$.

Lemma 3.6 Assume that $\Omega\subset X$ be an open bounded set such that $\mathrm{dom}L\cap\bar{\Omega}\neq\varnothing$, then N is L-compact on $\bar{\Omega}$.

Proof Since $\Omega\subset X$ is a bounded set, there exists $r>0$ such that $\bar{\Omega}\subset\{x\in X:||x||_X\le r\}$. By the condition ($\mathrm{A}_2$), there exists $\varphi_r\in L^1[0,+\infty)$ such that $|f(t,x(t),D_{0+}^{\alpha-1}x(t),D_{0+}^{\alpha}x(t)|\le\varphi_r(t)$, a.e. $t\in[0,+\infty)$. Then, for any $x\in\bar{\Omega}$, we have

$$\begin{aligned}|Nx(t)|&=\left|\phi_q\left[\frac{1}{\Gamma(\beta)w(t)}\int_0^t(t-s)^{\beta-1}f(s,x(s),D_{0+}^{\alpha-1}x(s),D_{0+}^{\alpha}x(s)\mathrm{d}s\right]\right|\\&\le\phi_q\left[\frac{t^{\beta-1}}{\Gamma(\beta)w(t)}\int_0^t|f(s,x(s),D_{0+}^{\alpha-1}x(s),D_{0+}^{\alpha}x(s)|\right]\\&\le\phi_q\left(\frac{||\varphi_r||_1t^{\beta-1}}{\Gamma(\beta)w(t)}\right).\end{aligned}$$

So,

$$\begin{aligned}|QNx(t)|&=\frac{\mathrm{e}^{-t}}{\Delta}\left|\int_0^{+\infty}Nx(t)\int_0^t g(s)\mathrm{d}s\mathrm{d}t\right|\le\frac{\mathrm{e}^{-t}}{\Delta}\int_0^{+\infty}|Nx(t)|\int_0^t g(s)\mathrm{d}s\mathrm{d}t\\&\le\frac{\mathrm{e}^{-t}}{\Delta}\int_0^{+\infty}\phi_q\left(\frac{||\varphi_r||_1t^{\beta-1}}{\Gamma(\beta)w(t)}\right)\mathrm{d}t\le\frac{m\phi_q(||\varphi_r||_1)}{\Delta\phi_q(\Gamma(\beta))}\mathrm{e}^{-t}.\end{aligned}$$

Thus,

$$||QNx||_Z\ \le\frac{m\phi_q(||\varphi_r||_1)}{\Delta\phi_q(\Gamma(\beta))}<+\infty.$$

Therefore, $QN(\bar{\Omega})$ is bounded. Now we divided three steps to show that $K_p(I-Q)N:\bar{\Omega}\to X$ is compact. For convenience, we let

$$h(t)=Nx(t)-QNx(t),\ \ l(t,x)=\frac{1}{w(t)}I_{0+}^{\beta}f(t,x(t),D_{0+}^{\alpha-1}x(t),D_{0+}^{\alpha}x(t)),\ x\in\bar{\Omega}.$$

Then, for $x \in \bar{\Omega}$, we have

$$||h||_1 = ||Nx - QNx||_1 \le ||Nx||_1 + ||QNx||_1 \le \frac{\phi_q(||\varphi_r||_1)m(1+\Delta)}{\phi_q(\Gamma(\beta))\Delta} := m', \tag{3.5}$$

and

$$\begin{aligned}|h(t)| &= |Nx(t) - QNx(t)| \le |Nx(t)| + |QNx(t)| \\ &\le \phi_q\left(\frac{t^{\beta-1}}{w(t)}\right)\frac{\phi_q(||\varphi_r||_1)}{\phi_q(\Gamma(\beta))} + \frac{\phi_q(||\varphi_r||_1)m}{\phi_q(\Gamma(\beta))\Delta}\mathrm{e}^{-t}.\end{aligned} \tag{3.6}$$

Step 1 We assert that $K_p(I-Q)N(\bar{\Omega})$ is uniformly bounded. For $x \in \bar{\Omega}$, we have

$$\left|\frac{K_p(I-Q)Nx(t)}{1+t^{\alpha-1}}\right| = \left|\frac{1}{\Gamma(\alpha)}\int_0^t \frac{(t-s)^{\alpha-1}}{1+t^{\alpha-1}}h(s)\mathrm{d}s\right| \le \frac{||h||_1}{\Gamma(\alpha)} \le \frac{m'}{\Gamma(\alpha)},$$

$$\left|D_{0+}^{\alpha-1}K_p(I-Q)Nx(t)\right| = \left|\int_0^t h(s)\mathrm{d}s\right| \le ||h||_1 \le m',$$

and

$$\begin{aligned}\left|D_{0+}^{\alpha}K_p(I-Q)Nx(t)\right| = |h(t)| &\le \phi_q\left(\frac{t^{\beta-1}}{w(t)}\right)\frac{\phi_q(||\varphi_r||_1)}{\phi_q(\Gamma(\beta))} + \frac{\phi_q(||\varphi_r||_1)m}{\phi_q(\Gamma(\beta))\Delta}\mathrm{e}^{-t} \\ &\le \sup_{t\in[0,+\infty)}\phi_q\left(\frac{t^{\beta-1}}{w(t)}\right)\frac{\phi_q(||\varphi_r||_1)}{\phi_q(\Gamma(\beta))} + \sup_{t\in[0,+\infty)}\frac{\phi_q(||\varphi_r||_1)m}{\phi_q(\Gamma(\beta))\Delta}\mathrm{e}^{-t} := m''.\end{aligned}$$

So, $K_p(I-Q)N(\bar{\Omega})$ is uniformly bounded.

Step 2 We prove that $K_p(I-Q)N(\bar{\Omega})$ is equicontinuous on any compact subinterval of $[0,+\infty)$. In fact, for any $t_1, t_2 \in [0,T]$ with T is a positive constant. It follows from the uniform continuity of $\frac{(t-s)^{\alpha-1}}{1+t^{\alpha-1}}$, $\frac{(t-s)^{\beta-1}}{w(t)}$ on $[0,T]\times[0,T]$ and the absolute continuity of integral that

$$\begin{aligned}&\left|\frac{K_p(I-Q)Nx(t_1)}{1+t_1^{\alpha-1}} - \frac{K_p(I-Q)Nx(t_2)}{1+t_2^{\alpha-1}}\right| \\ =&\frac{1}{\Gamma(\alpha)}\left|\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}h(s)\mathrm{d}s - \int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}h(s)\mathrm{d}s\right| \\ \le&\frac{1}{\Gamma(\alpha)}\left[\int_0^{t_1}\left|\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}} - \frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\right||h(s)|\,\mathrm{d}s + \int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}|h(s)|\,\mathrm{d}s\right]\end{aligned}$$

$\to 0$, as $t_1 \to t_2$, and

$$\begin{aligned}&\left|D_{0+}^{\alpha-1}K_p(I-Q)Nx(t_1) - D_{0+}^{\alpha-1}K_p(I-Q)Nx(t_2)\right| \\ =&\left|\int_{t_1}^{t_2}h(s)\mathrm{d}s\right| \le \int_{t_1}^{t_2}|h(s)|\mathrm{d}s \to 0, \text{ as } t_1 \to t_2 .\end{aligned}$$

Since

$$\begin{aligned}|l(t,x)| &= \left|\frac{1}{w(t)}I_{0+}^{\beta}f(t,x(t),D_{0+}^{\alpha-1}x(t),D_{0+}^{\alpha}x(t))\right| \\ &\le \frac{1}{\Gamma(\beta)}\int_0^t\frac{(t-s)^{\beta-1}}{w(t)}|f(t,x(t),D_{0+}^{\alpha-1}x(t),D_{0+}^{\alpha}x(t))| \\ &\le \frac{||\varphi_r||_1}{\Gamma(\beta)}\sup_{t\in[0,+\infty)}\frac{t^{\beta-1}}{w(t)} := \ell,\end{aligned}$$

$$
\begin{aligned}
&|l(t_1,x)-l(t_2,x)|\\
&=\frac{1}{\Gamma(\beta)}\bigg|\int_0^{t_1}\frac{(t_1-s)^{\beta-1}}{w(t_1)}f(t,x(t),D_{0+}^{\alpha-1}x(t),D_{0+}^{\alpha}x(t))\mathrm{d}s\\
&\quad-\int_0^{t_2}\frac{(t_2-s)^{\beta-1}}{w(t_2)}f(t,x(t),D_{0+}^{\alpha-1}x(t),D_{0+}^{\alpha}x(t))\mathrm{d}s\bigg|\\
&\le\frac{1}{\Gamma(\beta)}\bigg[\int_0^{t_1}\left|\frac{(t_1-s)^{\beta-1}}{w(t_1)}-\frac{(t_2-s)^{\beta-1}}{w(t_2)}\right|\left|f(t,x(t),D_{0+}^{\alpha-1}x(t),D_{0+}^{\alpha}x(t))\right|\mathrm{d}s\\
&\quad+\int_{t_1}^{t_2}\frac{(t_2-s)^{\beta-1}}{w(t_2)}\left|f(t,x(t),D_{0+}^{\alpha-1}x(t),D_{0+}^{\alpha}x(t))\right|\mathrm{d}s\bigg]\to 0,\quad \text{as } t_1\to t_2.
\end{aligned}
$$

Noting that $\phi_q(\cdot)$ is uniform continuous on $[-\ell,\ell]$, we derive

$$|Nx(t_1)-Nx(t_2)|=|\phi_q(l(t_1,x))-\phi_q(l(t_2,x))|\to 0,\quad \text{as } t_1\to t_2.$$

In addition,

$$
\begin{aligned}
|QNx(t_1)-QNx(t_2)|&\le\frac{|\mathrm{e}^{-t_1}-\mathrm{e}^{-t_2}|}{\Delta}\left|\int_0^{+\infty}Nx(t)\int_0^t g(s)\mathrm{d}s\mathrm{d}t\right|\\
&\le\frac{|\mathrm{e}^{-t_1}-\mathrm{e}^{-t_2}|}{\Delta}\int_0^{+\infty}|Nx(t)|\int_0^t g(s)\mathrm{d}s\mathrm{d}t\\
&\le\frac{m\phi_q(||\varphi_r||_1)}{\Delta\phi_q(\Gamma(\beta))}\left|\mathrm{e}^{-t_1}-\mathrm{e}^{-t_2}\right|\to 0,\quad \text{as } t_1\to t_2.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&|D_{0+}^{\alpha}K_p(I-Q)Nx(t_1)-D_{0+}^{\alpha}K_p(I-Q)Nx(t_2)|\\
&=|h(t_1)-h(t_2)|\le|Nx(t_1)-Nx(t_2)|+|QNx(t_1)-QNx(t_2)|\\
&\to 0,\quad \text{as } t_1\to t_2.
\end{aligned}
$$

From above, we get $K_p(I-Q)N(\bar{\Omega})$ is equicontinuous on $[0,T]$.

Step 3 We show a fact that $K_p(I-Q)N(\bar{\Omega})$ is equiconvergent at infinity. In fact, for any $\varepsilon>0$, from (3.5) there exists $L_1>0$ such that

$$\int_{L_1}^{+\infty}|h(s)|\,\mathrm{d}s<\varepsilon.$$

From $\lim\limits_{t\to+\infty}\frac{(t-L_1)^{\alpha-1}}{1+t^{\alpha-1}}=1$, $\lim\limits_{t\to+\infty}\phi_q(t^{\beta-1}/w(t))=0$ and (3.6) that there exists a constant $L>L_1$ such that, for any $t>L$, we have

$$1-\frac{(t-L_1)^{\alpha-1}}{1+t^{\alpha-1}}<\frac{\varepsilon}{m'},\quad |h(t)|<\varepsilon.$$

Thus, for any $t_2>t_1\ge L$, we get

$$
\begin{aligned}
&\left|\frac{K_p(I-Q)Nx(t_1)}{1+t_1^{\alpha-1}}-\frac{K_p(I-Q)Nx(t_2)}{1+t_2^{\alpha-1}}\right|\\
&=\frac{1}{\Gamma(\alpha)}\left|\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}h(s)\mathrm{d}s-\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}h(s)\mathrm{d}s\right|\\
&\le\frac{1}{\Gamma(\alpha)}\bigg[\int_0^{L_1}\left|\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}-\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\right||h(s)|\,\mathrm{d}s+\int_{L_1}^{t_1}\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}|h(s)|\,\mathrm{d}s\\
&\quad+\int_{L_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}|h(s)|\,\mathrm{d}s\bigg]
\end{aligned}
$$

$$\leq\frac{1}{\Gamma(\alpha)}\left[\left(1-\frac{(t_1-L_1)^{\alpha-1}}{1+t_1^{\alpha-1}}\right)+\left(1-\frac{(t_2-L_1)^{\alpha-1}}{1+t_2^{\alpha-1}}\right)\right]\int_0^{+\infty}|h(s)|\,\mathrm{d}s+\frac{2}{\Gamma(\alpha)}\int_{L_1}^{+\infty}|h(s)|\,\mathrm{d}s$$
$$<\frac{4\varepsilon}{\Gamma(\alpha)},$$
$$\left|D_{0+}^{\alpha-1}K_p(I-Q)Nx(t_1)-D_{0+}^{\alpha-1}K_p(I-Q)Nx(t_2)\right|=\left|\int_{t_1}^{t_2}h(s)ds\right|\leq\int_{L_1}^{+\infty}|h(s)|ds<\varepsilon,$$
$$\left|D_{0+}^{\alpha}K_p(I-Q)Nx(t_1)-D_{0+}^{\alpha}K_p(I-Q)Nx(t_2)\right|=|h(t_1)-h(t_2)|\leq|h(t_1)|+|h(t_2)|<2\varepsilon.$$

By Lemma 3.3, we obtain that $K_p(I-Q)N(\bar{\Omega})$ is compact.

In order to obtain our main results, we suppose that the following conditions are satisfied:

(A_3) There exists a constant $A>0$ such that if $\left|D_{0+}^{\alpha-1}x(t)\right|>A$, $x\in\mathrm{dom}L\backslash\mathrm{Ker}L$, $t\in[0,+\infty)$, then

$$\int_0^{+\infty}\phi_q\left(\frac{1}{w(t)\Gamma(\beta)}\int_0^t(t-\tau)^{\beta-1}f(\tau,x(\tau),D_{0+}^{\alpha-1}x(\tau),D_{0+}^{\alpha}x(\tau))\mathrm{d}\tau\right)\int_0^t g(s)\mathrm{d}s\mathrm{d}t\neq 0.$$

(A_4) There exist nonnegative functions $a(t)(1+t^{\alpha-1})^{p-1}$, $b(t)$, $c(t)$, $d(t)\in L^1[0,+\infty)$ with $(\Gamma(\beta)-m_1)>c\ell$, if $1<p<2$ and $(\Gamma(\beta)-2^{p-1}m_1)>c\ell$, if $p\geq 2$ such that for all $t\in[0,+\infty)$, $(x,y,z)\in\mathbb{R}^3$,

$$|f(t,x,y,z)|\leq a(t)|x|^{p-1}+\ b(t)|y|^{p-1}+c(t)|z|^{p-1}+d(t),$$

where

$$m_1=\phi_p(m)\left[a\phi_p(\tfrac{1}{\Gamma(\alpha)})+b\right],\ a=||a(t)\phi_p(1+t^{\alpha-1})||_1,\ b=||b(t)||_1,\ c=||c(t)||_1,\ d=||d(t)||_1.$$

(A_5) For $ct^{\alpha-1}\in\mathrm{Ker}L$, there exists a constant $G>0$ such that either

$$c\int_0^{+\infty}\phi_q\left(\frac{1}{w(t)\Gamma(\beta)}\int_0^t(t-\tau)^{\beta-1}f(\tau,c\tau^{\alpha-1},c\Gamma(\alpha),0)\mathrm{d}\tau\right)\int_0^t g(s)\mathrm{d}s\mathrm{d}t>0 \qquad (3.7)$$

or

$$c\int_0^{+\infty}\phi_q\left(\frac{1}{w(t)\Gamma(\beta)}\int_0^t(t-\tau)^{\beta-1}f(\tau,c\tau^{\alpha-1},c\Gamma(\alpha),0)\mathrm{d}\tau\right)\int_0^t g(s)\mathrm{d}s\mathrm{d}t<0 \qquad (3.8)$$

for all $c\in\mathbb{R}$ with $|c|>G$.

Lemma 3.7 Set $\Omega_1=\{x\in\mathrm{dom}L\backslash\mathrm{Ker}L: Lx=\lambda Nx,\ \lambda\in(0,1)\}$. Suppose that ($A_1$)-($A_4$) hold. Then, Ω_1 is bounded.

Proof By Lemma 2.4, for $x\in\Omega_1$, $D_{0+}^{\alpha-1}x\in C[0,+\infty)$, we have

$$x(t)=I_{0+}^{\alpha-1}D_{0+}^{\alpha-1}x(t)+c_0t^{\alpha-2}.$$

Since $I_{0+}^{2-\alpha}x(0)=0$, we derive $c_0=0$. So,

$$\frac{|x(t)|}{1+t^{\alpha-1}}=\left|\frac{1}{\Gamma(\alpha-1)}\int_0^t\frac{(t-s)^{\alpha-2}}{1+t^{\alpha-1}}D_{0+}^{\alpha-1}x(s)\mathrm{d}s\right|\leq\frac{||D_{0+}^{\alpha-1}x||_\infty}{\Gamma(\alpha)},$$

that is,

$$\left\|\frac{x}{1+t^{\alpha-1}}\right\|_\infty\leq\frac{||D_{0+}^{\alpha-1}x||_\infty}{\Gamma(\alpha)}. \qquad (3.9)$$

Also, since $Nx\in\mathrm{Im}L=\mathrm{Ker}Q$ for $x\in\Omega_1$, then $QNx=0$. It follows from (A_3) that there exists $t_0\in[0,+\infty)$ such that $|D_{0+}^{\alpha-1}x(t_0)|\leq A$. Thus,

$$|D_{0+}^{\alpha-1}x(t)|=\left|D_{0+}^{\alpha-1}x(t_0)+\int_{t_0}^t D_{0+}^{\alpha}x(s)\mathrm{d}s\right|\leq A+||D_{0+}^{\alpha}x||_1.$$

Therefore, we can obtain that

$$||D_{0+}^{\alpha-1}x||_\infty\leq A+||D_{0+}^{\alpha}x||_1. \qquad (3.10)$$

Then, from (A_4), for $x\in \Omega_1, t\in[0,+\infty)$, we have

$$\begin{aligned}\left|\phi_p(D^\alpha_{0+}x(t))\right| &= \left|\frac{1}{w(t)\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\lambda f(s,x(s),D^{\alpha-1}_{0+}x(s),D^\alpha_{0+}x(s))\mathrm{d}s\right|\\
&\le \frac{t^{\beta-1}}{w(t)\Gamma(\beta)}\int_0^t |f(s,x(s),D^{\alpha-1}_{0+}x(s),D^\alpha_{0+}x(s))|\mathrm{d}s\\
&\le \frac{t^{\beta-1}}{w(t)\Gamma(\beta)}\left(a\left\|\frac{x}{1+t^{\alpha-1}}\right\|_\infty^{p-1}+b\left\|D^{\alpha-1}_{0+}x\right\|_\infty^{p-1}+c\left\|D^\alpha_{0+}x\right\|_\infty^{p-1}+d\right)\\
&\le \frac{\ell}{\Gamma(\beta)}\left(a\left\|\frac{x}{1+t^{\alpha-1}}\right\|_\infty^{p-1}+b\left\|D^{\alpha-1}_{0+}x\right\|_\infty^{p-1}+c\left\|D^\alpha_{0+}x\right\|_\infty^{p-1}+d\right)\\
&\le \frac{\ell}{\Gamma(\beta)}\left(a\phi_p(\tfrac{1}{\Gamma(\alpha)})\left\|D^{\alpha-1}_{0+}x\right\|_\infty^{p-1}+b\left\|D^{\alpha-1}_{0+}x\right\|_\infty^{p-1}+c\left\|D^\alpha_{0+}x\right\|_\infty^{p-1}+d\right)\\
&\le \frac{\ell}{\Gamma(\beta)}\left[a\phi_p(\tfrac{1}{\Gamma(\alpha)})(A+||D^\alpha_{0+}x||_1)^{p-1}+b(A+||D^\alpha_{0+}x||_1)^{p-1}+c\left\|D^\alpha_{0+}x\right\|_\infty^{p-1}+d\right]\\
&\le \frac{\ell}{\Gamma(\beta)}\left[(a\phi_p(\tfrac{1}{\Gamma(\alpha)})+b)(A+||D^\alpha_{0+}x||_1)^{p-1}+c\left\|D^\alpha_{0+}x\right\|_\infty^{p-1}+d\right].\end{aligned}$$

In the case $1<p<2$, by Proposition 2.1, one gets

$$\left\|\phi_p(D^\alpha_{0+}x)\right\|_\infty \le \frac{\ell}{\Gamma(\beta)}\left[(a\phi_p(\tfrac{1}{\Gamma(\alpha)})+b)(A^{p-1}+||D^\alpha_{0+}x||_1^{p-1})+c\left\|D^\alpha_{0+}x\right\|_\infty^{p-1}+d\right]. \tag{3.11}$$

In the case $p\ge 2$, by Proposition 2.1, one has

$$\left\|\phi_p(D^\alpha_{0+}x)\right\|_\infty \le \frac{\ell}{\Gamma(\beta)}\left[(a\phi_p(\tfrac{1}{\Gamma(\alpha)})+b)2^{p-1}(A^{p-1}+||D^\alpha_{0+}x||_1^{p-1})+c\left\|D^\alpha_{0+}x\right\|_\infty^{p-1}+d\right]. \tag{3.12}$$

It has been known that

$$\begin{aligned}\left\|D^\alpha_{0+}x\right\|_1^{p-1} &= \phi_p\left\{\int_0^{+\infty}\left|\phi_q\left[\frac{1}{w(t)\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\lambda f(s,x(s),D^{\alpha-1}_{0+}x(s),D^\alpha_{0+}x(s))\mathrm{d}s\right]\right|\mathrm{d}t\right\}\\
&\le \frac{\phi_p(m)}{\Gamma(\beta)}\left[(a\phi_p(\tfrac{1}{\Gamma(\alpha)})+b)(A+||D^\alpha_{0+}x||_1)^{p-1}+c||D^\alpha_{0+}x||_\infty^{p-1}+d\right].\end{aligned}$$

If $1<p<2$, by Proposition 2.1, then

$$\left\|D^\alpha_{0+}x\right\|_1^{p-1} \le \frac{\phi_p(m)}{\Gamma(\beta)}\left[(a\phi_p(\tfrac{1}{\Gamma(\alpha)})+b)(A^{p-1}+||D^\alpha_{0+}x||_1^{p-1})+c||D^\alpha_{0+}x||_\infty^{p-1}+d\right],$$

that is,

$$\left\|D^\alpha_{0+}x\right\|_1^{p-1} \le \frac{m_1A^{p-1}}{\Gamma(\beta)-m_1}+\frac{c\phi_p(m)}{\Gamma(\beta)-m_1}||D^\alpha_{0+}x||_\infty^{p-1}+\frac{d\phi_p(m)}{\Gamma(\beta)-m_1}. \tag{3.13}$$

If $p\ge 2$, by Proposition 2.1, then

$$\left\|D^\alpha_{0+}x\right\|_1^{p-1} \le \frac{\phi_p(m)}{\Gamma(\beta)}\left[(a\phi_p(\tfrac{1}{\Gamma(\alpha)})+b)2^{p-1}(A^{p-1}+||D^\alpha_{0+}x||_1^{p-1})+c||D^\alpha_{0+}x||_\infty^{p-1}+d\right],$$

that is,

$$\left\|D^\alpha_{0+}x\right\|_1^{p-1} \le \frac{m_1(2A)^{p-1}}{\Gamma(\beta)-2^{p-1}m_1}+\frac{c\phi_p(m)}{\Gamma(\beta)-2^{p-1}m_1}||D^\alpha_{0+}x||_\infty^{p-1}+\frac{d\phi_p(m)}{\Gamma(\beta)-2^{p-1}m_1}. \tag{3.14}$$

Then, if $1<p<2$, by (3.11), (3.13), we derive

$$\left\|\phi_p(D^\alpha_{0+}x)\right\|_\infty \le \frac{\ell[(a\phi_p(\frac{1}{\Gamma(\alpha)})+b)\Gamma(\beta)A^{p-1}+d\Gamma(\beta)]}{\Gamma(\beta)(\Gamma(\beta)-m_1)-\Gamma(\beta)c\ell} := m_2,$$

$$\left\|D^\alpha_{0+}x\right\|_1^{p-1} \le \frac{m_1A^{p-1}}{\Gamma(\beta)-m_1}+\frac{c\phi_p(m)}{\Gamma(\beta)-m_1}m_2+\frac{d\phi_p(m)}{\Gamma(\beta)-m_1} := m_3,$$

that is,
$$\left\|D_{0+}^{\alpha}x\right\|_{\infty} \le \phi_q(m_2),\ \left\|D_{0+}^{\alpha}x\right\|_{1} \le \phi_q(m_3). \tag{3.15}$$
In the same way, if $p \ge 2$, by (3.12), (3.14), we derive
$$\left\|\phi_p(D_{0+}^{\alpha}x)\right\|_{\infty} \le \frac{\ell\left[2^{p-1}(a\phi_p(\frac{1}{\Gamma(\alpha)})+b)\Gamma(\beta)A^{p-1}+d\Gamma(\beta)\right]}{\Gamma(\beta)(\Gamma(\beta)-2^{p-1}m_1-c\ell)} := m_4,$$
$$\left\|D_{0+}^{\alpha}x\right\|_{1}^{p-1} \le \frac{m_1(2A)^{p-1}}{\Gamma(\beta)-2^{p-1}m_1}+\frac{c\phi_p(m)}{\Gamma(\beta)-2^{p-1}m_1}m_4+\frac{d\phi_p(m)}{\Gamma(\beta)-2^{p-1}m_1} := m_5,$$
that is,
$$\left\|D_{0+}^{\alpha}x\right\|_{\infty} \le \phi_q(m_4),\ \left\|D_{0+}^{\alpha}x\right\|_{1} \le \phi_q(m_5). \tag{3.16}$$
By (3.9), (3.10), (3.15), (3.16), we can get Ω_1 is bounded.

Lemma 3.8 Set $\Omega_2 = \{x \in \mathrm{Ker}L : QNx = 0\}$. Suppose that ($A_1$), ($A_2$) and ($A_5$) hold, then Ω_2 is bounded.

Proof For $x \in \Omega_2$, we have $x = ct^{\alpha-1}, c \in \mathbb{R}$, and $QNx = 0$, that is,
$$\int_0^{+\infty}\phi_q\left(\frac{1}{w(t)\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}f(\tau, c\tau^{\alpha-1}, c\Gamma(\alpha), 0)\mathrm{d}\tau)\right)\int_0^t g(s)\mathrm{d}s\mathrm{d}t = 0$$
By (A_5),we get $|c| \le G$. That is, Ω_2 is bounded.

Lemma 3.9 Set $\Omega_3 = \{x \in \mathrm{Ker}L : \lambda x + (1-\lambda)\vartheta JQNx = 0, \lambda \in [0,1]\}$. where $J : \mathrm{Im}Q \to \mathrm{Ker}L$ is a homeomorphism given by
$$J(c\mathrm{e}^{-t}) = ct^{\alpha-1}.$$
Suppose (A_1), (A_2) and (A_5) hold, then Ω_3 is bounded, where $\vartheta = 1$ if (3.7) holds and $\vartheta = -1$ if (3.8) holds.

Proof For $ct^{\alpha-1} \in \Omega_3$, without loss of generality, we suppose that (3.8) holds, then there exists $\lambda \in [0,1]$ such that
$$\lambda ct^{\alpha-1} = (1-\lambda)JQN(ct^{\alpha-1}),$$
that is,
$$\lambda c=\frac{(1-\lambda)}{\Delta}\int_0^{+\infty}\phi_q\left(\frac{1}{w(t)\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}f(\tau, c\tau^{\alpha-1}, c\Gamma(\alpha), 0)\mathrm{d}\tau\right)\int_0^t g(s)\mathrm{d}s\mathrm{d}t,$$
If λ=0, by (A_5), we obtain $|c|\le G$. If λ=1, then c=0. Moreover, for $\lambda\in(0,1)$, if $|c|>G$, we have
$$0\le\lambda c^2=\frac{(1-\lambda)}{\Delta}c\int_0^{+\infty}\phi_q\left(\frac{1}{w(t)\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}f(\tau, c\tau^{\alpha-1}, c\Gamma(\alpha), 0)\mathrm{d}\tau\right)\int_0^t g(s)\mathrm{d}s\mathrm{d}t<0.$$
It is a contradiction. So, Ω_3 is bounded.

Theorem 3.1 Suppose (A_1)-(A_5) hold. Then the problem (3.1) has at least one solution in dom$L$.

Proof Let $\Omega\supset \bigcup_{i=1}^{3} \bar{\Omega}_i$ be a bounded and open set. From Lemma 3.6, we get N is L-compact on $\bar{\Omega}$. By Lemmas 3.7 and 3.8, we obtain (i) and (ii) of Lemma 2.1 hold. So, we only need to show (iii) holds. Take
$$H(x,\lambda) = \lambda x + (1-\lambda)\vartheta JQNx.$$
According to Lemma 3.9, we derive $H(x,\lambda) \neq 0$ for $x\in\mathrm{Ker}L\cap\partial\Omega$. It follows from the homotopy of degree that
$$\begin{aligned}\deg\{JQN|_{\bar{\Omega}\cap\mathrm{Ker}L}, \Omega\cap\mathrm{Ker}L, 0\} &= \deg\{H(\cdot\,,0), \Omega\cap\mathrm{Ker}L, 0\}\\ &= \deg\{H(\cdot\,,1), \Omega\cap\mathrm{Ker}L, 0\}\\ &= \deg\{I, \Omega\cap\mathrm{Ker}L, 0\} \neq 0.\end{aligned}$$

By Lemma 2.1, we can get that operator function $Lx = Nx$ has at least one solution in $\mathrm{dom}L \cap \bar{\Omega}$, which equivalent to the problem (1.1) has at least one solution in X.

4. Example

Example 4.1 Consider the boundary value problems

$$\begin{cases} D_{0+}^{3/2}(w(t)\phi_{3/2}(D_{0+}^{3/2}x(t))) = f(t, x(t), D_{0+}^{\alpha-1}x(t), D_{0+}^{\alpha}x(t)), \quad \text{a.e. } t \in [0, +\infty), \\ D_{0+}^{1/2}(w(t)\phi_{3/2}(D_{0+}^{3/2}x(t)))\Big|_{t=0} = I_{0+}^{1/2}x(0) = D_{0+}^{3/2}x(0) = 0, \\ D_{0+}^{1/2}x(+\infty) = \displaystyle\int_0^{+\infty} \mathrm{e}^{-t} D_{0+}^{1/2}x(t)\mathrm{d}t. \end{cases} \tag{4.1}$$

Choose $\alpha = \beta = 3/2,\ p = 3/2,\ w(t) = \mathrm{e}^t(1+t)^{1/2},\ g(t) = \mathrm{e}^{-t},\ f(t, x(t), D_{0+}^{\alpha-1}x(t), D_{0+}^{\alpha}x(t)) = a(t)\sin\sqrt{|x(t)|} + b(t)\phi_{3/2}(D_{0+}^{1/2}x(t)) + c(t)\sin\sqrt{|D_{0+}^{3/2}x(t)|} + d(t)$, where $a(t) = \frac{\sqrt{\Gamma(3/2)}}{8\mathrm{e}^t\sqrt{1+t^{1/2}}},\ b(t) = \frac{1}{8\mathrm{e}^t},\ c(t) = \frac{1}{4\mathrm{e}^t},\ d(t) = \frac{1}{8\mathrm{e}^t}$. Clearly, $|f(t, x(t), D_{0+}^{\alpha-1}x(t), D_{0+}^{\alpha}x(t))| \le a(t)\sqrt{|x(t)|} + b(t)\sqrt{|D_{0+}^{1/2}x(t)|} + c(t)\sqrt{|D_{0+}^{3/2}x(t)|} + d(t)$. By simple calculation, we can get

$$a = \frac{\sqrt{\Gamma(3/2)}}{8},\ b = \frac{1}{8},\ c = \frac{1}{4},\ d = \frac{1}{8},\ m \le \frac{1}{2},\ \ell \le 1,$$

$$m_1 = \phi_p(m)[a\phi_p(\tfrac{1}{\Gamma(\alpha)}) + b] \le \frac{\sqrt{2}}{8},\ \Gamma(\beta) - m_1 \ge \Gamma(3/2) - \frac{\sqrt{2}}{8} > \frac{1}{4} \ge c\ell.$$

Take A=G=16, then we can easily check that (A_1)-(A_4) and (3.7) hold. By Theorem 3.1, the problem (4.1) has at least one solution.

References:

[1] MONGIOVI M S, ZINGALES M. A non-local model of thermal energy transport: The fractional temperature equation[J]. Int. J. Heat Mass Transfer, 2013, 67: 593-601.

[2] PAOLA M D, ZINGALES M. The multiscale stochastic model of fractional hereditary materials (FHM)[J]. Procedia IUTAM, 2013, 6: 50-59.

[3] LESZCZYNSKI J S, BLASZCZYK T. Modeling the transition between stable and unstable operation while emptying a silo[J]. Granular Matter, 2011, 13: 429-438.

[4] KILBAS A A, SRIVASTAVA H M, TRUJILLO J J. Theory and Applications of Fractional Differential Equations[M]. Amsterdam: Elsevier, 2006.

[5] MAGIN R L. Fractional calculus models of complex dynamics in biological tissues[J]. Comput. Math. Appl., 2010, 59: 1586-1593.

[6] MILLER K S, ROSS B. An Introduction to the Fractional Calculus and Fractional Differential Equations[M]. New York: Wiley, 1993.

[7] PODLUBNY I. Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications[M]. San Diego: Academic Press, 1999.

[8] SABATIER J, AGRAWAL O P, MACHADO J A T. Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering[M]. Dordrecht: Springer, 2007.

[9] AHMAD B, ALSAEDI A, ALGHAMDI B S. Analytic approximation of solutions of the forced Duffing equation with integral boundary conditions[J]. Nonlinear Analysis: Real World Appl., 2008, 9: 1727-1740.

[10] ZHANG X, FENG M, GE W. Existence result of second-order differential equations with integral boundary conditions at resonance[J]. J. Math. Anal. Appl., 2009, 353: 311-319.

[11] YANG A, MIAO C, GE W. Solvability for second-order nonlocal boundary value problems with a p-Laplacian at resonance on a half-line[J]. Electr. J. Qual. Theo., 2009, 19: 1-15.

[12] CABADA A, WANG G. Positive solutions of nonlinear fractional differential equations with integral boundary value conditions[J]. J. Math. Anal. Appl., 2012, 389: 403-411.
[13] JIANG W. Solvability for fractional differential equations at resonance on the half line[J]. Appl. Math. Comput., 2014, 247: 90-99.
[14] JANKOWSKI T. Positive solutions for fourth-order differential equations with deviating arguments and integral boundary conditions[J]. Nonlinear Anal., Theory Methods Appl., 2010, 73: 1289-1299.
[15] VONG S W. Positive solutions of singular fractional differential equations with integral boundary conditions[J]. Math. Comput. Model., 2013, 57, 1053-1059.
[16] WANG G. Explicit iteration and unbounded solutions for fractional integral boundary value problem on an infinite interval[J]. Appl. Math. Lett., 2015, 47: 1-7.
[17] ZHANG X. Existence and iteration of positive solutions for high-order fractional differential equations with integral conditions on a half-line[J]. J. Appl. Math. Comput., 2014, 45: 137-150.
[18] BOUCENNA A, MOUSSAOUI T, O'REGAN D. Existence of solutions for a boundary value problem of fractional order on the half-line via monotone theory[J]. Int. J. Differ. Equ. Appl., 2016, 15: 55-68.
[19] LI B, SUN S, HAN Z. Successively iterative method for a class of high-order fractional differential equations with multi-point boundary value conditions on half-line[J]. Bound. Value Probl., 2016, 5: 1-16.
[20] LIU Y, AHMAD B, AGARWAL R P. Existence of solutions for a coupled system of nonlinear fractional differential equations with fractional boundary conditions on the half-line[J]. Adv. Difference Equ., 2013, 46: 1-19.
[21] SHEN C, ZHOU H, YANG L. On the existence of solution to a boundary value problem of fractional differential equation on the infinite interval[J]. Bound. Value Probl., 2015, 241: 1-13.
[22] THIRAMANUS P, NTOUYAS S K, TARIBOON J. Existence of solutions for Riemann-Liouville fractional differential equations with nonlocal Erdélyi-Kober integral boundary conditions on the half-line[J]. Bound. Value Probl., 2015, 196: 1-15.
[23] ZHANG W, LIU W, CHEN T. Solvability for a fractional p-Laplacian multipoint boundary value problem at resonance on infinite interval[J]. Adv. Difference Equ., 2016, 183: 1-14.
[24] MAWHIN J. Topological Degree and Boundary Value Problems for Nonlinear Differential Equations in Topological Methods for Ordinary Differential Equations[M]. Lecture Notes in Math., Berlin: Springer, 1993.
[25] 葛渭高. 非线性常微分方程边值问题[M]. 北京: 科学出版社, 2007.

无穷区间上分数阶带 p-Laplacian算子微分方程积分共振边值问题解的存在性

刘宗宝[1], 刘文斌[2], 张伟[2]
(1. 无锡职业技术学院基础部, 江苏 无锡 214121; 2. 中国矿业大学数学学院, 江苏 徐州 221116)

摘要: 利用Mawhin连续性定理, 讨论一类分数阶p-Laplacian微分方程积分共振边值问题在无穷区间上解的存在性, 并举例说明主要结果.

关键词: 分数阶边值问题; p-Laplacian算子; 共振; 无穷区间; Mawhin连续性定理