Long-Time Dynamics of Solutions for a Class of Coupling Beam Equations with Nonlinear Boundary Conditions

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Abstract: In this paper, we study long-time dynamics of solutions for a class of coupling beam equations with strong dampings under nonlinear boundary conditions. Firstly, we prove the existence and uniqueness of global solutions by some inequalities and prior estimates methods. Secondly, by an absorbing set and asymptotic compactness of the related solution semigroup, we prove the existence of a global attractor.

Key words: Coupling beam equation; Nonlinear boundary condition; Global attractor

1. Introduction

This problem is derived from the equation

$$u_{tt} + u_{xxxx} - \left( \alpha + \beta \int_0^L |u_x(s,t)|^2 \, ds \right) u_{xx} = 0,$$

which was proposed by Woinowsky-Krieger\cite{1} as a model for vibrating beams with hinged ends. WANG et al.\cite{2} proved the strong global attractor for the Kirchhoff beam equations

$$u_{tt} + \alpha u_{xxxx} + \gamma u_{xxxxx} - \left( \beta + M(\int_0^L (u_x)^2 \, dx) + N(\int_0^L u_x u_{xx} \, dx) \right) u_{xx} + g(u) + f(u_t) = h(x),$$

subjected to the boundary conditions $u = u_{xx}$ on $\partial \Omega \times \mathbb{R}^+$. Chueshov\cite{3} studied well-posedness and long time dynamical of a class of quasilinear wave equation with a strong damping. The attractor on extensible beams with null boundary conditions were studied by some authors\cite{4-8}. In the following, we make some comments about previous works for the long-time dynamics of the beam equation with nonlinear boundary conditions. MA\cite{4} studied the existence and decay rates for the solution of the Kirchhoff-type beam equation with non-linear boundary conditions and proved long-time behavior of a model of extensible beam

$$u_{tt} + u_{xxxx} - M(||u_x||^2_2) u_{xx} = h(x)$$

with the absence of the structural damping and the rotational inertia, subjected to the nonlinear boundary conditions

$$\begin{align*}
&u(0, t) = u_x(0, t) = u_{xx}(L, t) = 0, \\
&u_{xxx}(L, t) - M(||u_x||^2_2) u_x(L, t) = f(u(L, t)) + g(u_t(L, t))
\end{align*}$$

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in [9] and [10] respectively. WANG et al.[11] considered a kind of more general nonlinear Kirchhoff-type beam equation

\[ u_{tt} - u_{xxxx} + u_{xxtt} - \sigma \left( \int_0^L (u_x)^2 \, dx \right) u_{xx} - \phi \left( \int_0^L (u_x)^2 \, dx \right) u_{xxt} = q(x), \]

subjected to the nonlinear boundary conditions

\[
\begin{align*}
    u(0, t) &= u_x(L, t) = u_{xx}(0, t) = 0, \\
    u_{xxxx}(L, t) &= f(u(L, t)) + g(u_t(L, t)).
\end{align*}
\]

In addition, we mentioned some results on solution of coupling equations. Choo and Chung[12] considered the solution of the nonplanar oscillatory beam equations

\[
\begin{align*}
    u_{tt} + \alpha_1 u_{xxxx} + \delta u_t - \psi(u_x, v_x, u_{xx}) &= p(x, t), \\
    v_{tt} + \alpha_2 v_{xxxx} + \delta v_t - \psi(u_x, v_x, v_{xx}) &= q(x, t)
\end{align*}
\]

under planar external force, where \(\psi(\omega_1, \omega_2, \omega_3) = \{\beta + \gamma (||\omega_1||^2 + ||\omega_2||^2)\} \omega_3\).

We also mentioned some results on global attractor of coupling beam equations. Giorgi et al.[13] considered a class of thermoelastic coupled beam equations and proved the existence of weak solution and global attractor. However, the long time behavior of the coupling beam equations with nonlinear boundary conditions was paid little attention.

In this paper, we will study the global attractor of coupling beam equations

\[
\begin{align*}
    u_{tt} + u_{xxxx} - u_{xxtt} - (||u_x||^2 + ||v_x||^2) u_{xx} - N(\|u_x\|^2)u_{xxt} &= p(x), \\
    v_{tt} + v_{xxxx} - v_{xxtt} - (||u_x||^2 + ||v_x||^2) v_{xx} - N(\|v_x\|^2)v_{xxt} &= q(x)
\end{align*}
\]

(1.1)

with strong dampings, subjected to the nonlinear boundary conditions

\[
\begin{align*}
    u(0, t) &= u_x(L, t) = u_{xx}(0, t) = 0, \\
    u_{xxxx}(L, t) &= f_1(u(L, t)) + g_1(u_t(L, t)) \\
    v(0, t) &= v_x(L, t) = v_{xx}(0, t) = 0, \\
    v_{xxxx}(L, t) &= f_2(v(L, t)) + g_2(v_t(L, t))
\end{align*}
\]

(1.2)

and the initial conditions

\[
\begin{align*}
    u(x, 0) &= u^0(x), \quad u_t(x, 0) = u^1(x), \quad x \in \Omega, \\
    v(x, 0) &= v^0(x), \quad v_t(x, 0) = v^1(x), \quad x \in \Omega.
\end{align*}
\]

(1.3)

Here \(\Omega = (0, L)\) is a bounded domain of \(\mathbb{R}\); \(N(\cdot)\) is continuous nonnegative nonlinear real function. \(-u_{xxtt}\) and \(-v_{xxtt}\) represent the rotational inertias; \(N(||u_x||^2)u_{xxt}\) and \(N(||v_x||^2)v_{xxt}\) express the strong dampings; \(p(x)\) and \(q(x)\) are external force terms.

The paper is organized as follows. In Section 2, we introduce some function assumptions and Sobolev spaces. In Section 3, we prove the existence of global weak and strong solution of coupling beam equations with strong dampings. In Section 4, we discuss the existence of a global attractor of the system (1.1)-(1.3).

2. Some Spaces and Functions Assumptions

Our analysis is based on the following Sobolev spaces. Let

\[
\begin{align*}
    U &= \{u \in H^1(0, L) : u(0) = 0\}, \\
    V &= \{u \in H^2(0, L) : u(0) = u_x(L) = 0\}, \\
    W &= \{u \in H^4(0, L) \cap V : u_{xx}(0) = 0\}.
\end{align*}
\]
Motivated by the boundary conditions (1.2) we assume, for regular solutions, that data $(u^0, u^1, v^0, v^1)$ satisfies the compatibility conditions

$$u_{xx}^0(L) = f_1(u^0(L)) + g_1(u^1(L)), \quad v_{xx}^0(L) = f_2(v^0(L)) + g_2(v^1(L)).$$  \hfill (2.1)

For regular solutions we consider the phase space $H_1 = \{(u^0, u^1, v^0, v^1) \in W \times W \times W \times W; \text{satisfies conditions (1.2)}\}$. For weak solutions we consider the phase space $H_0 = V \times U \times V \times U$, which guarantees that for regular data, the compatibility conditions (1.2) hold. In $H_0$ we adopt the norm defined by

$$|| (u(t), v_z(t), v(t), v_i(t)) ||_{H_0}^2 = ||u_{xx}||^2 + ||u_{xt}||^2 + ||v_{xx}||^2 + ||v_{xt}||^2.$$  \hfill (2.2)

**Assumption 1** We assume that $f_i, g_i, i = 1, 2 : \mathbb{R} \to \mathbb{R}$ are of class $C^1(\mathbb{R})$, satisfy $f_i(0) = g_i(0) = 0, i = 1, 2$, and there exist constants $k_i, p_i, m_i > 0, L_0, L_i > 0, p, r \geq 0, \forall u, v \in R$

$$-L_0 \leq \hat{f}_i(u) \leq \frac{1}{2} f_i(u)u + L_i, \quad i = 1, 2,$$

$$|f_i(u) - f_i(v)| \leq k_i (1 + |u|^p + |v|^p) |u - v|, \quad i = 1, 2,$$

$$(g_i(u) - g_i(v))(u - v) \geq p_i |u - v|^2, \quad i = 1, 2,$$

$$|g_i(u) - g_i(v)| \leq m_i (1 + |u|^r + |v|^r) |u - v|, \quad i = 1, 2,$$

where $\hat{f}_i = \int_0^t f_i(s)ds \geq 0, \quad i = 1, 2,$

**Assumption 2** The function $N(\cdot) \in C^1(\mathbb{R})$ is nondecreasing and satisfies

$$N(s) \geq \alpha + \beta s^\gamma \quad (\alpha, \beta > 0, \gamma \geq 1), \quad \forall s \in \mathbb{R}^+, \quad N(0) = 0.$$  \hfill (2.6)

**Assumption 3** $p(x), q(x) \in L^2(\Omega)$.

3. The Existence of Global Solution

**Theorem 3.1** Assume Assumptions 1-3 hold, for any initial data $(u^0, u^1, v^0, v^1) \in H_1$, there exists a unique regular solution $(u(t), v(t))$ of the problem (1.1)-(1.3) such that

$$u, v \in L^\infty_{\text{loc}}(\mathbb{R}^+, W) \cap C^0([0, \infty); V) \cap C^1([0, \infty); U),$$  \hfill (3.1)

$$||u_i||^2 + ||v_i||^2 + ||u_{xt}||^2 + ||v_{xt}||^2 + ||u_{xx}||^2 + ||v_{xx}||^2 + \frac{1}{2} (||u_{xx}||^2 + ||v_{xx}||^2)^2 \leq M_1,$$  \hfill (3.2)

where $M_1 > 0$ only depends on the initial data, $p$ and $q$.

**Proof** Let us consider the variational problem associated with (1.1)-(1.3): find $u(t), v(t) \in W, \forall \omega, \tilde{\omega} \in \mathcal{V}$ such that

$$\begin{cases}
\int_0^L u_{tt} \omega dx + \int_0^L u_{xt} \omega_x dx + \int_0^L u_{xx} \omega_{xx} dx + (||u_x||^2 + ||v_x||^2) \int_0^L u_x \omega_x dx \\
+ N(||u_x||^2) \int_0^L u_x \omega_x dx \to f_1(u(L, t)) \omega(L, t) + g_1(u(L, t)) \omega(L, t) - f_0 p(x) \omega dx,
\end{cases}$$

$$\\begin{cases}
\int_0^L v_{tt} \tilde{\omega} dx + \int_0^L v_{xt} \tilde{\omega}_x dx + \int_0^L v_{xx} \tilde{\omega}_{xx} dx + (||u_x||^2 + ||v_x||^2) \int_0^L v_x \tilde{\omega}_x dx \\
+ N(||u_x||^2) \int_0^L v_x \tilde{\omega}_x dx \to f_2(v(L, t)) \tilde{\omega}(L, t) + g_2(v(L, t)) \tilde{\omega}(L, t) - f_0 q(x) \tilde{\omega} dx.
\end{cases}$$  \hfill (3.3)

Estimate 1 In the first approximate equation and second approximate equation of (3.3), respectively putting $u = u^m(t)$ and $\tilde{\omega} = v^m(t)$, using the Schwarz inequality, then integrating from 0 to $t < t_m$, we get that

$$||u^m||^2 + ||v^m||^2 + ||u_{xt}^m||^2 + ||v_{xt}^m||^2 + ||u_{xx}^m||^2 + ||v_{xx}^m||^2 + \frac{1}{2} (||u_{xx}^m||^2 + ||v_{xx}^m||^2)^2$$

$$+ 2 \int_0^t N(||u_x^m||^2) ||u_x^m||^2 dx + 2 \int_0^t N(||v_x^m||^2) ||v_x^m||^2 dx + 2 \int_0^t N(||u_x^m||^2) ||u_x^m||^2 dx + 2 \int_0^t N(||v_x^m||^2) ||v_x^m||^2 dx + 2 \hat{f}_1(u^m(L, t))$$
Estimate 3, we can find a constant using the mean value theorem and the Young inequalities combined with Estimate 1 and \( z \) of problem (1.1)-(1.3) which depends continuously on initial data with respect to the norms.

M compatibility conditions (2.1) and the value inequality, we see that there exists \( \omega \) such that
\[
M \leq 2\hat{f}_1(u''(L,0)) + 2\hat{f}_2(v''(L,0)).
\]

Taking into account the assumption (3.4), and \( N(\cdot), \hat{f}_1, \hat{f}_2 \), we see that there exists \( M_1 > 0 \) such that \( \forall t \in [0,T], \forall m \in \mathbb{N} \),
\[
||u''_t||^2 + ||v''_t||^2 + ||u''_{xx}||^2 + ||v''_{xx}||^2 + ||u''_{xxt}||^2 + \frac{1}{2}(||u''_t||^2 + ||v''_t||^2) \leq M_1.
\]

Estimate 2 In the first approximate equation and second approximate equation of (3.3), respectively putting \( \omega = u''_m(0), t = 0 \) and \( \tilde{\omega} = v''_m(0), t = 0 \), using the Schwarz inequality, compatibility conditions (2.1) and the value inequality, we see that there exists \( M_2 > 0 \) such that \( \forall t \in [0,T], \forall m \in \mathbb{N} \),
\[
||u''_m(0)||^2 + ||v''_m(0)||^2 + ||u''_{xxt}(0)||^2 + ||v''_{xxt}(0)||^2 \leq M_2.
\]

Estimate 3 Let us fix \( t, \xi > 0 \) such that \( \xi < T - t \). Respectively taking the difference of the first approximate equation and the second approximate equation of (3.3) with \( t = t + \xi \) and \( t = t \) and putting \( \omega = u''(t + \xi) - u''(t) \) and \( \tilde{\omega} = v''(t + \xi) - v''(t) \), we can find a constant \( M_3 > 0 \), depending only on \( T \), such that \( \forall t \in [0,T], \forall m \in \mathbb{N} \),
\[
||u''_t||^2 + ||v''_t||^2 + ||u''_{xx}||^2 + ||v''_{xx}||^2 + ||u''_{xxt}||^2 + ||v''_{xxt}||^2 \leq M_3.
\]

Uniqueness Let \((u,v),(\tilde{u},\tilde{v})\) be two solutions of (1.1)-(1.3) with the same initial data. Then writing \( z = u - \tilde{u}, \tilde{z} = v - \tilde{v} \), taking the difference (3.3) with \( u = v = v \) and \( u = \tilde{u}, v = \tilde{v} \), respectively replacing \( \omega, \tilde{\omega} \) by \( z, \tilde{z} \) and then making a computation of addition, using the mean value theorem and the Young inequalities combined with Estimate 1 and Estimate 3, we can find a constant \( C > 0, \forall t \in (0,T) \),
\[
\frac{d}{dt}(||z_t||^2 + ||\tilde{z}_t||^2 + ||z_{xxt}||^2 + ||\tilde{z}_{xxt}||^2 + ||z_{xx}||^2 + ||\tilde{z}_{xx}||^2) 
\leq C(||z_t||^2 + ||\tilde{z}_t||^2 + ||z_{xxt}||^2 + ||\tilde{z}_{xxt}||^2 + ||z_{xx}||^2 + ||\tilde{z}_{xx}||^2).
\]

From Gronwall’s Lemma, we see that \( u = \tilde{u}, v = \tilde{v} \).

Since \( u_{xx}, v_{xx}, u_{xxt}, v_{xxt} \in L^2(0,\infty;L^2(\Omega)) \), we get \( u, v \in C^0([0,\infty);V) \). Similarly, \( u, v \in C^1([0,\infty);U) \). Consider \( u_n, v_n \) satisfying (1.1)-(1.3) where \( |u_{n1} - u_1|_{H^1(\Omega)} \rightarrow 0, |v_n - v_0|_{H^2(\Omega)} \rightarrow 0, |u_{n1} - u_1|_{H^1(\Omega)} \rightarrow 0, |v_n - v_1|_{H^1(\Omega)} \rightarrow 0, u_{mn}, v_n \in H^4(\Omega), u_1, v_1 \in H^4(\Omega) \) and satisfy the appropriate compatibility conditions on the boundary. By the standard linear semigroup method, it can be shown that the problem (1.1)-(1.3) admits a solution \((u_n, v_n)\) in \( W \times W \). The proof of Theorem 3.1 is completed.

**Theorem 3.2** If the initial data \((v^0, u^1, v^0, v^1) \in H_0 \), there exists a unique weak solution of problem (1.1)-(1.3) which depends continuously on initial data with respect to the norms of \( H_0 \).

**Proof** Let us consider \((u^0, u^1, v^0, v^1) \in H_0 \). Since \( H_1 \) is dense in \( H_0 \), then exists \((u^0_\mu, u^1_\mu, v^0_\mu, v^1_\mu) \in H_1 \), such that
\[
u^0_\mu \rightarrow u^0 \text{ in } V; \ u^1_\mu \rightarrow u^1 \text{ in } U; \\
v^0_\mu \rightarrow v^0 \text{ in } V; \ v^1_\mu \rightarrow v^1 \text{ in } U.
\]
For each \( \mu \in \mathbb{N} \), there exists \((u_\mu, v_\mu)\), smooth solution of the initial boundary value problem (1.1)-(1.3), which satisfies
\[
\begin{aligned}
&u_{t\mu} + v_{xx\mu} - u_{xxt\mu} - (||u_{x\mu}||^2 ||v_{x\mu}||_2 + ||v_{x\mu}||^2)u_{xx\mu} - N(||u_{x\mu}||^2)u_{xxt\mu} = p(x), \\
v_{t\mu} + v_{xxx\mu} - v_{xxt\mu} - (||u_{x\mu}||^2 ||v_{x\mu}||_2 + ||v_{x\mu}||^2)v_{xx\mu} - N(||v_{x\mu}||^2)v_{xxt\mu} = q(x), \\
u_\mu(x,0) = u_\mu^0(x), \quad u_\mu(x,0) = u_\mu^1(x), \\
u_\mu(0,t) = u_{xx\mu}(L,t) = u_{xx\mu}(0,t) = 0, \\
v_\mu(x,0) = v_\mu^0(x), \quad v_\mu(x,0) = v_\mu^1(x), \\
v_\mu(0,t) = v_{xx\mu}(L,t) = v_{xx\mu}(0,t) = 0, \\
u_{xx\mu}(L,t) = f_1(u_\mu(L,t)) + g_1(u_\mu(L,t)), \\
u_{xx\mu}(L,t) = f_2(v_\mu(L,t)) + g_2(v_\mu(L,t)).
\end{aligned}
\tag{3.5}
\]

Respectively multiplying the first equation by \( u_{\mu} \) and the second equation by \( v_{\mu} \) in (3.5), integrating over \( \Omega \), taking the sum, considering the argument used in the estimate of the existence of solution, we can find a constant \( C_0 \) independent of \( \mu \in \mathbb{N} \) such that
\[
||u_{t\mu}||^2 + ||u_{xxt\mu}||^2 + ||v_{xx\mu}||^2 + ||v_{xxt\mu}||^2 + ||v_{xx\mu}||^2 \leq C_0.
\]

Defining \( Z_{\mu,\sigma} = u_\mu - u_\sigma, \tilde{Z}_{\mu,\sigma} = v_\mu - v_\sigma, \mu, \sigma \in \mathbb{N} \), there exists \((u, v)\) such that
\[
u_\mu \rightarrow u \text{ strongly in } C([0,T); V), \quad u_{t\mu} \rightarrow u_t \text{ strongly in } C([0,T); U), \\
v_\mu \rightarrow v \text{ strongly in } C([0,T); V), \quad v_{t\mu} \rightarrow v_t \text{ strongly in } C([0,T); U).
\]

We can pass to the limit using standard arguments in order to obtain
\[
\begin{aligned}
&u_{tt} + u_{xxx} - u_{xxt} - (||u_x||^2 ||v_x||_2)u_{xx} - N(||u_x||^2)u_{xxt} = p(x), \\
v_{tt} + v_{xxx} - v_{xxt} - (||u_x||^2 ||v_x||_2)v_{xx} - N(||v_x||^2)v_{xxt} = q(x).
\end{aligned}
\]

Theorem 3.2 is proved.

**Remark 3.1** Let us set \( S(t)(u^0, u^1, v^0, v^1) \rightarrow (u, u_t, v, v_t), \forall t \geq 0. \) The operator \( S(t) \) defined in \( H_0 \) maps \( H_0 \) into itself, we see that \( S(t) \) is a nonlinear \( C_0 \)-semigroup on \( H_0 \).

### 4. The Existence of Global Attractor

**Lemma 4.1** Assume that for any bounded positive invariant set \( B \in H \) and any \( \varepsilon > 0 \), there exists \( T = T(\varepsilon, B) \) such that
\[
||S(T)x - S(T)y|| \leq \varepsilon + \varphi_T(x,y), \quad \forall x, y \in B,
\]
where \( \varphi_T : H \times H \rightarrow \mathbb{R} \), satisfies for any sequence \( \{z_n\} \in B \),
\[
\liminf_{n \rightarrow \infty} \liminf_{n \rightarrow \infty} \varphi_T(z_n, z_m) = 0.
\]

Then \( S(t) \) is asymptotically smooth.

**Lemma 4.2** Let \( S(t) \) be a dissipative \( C_0 \)-semigroup defined on a metric space \( H \); then \( S(t) \) has a compact global attractor in \( H \) if and only if it is asymptotically smooth in \( H \).

**Theorem 4.1** Assume the hypotheses of Theorem 3.2 and \( \rho = r = 0 \) hold, the corresponding semigroup \( S(t) \) of problem (1.1)-(1.3) has an absorbing set in \( H_0 \).

**Proof** Let us fix an arbitrary bounded set \( B \in H_0 \) and consider the solutions of problem (1.1)-(1.3) given by \( (u(t), u_t(t), v(t), v_t(t)) = S(t)(u^0, u^1, v^0, v^1) \) with \((u^0, u^1, v^0, v^1) \in B\).

Since \( u(0,t) = u_x(L,t) = u_x(0,t) = 0, \) the following inequalities hold:
\[
||u||_2 \leq \sqrt{L}||u_x||_2, \quad ||u_x||_2 \leq L||u_{xx}||_2, \quad ||u||_2 \leq L^2||u_{xx}||_2.
\]
\[
\tag{4.1}
\]
We can calculate the total energy functional
\[
E(t) = \frac{1}{2} \left( ||u_t||^2_2 + ||v_t||^2_2 + ||u_{xx}||^2_2 + ||v_{xx}||^2_2 + ||u_{x}||^2_2 + ||v_{x}||^2_2 \right)
+ \frac{1}{2} \left( ||u_{x}||^2_2 + ||v_{x}||^2_2 \right) + f_1(u(L, t)) + f_2(v(L, t)) - \int_0^L p u d x - \int_0^L q v d x.
\] (4.2)

Let us define
\[
\varphi(t) = \int_0^L (u_t - u_{xxt}) u d x + \int_0^L (v_t - v_{xxt}) v d x.
\]

In the first equation and the second equation of (1.1), respectively multiplying by \(u_t + \varepsilon v\) and \(v_t + \varepsilon v\), then integrating over \(\Omega\) and taking the sum, we have
\[
d\frac{d}{dt} E(t) + \varepsilon \frac{d}{dt} \varphi(t) + \varepsilon E(t) + \frac{\varepsilon}{2} \left( ||u_t||^2_2 + ||v_t||^2_2 \right) + \frac{\varepsilon}{2} \left( ||u_{xx}||^2_2 + ||v_{xx}||^2_2 \right)
+ \frac{\varepsilon}{2} \left( ||u_{x}||^2_2 + ||v_{x}||^2_2 \right) + \frac{3\varepsilon}{4} \left( ||u_{xx}||^2_2 + ||v_{xx}||^2_2 \right)^2 + N \left( ||u_{xx}||^2_2 \right) ||u_{x}||^2_2
+ \frac{\varepsilon}{2} \left( ||u||^2_2 + ||v||^2_2 \right) + g_1(u(L, t)) u_t(L, t) + g_2(v(L, t)) v_t(L, t)
\]
\[
= 2\varepsilon \left( ||u_{xx}||^2_2 + ||v_{xx}||^2_2 \right) + 2\varepsilon \left( ||u_t||^2_2 + ||v_t||^2_2 \right) + \varepsilon f_1(u(L, t)) + \varepsilon f_2(v(L, t))
- \varepsilon u(L, t) f_1(u(L, t)) - \varepsilon v(L, t) f_2(v(L, t)) - \varepsilon u(L, t) g_1(u(L, t))
- \varepsilon v(L, t) g_2(v(L, t)) - \varepsilon N \left( ||u||^2_2 \right) \int_0^L u_{x} u_{x} d x - \varepsilon N \left( ||v||^2_2 \right) \int_0^L v_{x} v_{x} d x.
\] (4.3)

In the following, we estimate (4.3).

Using (2.2) and (4.1), we obtain
\[
\left\{ \begin{array}{l}
\varepsilon f_1(u(L, t)) u_t(L, t) \leq \frac{1}{2} f_1(u(L, t)) \leq \frac{1}{2} \varepsilon k_1 L^3 ||u_{x}||^2_2 + \varepsilon L_1, \\
\varepsilon f_2(v(L, t)) v_t(L, t) \leq \frac{1}{2} f_2(v(L, t)) \leq \frac{1}{2} \varepsilon k_2 L^3 ||v_{x}||^2_2 + \varepsilon L_2.
\end{array} \right.
\] (4.4)

Using (2.4), we obtain
\[
g_1(u(L, t)) u_t(L, t) \geq p_1 ||u_{x}||^2_2, \quad g_2(v(L, t)) v_t(L, t) \geq p_2 ||v_{x}||^2_2.
\] (4.5)

Using (2.5) and \(r = 0\), we obtain
\[
\varepsilon u(L, t) f_1(u(L, t)) \leq 3m_2 ||u_{x}||^2_2 ||u_t||^2_2 \leq 9m_2^2 L^3 ||u_{x}||^2_2 + \varepsilon N \left( ||u_{x}||^2_2 \right) ||u_{x}||^2_2,
\]
\[
\varepsilon v(L, t) g_2(v(L, t)) \leq 3m_2 ||v_{x}||^2_2 ||v_t||^2_2 \leq 9m_2^2 L^3 ||v_{x}||^2_2 + \varepsilon N \left( ||v_{x}||^2_2 \right) ||v_{x}||^2_2.
\] (4.6)

Using (2.6), we obtain
\[
\left\{ \begin{array}{l}
N \left( ||u||^2_2 \right) ||u_{x}||^2_2 \geq \alpha ||u_{x}||^2_2 + \beta ||u||^2_2 ||u_{x}||^2_2 \geq \frac{\alpha}{2 \beta} ||u_t||^2_2 + \frac{\beta}{2} ||u_{x}||^2_2, \\
N \left( ||v||^2_2 \right) ||v_{x}||^2_2 \geq \alpha ||v_{x}||^2_2 + \beta ||v||^2_2 ||v_{x}||^2_2 \geq \frac{\alpha}{2 \beta} ||v_t||^2_2 + \frac{\beta}{2} ||v_{x}||^2_2.
\end{array} \right.
\] (4.7)

Considering that \(N(\cdot) \in C^1(\Omega)\), there exists a constant \(C > 0\), such that
\[
\varepsilon N \left( ||u||^2_2 \right) \int_0^L u_{x} u_{x} d x \leq \frac{C}{2} \varepsilon \left( ||u||^2_2 \right) \int_0^L u_{x} u_{x} d x + \frac{\alpha}{2} ||u_{x}||^2_2 \int_0^L u_{x} u_{x} d x,
\]
\[
\varepsilon N \left( ||v||^2_2 \right) \int_0^L v_{x} v_{x} d x \leq \frac{C}{2} \varepsilon \left( ||v||^2_2 \right) \int_0^L v_{x} v_{x} d x + \frac{\alpha}{2} ||v_{x}||^2_2 \int_0^L v_{x} v_{x} d x.
\] (4.8)

Then inserting (4.4)-(4.8) into (4.3), we obtain
\[
d\frac{d}{dt} E(t) + \varepsilon \frac{d}{dt} \varphi(t) + \varepsilon E(t) + \frac{\varepsilon}{2} \left( ||u_t||^2_2 + ||v_t||^2_2 \right) + \frac{\varepsilon}{2} \left( ||u_{xx}||^2_2 + ||v_{xx}||^2_2 \right)
+ \frac{\alpha}{2} \left( ||u_t||^2_2 + ||v_t||^2_2 \right) + \frac{\alpha}{2} \left( ||u_{xx}||^2_2 + ||v_{xx}||^2_2 \right)
\leq 2\varepsilon \left( ||u_{xx}||^2_2 + ||v_{xx}||^2_2 \right) + 2\varepsilon \left( ||u_t||^2_2 + ||v_t||^2_2 \right) + \varepsilon (L_1 + L_2) + 9m_2^2 L^3 ||u_{x}||^2_2
\]
Applying Gronwall's inequality, we obtain

\[
\frac{d}{dt} E(t) + \varepsilon \frac{d}{dt} \phi(t) + \varepsilon E(t) \leq \varepsilon (L_1 + L_2).
\]

Let us define modified energy functional

\[
E(t) = E(t) + 2L_0 + L_1 + L_2 + L^4(||p||^2 + ||q||^2)\]

Then we obtain

\[
\frac{d}{dt} (\hat{E}(t) + \varepsilon \phi(t)) + \varepsilon \hat{E}(t) \leq \varepsilon (2L_0 + 2L_1 + 2L_2 + L^4(||p||^2 + ||q||^2)).
\]

Since

\[
\begin{align*}
- \int_0^t p u dx &\geq -L^2 ||p|| ||u||_2^2 \geq -L^4 ||p||^2 - 1/4 ||u||^2, \\
- \int_0^t q v dx &\geq -L^2 ||q|| ||v||_2^2 \geq -L^4 ||q||^2 - 1/4 ||v||^2, \\
\end{align*}
\]

and using (2.2), we obtain

\[
\hat{f}_1(u(L, t)) \geq -L_0, \quad \hat{f}_2(v(L, t)) \geq -L_0,
\]

then we obtain

\[
\hat{E}(t) \geq \frac{1}{4} ||(u(t), u_t(t), v(t), v_t(t))||_{H_0}^2.
\]

Let us set

\[
\hat{E}_\varepsilon(t) = \hat{E}(t) + \varepsilon \phi(t).
\]

From (4.1), we have

\[
|\hat{E}_\varepsilon(t) - \hat{E}(t)| = |\varepsilon \phi(t)| \leq 2\varepsilon (L_2^2 + L^4) \hat{E}(t),
\]

for \(\varepsilon\) sufficiently small enough,

\[
(1 - 2\varepsilon (L_2^2 + L^4)) \hat{E}(t) \leq \hat{E}_\varepsilon(t) \leq (1 + 2\varepsilon (L_2^2 + L^4)) \hat{E}(t).
\]

Let us define \(L_3 = 1 + 2\varepsilon (L_2^2 + L^4)\), \(L_4 = 2L_0 + L_1 + L_2 + L^4(||p||^2 + ||q||^2)\). Inserting (16) into (4.14), we have

\[
\frac{d}{dt} \hat{E}_\varepsilon(t) + \frac{\varepsilon}{L_3} \hat{E}_\varepsilon(t) \leq \varepsilon L_4.
\]

Applying Gronwall's inequality, we obtain

\[
\hat{E}_\varepsilon(t) \leq \hat{E}_\varepsilon(0) \exp\left(-\frac{t \varepsilon}{L_3}\right) + L_3 L_4 \left(1 - \exp\left(-\frac{t \varepsilon}{L_3}\right)\right).
\]
Since the given invariant set $B$ is bounded, $\tilde{E}_\varepsilon(0)$ is also bounded. Then there exists $t_B > 0$ large enough such that $\forall t > t_B$,  
$$
(1 - 2\varepsilon(L^2 + L^4))\dot{E}(t) \leq \dot{E}_\varepsilon(t) \leq L_3 L_4.
$$

Then from (4.15) we have $\forall t > t_B$,  
$$
||(u(t), u_t(t), v(t), v_t(t))||_{H_0}^2 \leq \frac{4L_3 L_4}{(1 - 2\varepsilon(L^2 + L^4))}.
$$

This shows that  
$$
B = \left\{(u, u_t, v, v_t) \in H_0 : ||(u(t), u_t(t), v(t), v_t(t))||_{H_0}^2 \leq \frac{4L_3 L_4}{(1 - 2\varepsilon(L^2 + L^4))}\right\}
$$
is an absorbing set for $S(t)$ in $H_0$. The proof of Theorem 4.1 is completed.

**Theorem 4.2** Assume the hypotheses of Theorem 3.2 and $\rho = r = 0$ hold, the corresponding semigroup $S(t)$ of problem (1.1)-(1.3) is asymptotic compactness in $H_0$.

**Proof** Given initial data $(u_0, u^1, v_0, v^1)$ and $(\tilde{u}_0, \tilde{u}^1, \tilde{v}_0, \tilde{v}^1)$ in a bounded invariant set $B \subset H_0$, let $(u, v, (\tilde{u}, \tilde{v}))$ be the corresponding weak solution of problem (1.1)-(1.3). Then the difference $\omega = u - \tilde{u}, \tilde{\omega} = v - \tilde{v}$ is a weak solution of  
$$
\begin{align*}
\omega_t + \omega_{xxxx} - \omega_{xxtt} - (||u_x||^2 + ||v_x||^2)\omega_{xx} - N(||u_x||^2)\omega_{xxt} - \Delta N\tilde{u}_{xxt} &= 0, \\
\tilde{\omega}_t + \tilde{\omega}_{xxxx} - \tilde{\omega}_{xxtt} - (||\tilde{u}_x||^2 + ||\tilde{v}_x||^2)\tilde{\omega}_{xx} - N(||\tilde{v}_x||^2)\tilde{\omega}_{xxt} - \Delta \tilde{\omega}_{xxt} &= 0, \\
\omega(0, t) &= \omega_x(L, t) = \omega_{xx}(0, t) = 0, \\
\tilde{\omega}(0, t) &= \tilde{\omega}_x(L, t) = \tilde{\omega}_{xx}(0, t) = 0,
\end{align*}
$$

(4.17)  

where  
$$
\begin{align*}
\Delta N = N(||u_x||^2) - N(||\tilde{u}_x||^2), \\
\Delta \tilde{\omega} = N(||v_x||^2) - N(||\tilde{v}_x||^2), \\
f_1(u(L, t)) - f_1(\tilde{u}(L, t)) &= \Delta f, \\
g_1(u_t(L, t)) - g_1(\tilde{u}_t(L, t)) &= \Delta g,
\end{align*}
$$

(4.18)  

We can calculate the total energy functional  
$$
F(t) = \frac{1}{2}(||\omega||^2 + ||\omega_{xx}||^2 + ||\omega_{xt}||^2 + ||\tilde{\omega}||^2 + ||\tilde{\omega}_x||^2 + ||\tilde{\omega}_{xt}||^2),
$$

and define  
$$
\psi(t) = \int_0^L (\omega_t - \omega_{xxt})\omega dx + \int_0^L (\tilde{\omega}_t - \tilde{\omega}_{xxt})\tilde{\omega} dx.
$$

Respectively multiply the first equation and the second equation of (4.17) by $\omega_t + \mu \omega$ and $\tilde{\omega}_t + \mu \tilde{\omega}$. Then integrating over $\Omega$ and taking the sum, we have  
$$
\frac{d}{dt} F(t) + \mu \frac{d}{dt} \psi(t) + 2\mu F(t) + N(||u_x||^2)||\omega_{xx}||^2 + ||\omega_{xt}||^2 + \frac{\Delta g \omega_t(L, t) + \Delta \tilde{\omega}_t(L, t)}{2}
$$

$$
= - (||u_x||^2 + ||v_x||^2) \int_0^L \omega_x \omega_{xxt} dx - (||u_x||^2 + ||v_x||^2) \int_0^L \omega_{xxt} \omega dx
$$

$$
- \Delta f \omega_t(L, t) - \Delta \tilde{f} \tilde{\omega}_t(L, t) + \Delta N \int_0^L \tilde{u}_{xxt} \omega dx + \Delta \tilde{\omega}_x \tilde{\omega}_t dx
$$

$$
+ 2\mu (||\omega||^2 + ||\omega_{xx}||^2 + ||\omega_{xt}||^2 - \mu \omega(L, t) \Delta f - \mu \omega(L, t) \Delta g
$$

$$
- \mu N(||u_x||^2) \int_0^L \omega_x \omega_{xxt} dx - \mu N(||v_x||^2) \int_0^L \tilde{\omega}_x \tilde{\omega}_{xxt} dx
$$
\[
+ \mu \Delta N \int_0^L \ddot{u}_{xx} \omega dx + \mu \Delta \tilde{N} \int_0^L \ddot{v}_{xx} \tilde{\omega} dx - \mu(||u_x||^2 + ||v_x||^2)||\omega_x||^2
- \mu(||u_x||^2 + ||v_x||^2)||\tilde{\omega}_x||^2 - \mu \tilde{\omega}(L, t)(\Delta \tilde{f} + \Delta \tilde{g}).
\]

(4.19)

In the following, let us estimate (4.19).

From the assumption (2.3) and \( \rho = 0 \), we have

\[
\begin{align*}
&|\mu \omega(L, t) \Delta f| \leq 3k_1 \mu |\omega(L, t)|^2 \leq C_0 |\omega_x||^2, \\
&|\mu \omega(L, t) \Delta f| \leq 3k_2 \mu |\omega(L, t)|^2 \leq C_0 |\tilde{\omega}_x||^2, \\
&|\Delta f \omega(L, t)| \leq 3k_1 |\omega(L, t)||\omega(L, t)| \leq C_0 |\omega_x||^2 + \frac{2\mu}{\gamma^2} |\omega(L, t)|^2, \\
&|\Delta \tilde{f} \omega(L, t)| \leq 3k_2 |\omega(L, t)||\tilde{\omega}(L, t)| \leq C_0 |\omega_x||^2 + \frac{2\mu}{\gamma^2} |\tilde{\omega}(L, t)|^2.
\end{align*}
\]

(4.20)

Using (2.4), we obtain

\[
\Delta g \omega_1(L, t) \geq p_1 |\omega_1(L, t)|^2, \quad \Delta \tilde{g} \tilde{\omega}(L, t) \geq p_2 |\tilde{\omega}(L, t)|^2.
\]

(4.21)

From the assumption (2.5) and \( r = 0 \), we have

\[
\begin{align*}
&|\mu \omega(L, t) \Delta g| \leq 3 \mu m_1 |\omega(L, t)||\omega_1(L, t)| \leq C_0 |\omega_x||^2 + \frac{2\mu}{\gamma^2} |\omega(L, t)|^2, \\
&|\mu \omega(L, t) \Delta \tilde{g}| \leq 3 \mu m_2 |\omega(L, t)||\tilde{\omega}(L, t)| \leq C_0 |\omega_x||^2 + \frac{2\mu}{\gamma^2} |\tilde{\omega}(L, t)|^2.
\end{align*}
\]

(4.22)

From the assumption (2.6), we get

\[
\begin{align*}
\begin{bmatrix} N(||u_x||^2)||\omega_x||^2 \geq (\alpha + \beta ||u_x||^2) ||\omega_x||^2, \\
N(||v_x||^2)||\omega_x||^2 \geq (\alpha + \beta ||v_x||^2) ||\omega_x||^2.
\end{bmatrix}
\end{align*}
\]

(4.23)

Considering that \( N(\cdot) \in C^1(\mathbb{R}) \), there exists a constant \( C_0 > 0 \) such that

\[
\begin{align*}
-\mu N(||u_x||^2) \int_0^L \omega_x \omega x dx \leq \frac{C_0}{2\mu} ||\omega_x||^2 + \frac{\gamma}{2} ||\omega_x||^2, \\
-\mu N(||v_x||^2) \int_0^L \omega_x \omega x dx \leq \frac{C_0}{2\mu} ||\omega_x||^2 + \frac{\gamma}{2} ||\omega_x||^2.
\end{align*}
\]

(4.24)

From the mean value theorem \( N(a^2) - N(b^2) \leq N'(\sup\{a^2, b^2\})|a - b||a + b| \), we get

\[
\begin{align*}
\Delta N \int_0^L \ddot{u}_{xx} \omega dx \leq C_0 ||\omega_x||^2 ||\omega_x||^2 \leq \frac{C_0}{2\mu} ||\omega_x||^2 + \frac{\gamma}{2} ||\omega_x||^2, \\
\Delta \tilde{N} \int_0^L \ddot{v}_{xx} \tilde{\omega} dx \leq C_0 ||\tilde{\omega}_x||^2 ||\omega_x||^2 \leq \frac{C_0}{2\mu} ||\tilde{\omega}_x||^2 + \frac{\gamma}{2} ||\omega_x||^2, \\
\mu \Delta N \int_0^L \ddot{u}_{xx} \omega dx = -\mu \Delta \tilde{N} \int_0^L \ddot{u}_{xx} \tilde{\omega} dx \leq C_0 ||\omega_x||^2, \\
\mu \Delta \tilde{N} \int_0^L \ddot{v}_{xx} \tilde{\omega} dx = -\mu \Delta N \int_0^L \ddot{v}_{xx} \omega dx \leq C_0 ||\tilde{\omega}_x||^2.
\end{align*}
\]

(4.25)

Obviously, we get

\[
\begin{align*}
-(||u_x||^2 + ||v_x||^2) \int_0^L \omega_x \omega x dx \leq \frac{C_0}{2\mu} ||\omega_x||^2 + \frac{\gamma}{2} ||\omega_x||^2, \\
-(||u_x||^2 + ||v_x||^2) \int_0^L \omega_x \omega x dx \leq \frac{C_0}{2\mu} ||\tilde{\omega}_x||^2 + \frac{\gamma}{2} ||\tilde{\omega}_x||^2, \\
-\mu(||u_x||^2 + ||v_x||^2)||\omega_x||^2 \leq C_0 ||\omega_x||^2, \\
-\mu(||u_x||^2 + ||v_x||^2)||\tilde{\omega}_x||^2 \leq C_0 ||\tilde{\omega}_x||^2.
\end{align*}
\]

(4.26)

Substituting (4.20)-(4.26) into (4.19), we get

\[
\frac{d}{dt} F(t) + \mu \frac{d}{dt} \psi(t) + 2\mu F(t) + (\alpha + \beta ||u_x||^2) ||\omega_x||^2 + (\alpha + \beta ||v_x||^2) ||\omega_x||^2 \\
\leq \left(6C_0 + \frac{3C_0^2}{2\mu}\right)(||\omega_x||^2 + ||\tilde{\omega}_x||^2) + \left(\frac{7\mu}{2} + 2\mu L\right)(||\omega_x||^2 + ||\tilde{\omega}_x||^2).
\]

(4.27)

With \( \mu \leq 2\alpha/(7 + 4L) \) sufficiently small enough, we have

\[
\frac{d}{dt} F(t) + \mu \frac{d}{dt} \psi(t) + 2\mu F(t) \leq \left(6C_0 + \frac{3C_0^2}{2\mu}\right)(||\omega_x||^2 + ||\tilde{\omega}_x||^2).
\]
Defining $F_\mu(t) = F(t) + \mu \psi(t)$, we get

$$|F_\mu(t) - F(t)| = |\mu \psi(t)| \leq 2\mu L^2 F(t).$$

For $0 < \mu \leq \min\{2\alpha/(7 + 4L), 1/2L^2\}$ sufficiently small enough, we get

$$(1 - 2\mu L^2)F(t) \leq F_\mu(t) \leq (1 + 2\mu L^2)F(t). \quad (4.28)$$

Inserting (4.28) into (4.27), there exists a constant $C_1 > 0$ such that

$$\frac{d}{dt}F_\mu(t) + \frac{2\mu}{1 + 2\mu L^2}F_\mu(t) \leq C_1(||\omega_s||_2^2 + ||\bar{\omega}_s||_2^2).$$

From Gronwall’s lemma, we obtain

$$F_\mu(t) \leq F_\mu(0) \exp(-\mu/(1 + 2\mu L^2)) t$$

$$+ C_1 \int_0^t \exp\left(-\mu/(1 + 2\mu L^2) \right) \left(||\omega_s||_2^2 + ||\bar{\omega}_s||_2^2 \right) ds \quad (4.29)$$

Combining (4.28) and (4.29), there exists a constant $C_B > 0$, only depending on the size of $B$, such that

$$||s,(\omega(t), \omega(t), \bar{\omega}(t), \bar{\omega}(t))||_{H_0}$$

$$\leq \frac{\sqrt{2}}{\sqrt{1 - 2\mu L^2}} C_B \exp(-\mu/(1 + 2\mu L^2)) t + \frac{\sqrt{2}C_1}{\sqrt{1 - 2\mu L^2}} \left( \int_0^t \left(||\omega_s||_2^2 + ||\bar{\omega}_s||_2^2\right) ds \right)^{1/2}. \quad (4.30)$$

Given $\varepsilon > 0$, we choose $T$ large such that

$$\frac{\sqrt{2}}{\sqrt{1 - 2\mu L^2}} C_B \exp(-\mu/(1 + 2\mu L^2)) T \leq \varepsilon, \quad (4.31)$$

and defined $\phi_T: H_0 \times H_0 \rightarrow \mathbb{R}$ as

$$\phi_T((u^0, u^1, v^0, v^1), (\tilde{u}^0, \tilde{u}^1, \tilde{v}^0, \tilde{v}^1)) = \frac{\sqrt{2}C_1}{\sqrt{1 - 2\mu L^2}} \left( \int_0^T \left(||\omega_s||_2^2 + ||\bar{\omega}_s||_2^2\right) ds \right)^{1/2}. \quad (4.32)$$

From (4.30)-(4.32), we have

$$||S(T)((u^0, u^1, v^0, v^1) - S(T)(\tilde{u}^0, \tilde{u}^1, \tilde{v}^0, \tilde{v}^1))||_{H_0} \leq \varepsilon + \phi_T((u^0, u^1, v^0, v^1), (\tilde{u}^0, \tilde{u}^1, \tilde{v}^0, \tilde{v}^1)), \quad (4.33)$$

for all $(u^0, u^1, v^0, v^1), (\tilde{u}^0, \tilde{u}^1, \tilde{v}^0, \tilde{v}^1) \in B$. Given a sequence $(u^0_n, u^1_n, v^0_n, v^1_n) \in B$, since $B$ is bounded and positive invariant, the corresponding sequence $(u_n(t), u_{tn}(t), v_n(t), v_{tn}(t))$ of solutions of problem (1.1)-(1.3) is uniformly bounded in $H_0$. So $(u_n)$ is bounded in $C([0, \infty), V) \cap C^1([0, \infty), U)$. $V \rightarrow H_0^1$ compactly, and there exists a subsequence $(u_{nk})$ which converge strongly in $C([0, T], H_0^1(\Omega))$. Therefore

$$\lim_{k \rightarrow \infty} \lim_{l \rightarrow \infty} \int_0^T (||u_{xnk}(s) - u_{xnl}(s)||_2^2 + ||\bar{u}_{xnk}(s) - \bar{u}_{xnl}(s)||_2^2) ds = 0,$$

and consequently

$$\lim_{k \rightarrow \infty} \lim_{l \rightarrow \infty} \phi_T((u^0_{nk}, u^1_{nk}, v^0_{nk}, v^1_{nk}), (\tilde{u}^0_{nk}, \tilde{u}^1_{nk}, \tilde{v}^0_{nk}, \tilde{v}^1_{nk})) = 0.$$

So $S(t)$ is asymptotically smooth in $H_0$.

In view of Lemma 4.2, Theorem 4.1 and Theorem 4.2, we have the main result.

**Theorem 4.3** The corresponding semigroup $S(t)$ of the problem (1.1)-(1.3) has a compact global attractor in $H_0$.

**References:**

一类耦合梁方程组在非线性边界条件下解的长时间动力行为

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摘要: 本文研究一类具有强阻尼项的耦合梁方程组在非线性边界条件下的长时间动力行为, 首先利用一些常用不等式和先验估计证明该系统存在唯一的整体解, 其次通过证明系统存在有界吸收集得到整体吸引子的存在性.

关键词: 耦合梁方程组; 非线性边界条件; 整体; 吸引子