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Positive Solutions for Kirchhoff-Type Equations with an Asymptotically Nonlinearity

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Abstract: We focus on a class of nonlinear Kirchhoff-type equation. The nonlinear function $f(x,u)$ is either asymptotically linear or asymptotically nonlinear with respect to u at infinity. Under certain conditions on the potential function $V(x)$ and the nonlinear term $f(x,u)$, the existence of positive solutions is obtained without using the compactness of embedding of the working space.
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1. Introduction and main results

In this paper, we study the existence of positive solutions for the following nonlinear Kirchhoff-type problem:

$$\begin{cases} -(a+b\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x)\triangle u+\lambda V(x)u=f(x,u), \text{ in } \mathbb{R}^3,\\ u\in H^1(\mathbb{R}^3), \end{cases} \tag{1.1}$$

where a,b are positive constants and $\lambda\geq 1$ is a parameter. We assume that the functions $V(x)$ and $f(x,s)$ satisfy the following hypotheses.

(v_1) $V(x)\in C(\mathbb{R}^3,\mathbb{R})$ satisfies $V(x)\geq 0$ on $\mathbb{R}^3$;

(v_2) There exists $d>0$ such that the set $\{x\in\mathbb{R}^3 : V(x)\leq d\}$ has finite measure;

(f_1) $f(x,s)\in C(\mathbb{R},\mathbb{R}+)$, $f(x,s)\equiv 0$ for all $s<0$ and $\lim\limits_{s\to 0^+}\frac{f(x,s)}{s}=0$;

(f_2) $\lim\limits_{s\to+\infty}\frac{f(x,s)}{s}=l, 0<\frac{f(x,s)}{s}\leq l$ with $l\in(0,+\infty)$;

(A_1) There exists a constant $\beta\in(0,1)$ such that

$$(1-\beta)l>\mu^*:=\inf\left\{\int_{\mathbb{R}^3}(a|\nabla u|^2+\lambda V(x)u^2)\mathrm{d}x : u\in H^1(\mathbb{R}^3),\right.$$

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$$\int_{\mathbb{R}^3} F(x,u)\mathrm{d}x \geq \frac{l}{2}, b(\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x)^2 < 2\beta l\Big\},$$

where $F(x,u) := \int_0^u f(x,t)\mathrm{d}t$;

(f_3) $\lim\limits_{s\to+\infty} \frac{f(x,s)}{s^3} = l$ with $l \in (0,+\infty)$;

(f_4) There exist $0 < C_0 < \frac{\mu-2}{2\mu\eta_2}$ and $\mu \geq 4$ such that

$$F(x,s) - \frac{1}{\mu} f(x,s)s \leq C_0 s^2$$

for all $s \geq 0$ and $x \in \mathbb{R}^3$, where η_2 is defined by Lemma 2.1;

(A_2) There exists a constant $\beta \in (0,1)$ such that

$$(1-\beta)b > \mu^* := \inf\Big\{ (\int_{\mathbb{R}^3} (a|\nabla u|^2 + \lambda V(x)u^2)\mathrm{d}x)^2 : u \in H^1(\mathbb{R}^3), \int_{\mathbb{R}^3} lu^4 \mathrm{d}x \geq b^2 \Big\}.$$

The problem (1.1) is related to the stationary analogue of the following equation

$$u_{tt} - (a + b\int_{\Omega} |\nabla u|^2 \mathrm{d}x)\Delta u = f(x,u) \text{ in } \Omega \tag{1.2}$$

proposed by Kirchhoff in 1883 (see [1]) to describe the transversal oscillations of a stretched string, particularly, taking into account the subsequent change in string length produced by transverse vibrations. In (1.2), u denotes the displacement, $f(x,u)$ the external force, and b the initial tension while a is related to the intrinsic properties of the string (such as Young's modulus).

Such a class of problem is often referred to as being nonlocal because of the presence of the integral term $\int_\Omega |\nabla u|^2 \nabla u dx$, which means that (1.2) is no longer a pointwise identity. This makes the study of such a class of problem particularly difficulties and interesting. Similar nonlocal problems also model several physical and biological systems, where u describes a process which depends on the average of itself, for example, the population density(see [2-3]) and the references therein. Recently, assuming that the potential $V(x)$ satisfies:

(v_1') $V(x) \in C(\mathbb{R}^3, \mathbb{R})$ satisfies $\inf\limits_{x\in\mathbb{R}^3} V(x) \geq b_1 > 0$, where $b_1 > 0$ is a constant.

(v_2') meas$\{x \in \mathbb{R}^3 : V(x) \leq M\} < \infty$, for any $M > 0$, where meas(.) denotes the Lebesgue measure in $\mathbb{R}^3$. The authors[4-5] obtained the existence and multiplicity of nontrivial solutions of (1.1) with $\lambda = 1$. The hypotheses (v_1') and (v_2') were used in [6] to guarantee the compact embedding of the working space (see [7, Lemma 3.4]). Obviously, (v_2) is weaker than (v_2'), which cannot guarantee the compactness of the embedding. This situation becomes more delicate. Recently, some authors in [8-11] dealt with this cases. For example, LIANG and ZHANG[8] investigated the existence of solutions of Kirchhoff type problems with critical nonlinearity. If $f(x,u)$ in (1.1) is superlinear at infinity, the authors[9] proved two existence theorems of nontrivial weak solutions and a sequence of high energy weak solutions for (1.1). Particularly, SUN and WU[10] also studied the existence of ground state solutions. More recently, XU and CHEN[11] also investigated the existence and multiplicity results if $f(x,u)$ is either sublinear or superlinear at infinity. But to the author's knowledge, there are few works on the existence of positive solutions for (1.1), when $f(x,u)$ is asymptotically linear and $V(x)$ satisfies more general conditions.

The main results are the following theorems.

Theorem 1.1 Let (v_1)-(v_2), (f_1)-(f_2) and (A_1) hold, then the problem (1.1) possesses a positive solution for large $\lambda > 0$.

Remark 1.1 Condition like (f_1)-(f_2) and (A_1) on the nonlinear term f was employed in [11-13]. For example, if $V(x) = 1$, SUN et al.[12] obtained the ground state solutions of Schrödinger-Poisson equations. When $\lambda = 1$, using similar assumptions on the nonlinearity f, LIU et al.[13] proved the existence of positive solution of (1.1), assuming $V(x)$ satisfying (v_1) and $\lim\limits_{|x|\to+\infty} V(x) = V(\infty) \in (0, \infty)$. Theorem 1.1 extends the main results in [12] to the Kirchhoff-type equations.

Theorem 1.2 Let (v_1)-(v_2), (f_1), (f_3)-(f_4) and (A_2) hold, then for large $\lambda > 0$ the problem (1.1) possesses a positive solution.

Remark 1.2 Under the conditions (f_1), (f_3) and (f_4'): $\frac{F(x,u)}{u^4}$ is nondecreasing for $u > 0$, DING et al.[14] studied the existence of positive solutions for a class of nonhomogeneous Schrödinger-Poisson system. Note that (f_4') is stronger than (f_4). Theorem 1.2 also extends the main results in [14] to the Kirchhoff-type equations. To the best of our knowledge, little has been done for Kirchhoff-type equations with asymptotically linear or asymptotically nonlinear under relaxed assumptions (v_1)-(v_2).

The rest of the paper is given as follows: in Section 2, we present some preliminary results. In Section 3 and 4, we give the proofs of Theorem 1.1 and 1.2, respectively. In the latter parts of this paper, we use $C > 0$ to denote any positive constant.

2. Preliminaries

Let

$$E = \{u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)u^2 \mathrm{d}x < +\infty\}$$

be equipped with the inner product and norm

$$\langle u, v\rangle = \int_{\mathbb{R}^3} (a\nabla u \nabla v + V(x)uv)\mathrm{d}x,\ \|u\| = \langle u, u\rangle^{\frac{1}{2}}.$$

We also need the following inner product and norm

$$\langle u, v\rangle_\lambda = \int_{\mathbb{R}^3} (a\nabla u \nabla v + \lambda V(x)uv)\mathrm{d}x,\ \|u\|_\lambda = \langle u, u\rangle_\lambda^{\frac{1}{2}}.$$

Set $E_\lambda = (E, \|.\|_\lambda)$, then we have the following lemma.

Lemma 2.1 Under assumptions (v_1) and (v_2), for $2 \le r < 2^*$, the embedding $E_\lambda \hookrightarrow L^r(\mathbb{R}^3)$ is continuous. Hence, there is $\eta_r > 0$ (independent of λ) such that

$$\|u\|_r \le \eta_r \|u\|_\lambda,\ \forall u \in E_\lambda,$$

where $\|.\|_r$ denotes the usual norm on $L^r(\mathbb{R}^3)$.

Proof The proof is similar to that of Lemma 2.1 in [15], and is omitted here.

It follows from (v_1)-(v_2) and (f_1)-(f_2) that the function $I_\lambda : E_\lambda \to \mathbb{R}$ defined by

$$I_\lambda(u) = \frac{1}{2}\|u\|_\lambda^2 + \frac{b}{4}(\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x)^2 - \int_{\mathbb{R}^3} F(x, u)\mathrm{d}x \tag{2.1}$$

is of class $C^1(E_\lambda, \mathbb{R})$, and

$$\langle I_\lambda'(u), v\rangle_\lambda = \int_{\mathbb{R}^3} (a\nabla u \nabla v + \lambda V(x)uv)\mathrm{d}x + b\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x \int_{\mathbb{R}^3} \nabla u \nabla v \mathrm{d}x - \int_{\mathbb{R}^3} f(x, u)v\mathrm{d}x \tag{2.2}$$

for all u, $v \in E_\lambda$. Furthermore, the critical points of I_λ are weak solutions of the problem (1.1). Thus, we only need to look for critical points of I_λ on E_λ. To find the critical points of I_λ, we use the following mountain pass theorem, which is a very useful tool in dealing with the asymptotically linear case.

Lemma 2.2[15] Let E be a real Banach space with its dual space E^* and suppose that $I \in C^1(E, \mathbb{R})$ satisfies

$$\max\{I(0), I(e)\} \le \nu < \eta \le \inf_{\|u\|=\rho} I(u),$$

for some $\nu < \eta, \rho > 0$ and $e \in E$ with $\|e\| > \rho$. Let $c \ge \eta$ be characterized by

$$c = \inf_{\gamma \in \Gamma} \max_{0 \le \tau \le 1} I(\gamma(\tau)),$$

where $\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0, \gamma(1) = e\}$ is the set of continuous paths joining 0 and e, then there exists $\{u_n\} \subset E$ such that

$$I(u_n) \to c \ge \eta \text{ and } (1 + \|u_n\|)\|I'(u_n)\|_{E^*} \to 0, \text{ as } n \to \infty. \tag{2.3}$$

Set $B_R := \{x \in \mathbb{R}^3 : |x| < R\}$ and $B_R^C := \mathbb{R}^3 \setminus B_R = \{x \in \mathbb{R}^3 : |x| \ge R\}$. Next we provide a result which is due to [16-18].

Lemma 2.3 Assume (v_1) and (v_2) hold. Then for any $\varepsilon > 0$, there exists $\tau_\varepsilon > 0$ and $R_\varepsilon > 0$ such that $\|u\|^p_{L^p_{B^C_{R_\varepsilon}}} \le \varepsilon \|u\|^p_\lambda$ for all $u \in E_\lambda$ and $\lambda \ge \tau_\varepsilon$, where $2 \le p < 2^*$.

Proof The proof of this lemma is inspired by [18]. For the convenience of the reader we sketch it here. For any $R > 0$ define

$$A(R) := \{x \in \mathbb{R}^3 : |x| > R,\ V(x) \ge d\}, \quad B(R) := \{x \in \mathbb{R}^3 : |x| > R,\ V(x) < d\}.$$

Then

$$\int_{A(R)} u^2 \mathrm{d}x \le \frac{1}{\lambda d} \int_{A(R)} \lambda V(x) u^2 \mathrm{d}x \le \frac{1}{\lambda d} \int_{A(R)} (a|\nabla u|^2 + \lambda V(x) u^2) \mathrm{d}x \le \frac{1}{\lambda d} \|u\|^2_\lambda.$$

For $2 < p < 2^*$, using the Hölder inequality and Lemma 2.1, we obtain

$$\int_{B(R)} u^2 \mathrm{d}x \le |B(R)|^{\frac{1}{p'}} (\int_{B(R)} u^p \mathrm{d}x)^{\frac{2}{p}} \le |B(R)|^{\frac{1}{p'}} \|u\|^2_p \le \eta_p^2 |B(R)|^{\frac{1}{p'}} \|u\|^2_\lambda.$$

Setting $\theta = \frac{3(p-2)}{2p}$ and using the Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}\int_{B^C_{R_\varepsilon}} u^p \mathrm{d}x &\le C \|\nabla u\|^{\theta p}_{L^2_{B^C_{R_\varepsilon}}} \|u\|^{(1-\theta)p}_{L^2_{B^C_{R_\varepsilon}}} \le C \|u\|^{\theta p}_\lambda (\int_{A(R)} u^2 \mathrm{d}x + \int_{B(R)} u^2 \mathrm{d}x)^{\frac{(1-\theta)p}{2}} \\ &\le C \|u\|^{\theta p}_\lambda (\frac{1}{\lambda d} \|u\|^2_\lambda + \eta_p^2 |B(R)|^{\frac{1}{p'}} \|u\|^2_\lambda)^{\frac{(1-\theta)p}{2}} \le C \|u\|^p_\lambda (\frac{1}{\lambda d} + \eta_p^2 |B(R)|^{\frac{1}{p'}})^{\frac{(1-\theta)p}{2}}.\end{aligned}$$

According to (v_2), we obtain that $|B(R)| \to 0$ as $R \to \infty$. Then, if λ and R are large enough, the term in brackets above will be arbitrarily small. This concludes the proof the Lemma 2.3.

3. The Asymptotically Linear Case

In this section, we give the proof of Theorem 1.1. In what follows, we give several lemmas which are useful to the proof of the main results.

Lemma 3.1 Assume (v_1)-(v_2), (f_1)-(f_2) and (A_1) hold. Then the sequence $\{u_n\}$ defined in (2.3) is bounded in E_λ.

Proof Inspired by [17], we argue by contradiction and assume that $\|u_n\|_\lambda \to \infty$ as $n \to \infty$. Set $\omega_n = \frac{u_n}{\|u_n\|_\lambda}$, then there is a $\omega \in E_\lambda$ such that

$$\begin{cases} \omega_n \rightharpoonup \omega \text{ in } E_\lambda, \\ \omega_n \to \omega \text{ in } L^2_{\mathrm{loc}}(\mathbb{R}^3), \\ \omega_n \to \omega \text{ a.e. } x \in \mathbb{R}^3. \end{cases} \tag{3.1}$$

In what follows, we will obtain a contradiction by ruling out the vanishing and nonvanishing of $\{\omega_n\}$.

Assume that $\{\omega_n\}$ is vanishing. Then, suppose that, for every $R > 0$,

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} \omega_n^2 = 0. \tag{3.2}$$

By (f_1) and (f_2), there exists $C > 0$ such that

$$\frac{F(x,u)}{u^2} \le \frac{C}{2} \tag{3.3}$$

uniformly $x \in \mathbb{R}^3$. For any $0 < \varepsilon < 1$, by Lemma 2.3, there exists a $R_\varepsilon > 0$ such that

$$\lim_{n\to\infty} \int_{B_{R_\varepsilon}^C} \omega_n^2 \mathrm{d}x \le \frac{\varepsilon}{C}. \tag{3.4}$$

Then it follows from (3.2)-(3.4) that

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} \frac{F(x,u_n)}{u_n^2} \omega_n^2 \mathrm{d}x \le \frac{C}{2} \lim_{n\to\infty} \int_{\mathbb{R}^3} \omega_n^2 \mathrm{d}x \le \frac{\varepsilon}{2}. \tag{3.5}$$

By (2.3), one has

$$I_\lambda(u_n) \ge \frac{1}{2}\|u_n\|_\lambda^2 - \int_{\mathbb{R}^3} \frac{F(x,u_n)}{u_n^2} \omega_n^2 \|u_n\|_\lambda^2 \mathrm{d}x.$$

Combining this with (3.5), we obtain

$$c \ge \frac{1}{2}(1-\varepsilon)\|u_n\|_\lambda^2 \to +\infty \text{ as } n \to \infty,$$

which means a contradiction. Hence, non-vanishing must hold, that is, there exist $R, \alpha > 0$ and bounded sequence $\{y_n\} \subset \mathbb{R}^3$ such that

$$\lim_{n\to\infty} \int_{B_R(y_n)} \omega_n^2 \mathrm{d}x \ge \alpha > 0. \tag{3.6}$$

Using (3.1), we see that $\omega \ne 0$. By (f_2), one has

$$0 < \frac{F(x,u_n)}{u_n^2} \le \frac{l}{2}. \tag{3.7}$$

It follows from Lemma 2.1 that there exists $\eta_2^2 > l$ such that $\|u\|_2 \le \eta_2 \|u\|_\lambda$. Then, by (2.1), (3.6) and (3.7), we obtain that

$$\begin{aligned} I_\lambda(u_n) &\ge \big(\frac{1}{2}\|\omega_n\|_\lambda^2 - \int_{\mathbb{R}^3} \frac{F(x,u_n)}{u_n^2} \omega_n^2 \mathrm{d}x\big)\|u_n\|_\lambda^2 \ge \big(\frac{\eta_2^2}{2}\int_{\mathbb{R}^3} \omega_n^2 \mathrm{d}x - \frac{l}{2}\int_{\mathbb{R}^3} \omega_n^2 \mathrm{d}x\big)\|u_n\|_\lambda^2 \\ &\ge \big(\frac{\eta_2^2}{2} - \frac{l}{2}\big)\int_{B_R(y_n)} \omega_n^2 \mathrm{d}x \|u_n\|_\lambda^2. \end{aligned}$$

Thus, by (2.3) and (3.6)

$$c = \lim_{n\to\infty} I_\lambda(u_n) \ge \big[\big(\frac{\eta_2^2}{2} - \frac{l}{2}\big)\int_{B_R(y_n)} \omega_n^2 \mathrm{d}x\big] \lim_{n\to\infty} \|u_n\|_\lambda^2 = +\infty,$$

which is a contradiction. This concludes the proof of Lemma 3.1.

Lemma 3.2 Under the assumptions (v_1)-(v_2), (f_1)-(f_2), any bounded cerami sequence of I_λ defined in (2.3) has a convergent subsequence in E_λ.

Proof Let $\{u_n\}$ be a bounded sequence defined by (2.3). After a subsequence, we can assume that $u_n \rightharpoonup u$ in E_λ. Set $\omega_n = u_n - u$. By (f_1) and (f_2), for any $0 < \varepsilon_1 < \frac{2}{1+\eta_2}$ and $1 < p < 2^* - 1$, there exists a constant $C_{\varepsilon_1} > 0$ such that

$$f(x,\omega_n) \le \frac{\varepsilon_1}{2}|\omega_n| + C_{\varepsilon_1}|\omega_n|^p$$

and therefore

$$\int_{\mathbb{R}^3} f(x,\omega_n)\omega_n \mathrm{d}x \le \frac{\varepsilon_1}{2}\int_{\mathbb{R}^3}|\omega_n|^2\mathrm{d}x + C_{\varepsilon_1}\int_{\mathbb{R}^3}|\omega_n|^{p+1}\mathrm{d}x \le \frac{\varepsilon_1\eta_2}{2}\|\omega_n\|_\lambda^2 + C_{\varepsilon_1}\int_{\mathbb{R}^3}|\omega_n|^{p+1}\mathrm{d}x. \tag{3.8}$$

Since $\|\omega_n\|_\lambda$ is uniformly bounded in E_λ and $\lambda \ge 1$, we may fix a $\varepsilon_2 > 0$ such that

$$\frac{\varepsilon_1}{2}\|\omega_n\|_\lambda^2 \ge \varepsilon_2\|\omega_n\|_\lambda^{p+1} \text{ for all } n \text{ and } \lambda \ge 1. \tag{3.9}$$

By Lemma 2.3, there exist $\tau_{\varepsilon_2} > 0$ and $R_{\varepsilon_2} > 0$ such that

$$C_{\varepsilon_1}\int_{B^C_{R_{\varepsilon_2}}}|\omega_n|^{p+1}\mathrm{d}x \le \varepsilon_2\|\omega_n\|_\lambda^{p+1} \text{ for all } \lambda \ge \tau_{\varepsilon_2} > 1.$$

Using the fact that $\omega_n \to 0$ in $L^{p+1}_{loc}(\mathbb{R}^3)$ and (3.9), we have

$$\begin{aligned} C_{\varepsilon_1}\int_{\mathbb{R}^3}|\omega_n|^{p+1}\mathrm{d}x &= C_{\varepsilon_1}\Big(\int_{B_{R_{\varepsilon_2}}}|\omega_n|^{p+1}\mathrm{d}x + \int_{B^C_{R_{\varepsilon_2}}}|\omega_n|^{p+1}\mathrm{d}x\Big) \\ &\le \varepsilon_2\|\omega_n\|_\lambda^{p+1} + o(1) \le \frac{\varepsilon_1}{2}\|\omega_n\|_\lambda^2 + o(1), \end{aligned} \tag{3.10}$$

as $n \to \infty$. It follows from (2.3), (3.8) and (3.10) that

$$\begin{aligned} o(1) = \langle I'_\lambda(\omega_n), \omega_n\rangle_\lambda &\ge \|\omega_n\|_\lambda^2 - \frac{\varepsilon_1\eta_2}{2}\|\omega_n\|_\lambda^2 - \frac{\varepsilon_1}{2}\|\omega_n\|_\lambda^2 + o(1) \\ &= (1 - \frac{\varepsilon_1\eta_2}{2} - \frac{\varepsilon_1}{2})\|\omega_n\|_\lambda^2 + o(1), \end{aligned}$$

which means $\omega_n \to 0$ in E_λ by the value of ε_1. The proof of Lemma 3.2 is complete.

Proof of Theorem 1.1 The proof of this theorem is divided into three steps.

Step 1 There exist ρ, η, $m > 0$ such that $\inf\{I_\lambda(u) : u \in E \text{ with} \|u\|_\lambda = \rho\} > \eta$.

Fix any $2 < p < 2^*$. For any $\varepsilon > 0$, it follows from (f_1) and (f_2) that there exists $C_\varepsilon > 0$ such that

$$|f(x,u)| \le \varepsilon|u| + C_\varepsilon|u|^{p-1} \text{ for all } u \in E_\lambda,$$

and then

$$|F(x,u)| \le \frac{\varepsilon}{2}|u|^2 + \frac{C_\varepsilon}{p}|u|^p \text{ for all } u \in E_\lambda.$$

It follows from Lemma 2.1 that, for all $u \in E_\lambda$,

$$\int_{\mathbb{R}^3} F(x,u)\mathrm{d}x \le \frac{\varepsilon}{2}\int_{\mathbb{R}^3}|u|^2\mathrm{d}x + \frac{C_\varepsilon}{p}\int_{\mathbb{R}^3}|u|^p\mathrm{d}x \le \frac{\varepsilon\eta_2}{2}\|u\|_\lambda^2 + C\|u\|_\lambda^p. \tag{3.11}$$

Combining (2.1) with (3.11), one has

$$I_\lambda(u) \ge \frac{1}{2}\|u\|_\lambda^2 - \frac{\varepsilon\eta_2}{2}\|u\|_\lambda^2 - C\|u\|_\lambda^p = (\frac{1-\varepsilon\eta_2}{2})\|u\|_\lambda^2 - C\|u\|_\lambda^p.$$

Fix any $0 < \varepsilon < \frac{1}{\eta_2}$ and let $\|u\|_\lambda = \rho$ small enough. It is easy to see that there exists $\eta > 0$ such that $I_\lambda(u)|_{\|u\|_\lambda = \rho} \ge \eta > 0$.

Step 2 There exists $v^* \in E$ with $\|v^*\|_\lambda > \rho$ such that $I_\lambda(v^*) < 0$.

By (A_1), in view of the definition of μ^* and $(1-\beta)l > \mu^*$, there exists $v^* \in E$ such that

$$\int_{\mathbb{R}^3} F(x,v^*)\mathrm{d}x > \frac{l}{2},\ b(\int_{\mathbb{R}^3}|\nabla v^*|^2\mathrm{d}x)^2 < 2\beta l$$

and $\mu^* \le \|v^*\|_\lambda^2 < (1-\beta)l$. Then, by (2.1), we obtain

$$\begin{aligned} I_\lambda(v^*) &= \frac{1}{2}\|v^*\|_\lambda^2 + \frac{b}{4}(\int_{\mathbb{R}^3}|\nabla v^*|^2\mathrm{d}x)^2 - \int_{\mathbb{R}^3} F(x,v^*)\mathrm{d}x \\ &\le \frac{1}{2}\|v^*\|_\lambda^2 + \frac{1}{4}\times 2\beta l - \frac{l}{2} = \frac{1}{2}(\|v^*\|_\lambda^2 - (1-\beta)l) < 0. \end{aligned}$$

We choose $\rho > 0$ small enough such that $\|v^*\|_\lambda > \rho$, and Step 2 is proved.

Step 3 I_λ has a nontrivial positive critical point in E_λ.

By Step 1, Step 2 and Lemma 2.2, we see that there is a Cerami sequence $\{u_n\} \subset E$ satisfying (2.3). Thus, it follows from Lemma 3.1 and Lemma 3.2 that there exists a nontrivial $u_0 \in E_\lambda$ such that $I'_\lambda(u_0) = 0$. In what follows, we prove that $u_0 > 0$. By (f_1)-(f_2), we have

$$\begin{aligned} 0 &= \langle I'(u_0), u_0^- \rangle \\ &= \int_{\mathbb{R}^3} (a\nabla u \nabla u_0^- + \lambda V(x) u_0 u_0^-)\mathrm{d}x + b(\int_{\mathbb{R}^3} |\nabla u_0|^2 \mathrm{d}x) \int_{\mathbb{R}^3} \nabla u_0 \nabla u_0^- \mathrm{d}x - \int_{\mathbb{R}^3} f(x, u_0) u_0^- \mathrm{d}x \\ &\geq \int_{\mathbb{R}^3} (a\nabla u \nabla u_0^- + \lambda V(x) u_0 u_0^-)\mathrm{d}x + b(\int_{\mathbb{R}^3} |\nabla u_0|^2 \mathrm{d}x) \int_{\mathbb{R}^3} \nabla u_0 \nabla u_0^- \mathrm{d}x - \int_{\mathbb{R}^3} (l u_0) u_0^- \mathrm{d}x \\ &= -\|u_0^-\|^2 - (b \int_{\mathbb{R}^3} |\nabla u_0|^2 \mathrm{d}x) \int_{\mathbb{R}^3} |\nabla u_0^-|^2 \mathrm{d}x, \end{aligned}$$

where $u_0^- = \max\{-u_0, 0\}$. This shows that $u_0^- = 0$ and $u_0 \geq 0$. From the Harnacks inequality[19], we can infer that $u_0 > 0$ for all $x \in \mathbb{R}^3$. The nonzero critical point of I_λ is the positive solution for the problem (1.1). The proof is completed.

4. The Asymptotically 3-Linear Case

This section is devoted to the proof of Theorem 1.2. We consider the problem (1.1) with the case: asymptotically cubic case at infinity.

Lemma 4.1 Assume that (v_1)-(v_2) and (f_4) hold, then the sequence $\{u_n\}$ defined in (2.3) is bounded in E_λ.

Proof By (f_4) and Lemma 2.1, one has

$$\begin{aligned} \tilde{c} + 1 &\geq I_\lambda(u_n) - \frac{1}{\mu}\langle I'(u_n), u_n \rangle_\lambda \\ &= (\frac{1}{2} - \frac{1}{\mu})\|u_n\|_\lambda^2 + (\frac{b}{4} - \frac{b}{\mu})(\int_{\mathbb{R}^3} |\nabla u_n|^2 \mathrm{d}x)^2 + \int_{\mathbb{R}^3} [\frac{1}{\mu} f(u_n) u_n - F(u_n)]\mathrm{d}x \\ &\geq \frac{\mu - 2}{2\mu}\|u_n\|_\lambda^2 - C_0 \int_{\mathbb{R}^3} u_n^2 \mathrm{d}x \geq (\frac{\mu - 2}{2\mu} - C_0 \eta_2)\|u_n\|_\lambda^2 \end{aligned}$$

for n large enough. This implies that $\{u_n\}$ is bounded in E_λ since $0 < C_0 < \frac{\mu-2}{2\mu\eta_2}$ and $\mu \geq 4$.

Lemma 4.2 Under assumptions (v_1)-(v_2), (f_1), (f_3) and (f_4), any bounded cerami sequence of I_λ defined in (2.3) has a convergent subsequence in E_λ.

Proof Part of the proof is similar to Lemma 3.2. For the reader's convenience, we sketch the proof here. Let $\{u_n\}$ be a bounded sequence defined by (2.3). After a subsequence, we can assume that $u_n \rightharpoonup u$ in E_λ. Set $\omega_n = u_n - u$. By (f_1) and (f_3), for any $0 < \varepsilon_2 < \frac{2}{1+\eta_2}$, there exists a constant $C_{\varepsilon_1} > 0$ such that

$$f(x, \omega_n) \leq \frac{\varepsilon_2}{2}|\omega_n| + C_{\varepsilon_2}|\omega_n|^3$$

and therefore

$$\int_{\mathbb{R}^3} f(x, \omega_n)\omega_n \mathrm{d}x \leq \frac{\varepsilon_2}{2}\int_{\mathbb{R}^3} |\omega_n|^2 \mathrm{d}x + C_{\varepsilon_2}\int_{\mathbb{R}^3} |\omega_n|^4 \mathrm{d}x \leq \frac{\varepsilon_2 \eta_2}{2}\|\omega_n\|_\lambda^2 + C_{\varepsilon_2}\int_{\mathbb{R}^3} |\omega_n|^4 \mathrm{d}x. \quad (4.1)$$

Since $\|\omega_n\|_\lambda$ is uniformly bounded in E_λ and $\lambda \geq 1$, we may fix a $\varepsilon_3 > 0$ such that

$$\frac{\varepsilon_2}{2}\|\omega_n\|_\lambda^2 \geq \varepsilon_3 \|\omega_n\|_\lambda^4 \text{ for all } n \text{ and } \lambda \geq 1. \quad (4.2)$$

By Lemma 2.3 there exist $\tau_{\varepsilon_3} > 0$ and $R_{\varepsilon_3} > 0$ such that

$$C_{\varepsilon_2}\int_{B^C_{R_{\varepsilon_3}}} |\omega_n|^4 \mathrm{d}x \leq \varepsilon_3 \|\omega_n\|_\lambda^4 \text{ for all } \lambda \geq \tau_{\varepsilon_3} > 1.$$

Using the fact that $\omega_n \to 0$ in $L^4_{\text{loc}}(\mathbb{R}^N)$ and (4.2), we have

$$C_{\varepsilon_2}\int_{\mathbb{R}^3}|\omega_n|^4\mathrm{d}x = C_{\varepsilon_2}\Big(\int_{B_{R_{\varepsilon_3}}}|\omega_n|^4\mathrm{d}x + \int_{B^C_{R_{\varepsilon_3}}}|\omega_n|^4\mathrm{d}x\Big)$$
$$\le \varepsilon_3\|\omega_n\|^4_\lambda + o(1) \le \frac{\varepsilon_2}{2}\|\omega_n\|^2_\lambda + o(1) \text{ as } n\to\infty. \quad (4.3)$$

It follows from (2.3), (4.1) and (4.3)that

$$o(1) = \langle I'_\lambda(\omega_n),\omega_n\rangle_\lambda \ge \|\omega_n\|^2_\lambda - \frac{\varepsilon_2\eta_2}{2}\|\omega_n\|^2_\lambda - \frac{\varepsilon_2}{2}\|\omega_n\|^2_\lambda + o(1) = (1-\frac{\varepsilon_2\eta_2}{2}-\frac{\varepsilon_2}{2})\|\omega_n\|^2_\lambda + o(1),$$

which means $\omega_n \to 0$ in E_λ by the value of ε_2. The proof of Lemma 4.2 is complete.

Proof of Theorem 1.2 The proof of this theorem is divided into three steps.

Step 1 There exist ρ, η, $m > 0$ such that $\inf\{I_\lambda(u) : u \in E \text{ with} \|u\|_\lambda = \rho\} > \eta$. For any $\varepsilon > 0$, it follows from (f_1) and (f_3) that there exists $C_\varepsilon > 0$ such that

$$|f(x,u)| \le \varepsilon|u| + C_\varepsilon|u|^3 \text{ for all } u\in E_\lambda, \quad (4.4)$$

and

$$|F(x,u)| \le \frac{\varepsilon}{2}|u|^2 + \frac{C_\varepsilon}{4}|u|^4 \text{ for all } u\in E_\lambda. \quad (4.5)$$

Together (2.1) with Lemma 2.1, one has

$$I_\lambda(u) \ge \frac{1}{2}\|u\|^2_\lambda - \frac{\varepsilon\eta_2}{2}\|u\|^2_\lambda - C\|u\|^4_\lambda \ge \frac{1}{2}\|u\|^2_\lambda - \frac{\varepsilon\eta_2}{2}\|u\|^2_\lambda - C\|u\|^4_\lambda = (\frac{1-\varepsilon\eta_2}{2})\|u\|^2_\lambda - C\|u\|^4_\lambda.$$

Fix any $0 < \varepsilon < \frac{1}{\eta_2}$ and let $\|u\|_\lambda = \rho$ small enough. It is easy to see that there exists $\eta > 0$ such that $I_\lambda(u)|_{\|u\|=\rho} \ge \eta > 0$.

Step 2 There exists $\tilde{v}^* \in E_\lambda$ with $\|\tilde{v}^*\|_\lambda > \rho$ such that $I_\lambda(\tilde{v}^*) < 0$.

By (A_2), In view of the definition of μ^* and $(1-\beta)b > \mu^*$, there exists $\tilde{v}^* \in E$ such that $\int_{\mathbb{R}^3} l(\tilde{v}^*)^4\mathrm{d}x \ge b^2$ and $\mu^* \le \|\tilde{v}^*\|^4_\lambda < (1-\beta)b$. Then, by (2.1) and Fatou's lemma, we obtain

$$\lim_{t\to\infty}\frac{I_\lambda(t\tilde{v}^*)}{t^4} = \lim_{t\to\infty}[\frac{1}{2t^2}\|\tilde{v}^*\|^2_\lambda + \frac{b}{4}(\int_{\mathbb{R}^3}|\nabla\tilde{v}^*|^2\mathrm{d}x)^2 - \int_{\mathbb{R}^3}\frac{F(t\tilde{v}^*)}{(t\tilde{v^*})^4}\tilde{v^*}^4\mathrm{d}x]$$
$$\le \frac{b}{4}(\int_{\mathbb{R}^3}|\nabla\tilde{v}^*|^2\mathrm{d}x)^2 - \frac{1}{4}\int_{\mathbb{R}^3}l(\tilde{v}^*)^4\mathrm{d}x \le -\frac{b^2\beta}{4} < 0.$$

We choose $\rho > 0$ small enough such that $\|\tilde{v}^*\|_\lambda > \rho$, and Step 2 is proved.

Step 3 I_λ has a nontrivial positive critical point in E_λ.

By Step 1, Step 2 and Lemma 2.2, we see that there is a Cerami sequence $\{u_n\} \subset E$ satisfying (2.3). Thus, it follows from Lemma 4.1 and Lemma 4.2 that there exists a nontrivial $u_1 \in E_\lambda$ such that $I'_\lambda(u_1) = 0$. The proof of $u_1 > 0$ is similar to that of of Theorem 1.1, and is omitted here. The proof is complete.

5. Examples

It is not difficult to find examples of functions which satisfy our assumptions.

Example 5.1 Let $V(x) = c(>0)$, $\forall x \in \mathbb{R}^3$. And for any $\tau_0 > 0$, and $l = (a+b+1)\tau_0$, let

$$f(x,t) = \begin{cases}(a+b+1)\tau_0\frac{t^2}{\cos|x|+4+t},\ t\ge 0,\\ 0,\ t<0.\end{cases}$$

Clearly, (v_1)-(v_2) and (f_1) hold. Simple calculation shows that for $u \ge 0$

$$F(x,u) = (a+b+1)\tau_0[\frac{u^2}{2} - (4+\cos|x|)u + (4+\cos|x|)^2\ln(4+\cos|x|+u) - (4+\cos|x|)^2\ln(4+\cos|x|)].$$

Choose some $\tau_0 > 0$, $0 < R \leq 1$ such that $\tau_0 \geq 2R$, and take $\varphi \in C_0^\infty(\mathbb{R}^3, [0,4])$ such that $\varphi(x) = 4$ if $|x| \leq R$, $\varphi(x) = 0$ if $|x| \geq 2R$ and $|\nabla\varphi| \leq \frac{3}{R}$ for all $x \in \mathbb{R}^3$. Then

$$\begin{aligned}&\int_{\mathbb{R}^3} F(x,\varphi)\mathrm{d}x \geq \int_{|x|\leq R} F(x,\varphi)\mathrm{d}x \\ &= (a+b+1)\tau_0 \int_{|x|\leq R} [8 - 4(4+\cos|x|) + (4+\cos|x|)^2 \ln\frac{8+\cos|x|}{4+\cos|x|}] \\ &\geq 8(a+b+1)\tau_0 \int_{|x|\leq R} (2\ln 2 - 1) \geq \frac{32}{3}\pi R^3 (a+b+1)(2ln2-1) \geq \frac{l}{2}. \qquad (5.1)\end{aligned}$$

Taking $\tau_0 = 7056\pi^2$, $R = 1$ and $\beta = \frac{1}{2}$, we have

$$b(\int_{\mathbb{R}^3} |\nabla\varphi|^2 \mathrm{d}x)^2 \leq b(\int_{R\leq|x|\leq 2R} \frac{9}{R^2}\mathrm{d}x)^2 \leq 7056 b\pi^2 R^2 \leq 2\beta l, \qquad (5.2)$$

and in view of the definition of μ^*, we can find constants λ and $V(x) = c$ satisfying

$$\begin{aligned}\int_{\mathbb{R}^3} (a|\nabla\varphi|^2 + \lambda V(x)|\varphi|^2)\mathrm{d}x &\leq \int_{|x|\leq 2R} a\frac{9}{R^2}\mathrm{d}x + \int_{|x|\leq 2R} 16\lambda c \mathrm{d}x \\ &\geq \frac{32}{3}\pi R^3 (\frac{9a}{R^2} + 16\lambda c) \leq \frac{l}{2} = (1-\beta)l. \qquad (5.3)\end{aligned}$$

(5.1)-(5.3) means that (f_2) and (A_1) hold, and Theorem 1.1 applies.

Example 5.2 Let $V(x) = c(>0)$, $\forall x \in \mathbb{R}^3$. And for any $\tau_0 > 0$, and $l = \frac{b}{4}\tau_0$, let

$$f(x,t) = \begin{cases} \frac{b}{4}\tau_0 t^3, \ t \geq 0, \\ 0, \ t < 0. \end{cases}$$

It is easy to see that (v_1)-(v_2), (f_1) and (f_3) hold. Direct calculation shows that for $u \geq 0$, $F(x,u) = \frac{b}{16}\tau_0 u^4$. Then, for any $0 < C_0 < \frac{\mu-2}{2\mu\eta_2}$

$$F(x,u) - \frac{1}{\mu} f(x,u)u = (\frac{1}{4} - \frac{1}{\mu}) l u^4 \leq C_0 u^2,$$

since $\mu \geq 4$, which means (f_3) holds. For any $R > 0$, choose $\tau_0 = \frac{b}{\pi}$ and $\varphi \in C_0^\infty(\mathbb{R}^3, [0,1])$ such that $\varphi(x) = 1$ if $|x| \leq R$, $\varphi(x) = 0$ if $|x| \geq 2R$ and $|\nabla\varphi| \leq \frac{\tilde{C}}{R}$ for all $x \in \mathbb{R}^3$, where $\tilde{C}$ is an arbitrary constant independent of x. Then

$$\int_{\mathbb{R}^3} l|\varphi|^4 \mathrm{d}x \geq \int_{|x|\leq R} l|\varphi|^4 \mathrm{d}x \geq \frac{4l}{3}\pi R^3. \qquad (5.4)$$

Taking $R = 1$, (5.4) implies $\int_{\mathbb{R}^3} l|\varphi|^4 \mathrm{d}x \geq b^2$. Moreover

$$(\int_{\mathbb{R}^3} (a|\nabla\varphi|^2 + \lambda V(x)|\varphi|^2)\mathrm{d}x)^2 \leq (\int_{|x|\leq 2R} a\frac{\tilde{C}^2}{R^2}\mathrm{d}x + \int_{|x|\leq 2R} \lambda c \mathrm{d}x)^2 \leq \frac{1024}{9}\pi^2 (a\tilde{C}^2 + \lambda c R^4)^2 R^2, \qquad (5.5)$$

And choosing $\tilde{C}^2 = \frac{3\sqrt{b}}{128\pi a}$ and $\lambda = \frac{3\sqrt{b}}{128\pi c}$ and $\beta = \frac{1}{2}$, in view of the definition of μ^* and (5.5), we have

$$\mu^* \leq \frac{1024}{9}\pi^2 (a\tilde{C}^2 + \lambda c R^4)^2 R^2 = \frac{b}{4} < \frac{b}{2} = (1-\beta)b,$$

which impies (A_2) holds, and Theorem 1.2 applies.

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具有渐进非线性项的Kirchhoff型方程的正解

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摘要: 本文研究一类非线性Kirchhoff型方程. 非线性项函数$f(x,u)$ 在无穷远处关于u 是渐进线性或渐进非线性的. 若位势函数$V(x)$和非线性项$f(x,u)$满足给定的条件, 本文在工作空间缺乏紧性嵌入的情形下获得该方程正解的存在性.

关键词: Kirchhoff型方程; 渐进非线性; 变分法; 正解