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The Boundedness of Maximal Dyadic Derivative Operator on Dyadic Martingale Hardy Space with Variable Exponents

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Abstract: In this paper, we research dyadic martingale Hardy space with variable exponents. By the characterization of log-Hölder continuity, the Doob's inequality is derived. Moreover, we prove the boundedness of maximal dyadic derivative operator by the atomic decomposition of variable exponent martingale space, which generalizes the conclusion in classical case.
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1. Introduction

It's well known that Musielak-Orlicz spaces and their special case, variable exponent Lebesgue spaces have been got more and more attention in modern analysis and functional space theory. In particular, Musielak-Orlicz spaces were studied by Musielak[11]. Hudzik and Kowalewski[8] studied some geometry properties of Musielak-Orlicz spaces. Kovacik and Rakosnik[9], FAN and ZHAO[6] investigated various properties of variable exponent Lebesgue spaces and Sobolev spaces. Diening[5] and Cruz-Uribe[2−4] proved the boundedness of Hardy-Littlewood maximal operator on variable exponent Lebesgue function spaces $L^{p(\cdot)}(\mathbb{R}^n)$ under the conditions that the exponent $p(\cdot)$ satisfies so called log-Hölder continuity and decay restriction. Many other authors studied its applications to harmonic analysis and some other subjects.

The situation of martingale spaces is different from function spaces. For example, the good-λ inequality method used in classical martingale theory can not be used in variable exponent case. Recently, variable exponent martingale spaces have been paid more attention too. Aoyama[1] proved that, if $p(\cdot)$ is $\mathcal{F}_0$-measurable, then there exists a positive constant c such that $\|M(f)\|_{L_{p(\cdot)}} \leq c\|f\|_{L_{p(\cdot)}}$ for $f \in L_{p(\cdot)}$. Nakai and Sadasue[12] pointed out that

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the inverse is not true, namely, there exists a variable exponent $p(\cdot)$ such that $p(\cdot)$ is not $\mathcal{F}_0$-measurable, and the above inequality holds, under the assumption that every σ-algebra $\mathcal{F}_n$ is generated by countable atoms. HAO[7] established an atomic decomposition of a predictable martingale Hardy space with variable exponents defined on probability spaces. Motivated by them, we research dyadic martingale Hardy space with variable exponents. Firstly, we give the characterization of log-Hölder continuity. By this, the Doob's inequality is derived. Secondly, we give the atomic decomposition of variable exponent martingale space. Moreover, we prove the boundedness of maximal dyadic derivative operator.

2. Preliminaries and Notation

In this paper the unit interval $[0,1)$ and Lebesgue measure μ are to be considered. Through this paper, denote $\mathbb{Z}$, $\mathbb{N}$ the integer set and nonnegative integer set, respectivety. By a dyadic interval we mean one of the form $[k2^{-n},(k+1)2^{-n}) := I_n^k$ for some $k\in\mathbb{N}, 0\le k<2^n$. Given $n\in\mathbb{N}$ and $x\in[0,1)$, let $I_n(x)$ denote the dyadic interval of length 2^{-n} which contains x. The σ-algebra generated by the dyadic intervals $\{I_n(x): x\in[0,1)\}$ will be denoted by $\mathcal{F}_n$, more precisely,

$$\mathcal{F}_n=\sigma\{[k2^{-n},(k+1)2^{-n}) : 0\le k<2^n\}.$$

Obviously, $(\mathcal{F}_n)$ is regular. Define $\mathcal{F}=\sigma(\cup_n\mathcal{F}_n)$ and denote the set of dyadic intervals $\{[k2^{-n},(k+1)2^{-n}), 0\le k<2^n\}$ by $A(\mathcal{F}_n)$ and write $A=\cup_n A(\mathcal{F}_n)$. The conditional expectation operators relative to $\mathcal{F}_n$ are denoted by E_n. For a complex valued martingale $f=(f_n)_{n\ge0}$, denote $df_i=f_i-f_{i-1}$(with convention $df_{-1}=0$) and

$$M_n(f) = \sup_{0\le i\le n}|f_i|, \quad M(f)=\sup_{n\ge0}|f_n|.$$

It is easy to see that, in case $f\in L_1[0,1)$, the maximal function can also be given by

$$M(f)(x)=\sup_n\frac{1}{\mu(I_n(x))}\Big|\int_{I_n(x)}f(t)\mathrm{d}t\Big|.$$

Proposition 2.1[10] If $(\mathcal{F}_n)$ is regular, then for all nonnegative adapted processes $\gamma=(\gamma_n)$ and $\lambda\ge\|\gamma_0\|_\infty$, there exist a constant $c>0$ and a stopping time τ_λ such that

$$\{M(\gamma)>\lambda\}\subset\{\tau_\lambda<\infty\},$$

$$\mu(\{\tau_\lambda<\infty\})\le c\mu(\{M(\gamma)>\lambda\}),$$

$$\sup\nolimits_{n\le\tau_\lambda}\gamma_n=M_{\tau_\lambda}(\gamma)\le\lambda,$$

$$\lambda_2\ge\lambda_1\ge\|\gamma_0\|_\infty\Rightarrow\tau_{\lambda_1}\le\tau_{\lambda_2}.$$

Letting $p(\cdot):[0,1)\to(0,\infty)$ be an $\mathcal{F}$-measurable function, we define

$$p_B^-=\operatorname{ess\,inf}\{p(x):x\in B\},\quad p_B^+=\operatorname{ess\,sup}\{p(x):x\in B\},\ B\subset[0,1).$$

We use the abbreviations $p^+=p^+_{[0,1)}$ and $p^-=p^-_{[0,1)}$. Moreover when $p(\cdot)\ge1$, we also define the conjugate function $p'(\cdot)$ by $\frac{1}{p(x)}+\frac{1}{p'(x)}=1$. Let $\mathcal{P}([0,1))$ denote the collection of all $\mathcal{F}$-measurable functions $p(\cdot):[0,1)\to(0,\infty)$ such that $0<p^-\le p^+<\infty$.

We say that p is log-Hölder continuous if

$$|p(x)-p(y)|\le\frac{c}{-\log d(x,y)}, \tag{2.1}$$

when $d(x,y)\le1/2$.

The Lebesgue space with variable exponent $p(\cdot)$ denoted by $L_{p(\cdot)}$ is defined as the set of all $\mathcal{F}$-measurable functions f satisfying

$$\|f\|_{L_{p(\cdot)}}=\inf\Big\{\lambda>0:\rho_{p(\cdot)}(|f(x)|/\lambda)\le1\Big\}<\infty,$$

where

$$\rho_{p(\cdot)}(f)=\int_0^1 |f(x)|^{p(x)}\mathrm{d}\mu.$$

Then we define the variable exponent martingale Hardy space analogous to classical martingale Hardy space as follows

$$H_{p(\cdot)}=\left\{f=(f_n):M(f)\in L_{p(\cdot)}\right\},\|f\|_{H_{p(\cdot)}}=\|M(f)\|_{p(\cdot)}.$$

We always denote by c some positive constant, it may be different in each appearance.

3. Doob's Inequality on Dyadic Martingale Space

Lemma 3.1 Assume that $p^+<\infty$, the following two conditions are equivalent:

(i) p is log-Hölder continuous;

(ii) For all $t\in[0,1)$ and $n\in\mathbb{N}$, we have $\mu(I_n(t))^{p^-_{I_n(t)}-p^+_{I_n(t)}}\le c$.

Proof (i)⇒(ii) Since $x,y\in I_n(t)$, then $d(x,y)\le 2^{-n}$. Thus

$$p^-_{I_n(t)}-p^+_{I_n(t)}=\operatorname{ess}\inf_{x\in I_n(t)}p(x)-\operatorname{ess}\sup_{x\in I_n(t)}p(x)\ge\frac{c}{\log d(x,y)}\ge\frac{c}{-n\log 2}. \tag{3.1}$$

Moreover, we have

$$\mu(I_n(t))^{p^-_{I_n(t)}-p^+_{I_n(t)}}\le(2^{-n})^{\frac{c}{-n\log 2}}=2^{\frac{c}{\log 2}}. \tag{3.2}$$

(ii)⇒(i) Let $x,y\in[0,1)$ be points with $d(x,y)\le 1/2$, i.e. $y\in I_1(x)$ or $y\in I_1(x+1/2)$ which imply $y\in I_0(x)$. Thus we have

$$d(x,y)^{-|p(x)-p(y)|}\le\mu(I_0(x))^{p^-_{I_0(x)}-p^+_{I_0(x)}}\le c, \tag{3.3}$$

so p is log-Hölder continuous.

Remark 3.1[8] If p is log-Hölder continuous, then we have

$$\mu(I_n)^{1/p^-_{I_n}}\approx\mu(I_n)^{1/p^+_{I_n}}\approx\mu(I_n)^{1/p(x)}\approx\|\chi_{I_n}\|_{p(\cdot)},\quad x\in I_n.$$

Lemma 3.2 Suppose that p is log-Hölder continuous with $1<p^-\le p^+<\infty$. Let $f\in L_{p(\cdot)}$ such that $\|f\|_{p(\cdot)}\le 1/2$. Then for every $x\in[0,1)$ we have

$$(Mf(x))^{p(x)}\le c(M(f^{p(x)})(x)+1).$$

Proof Fix $x\in[0,1)$ and let I_n be a dyadic interval which contains x. By Theorem 2.8 in [9], we have $\|f\|_{p^-_{I_n(x)}}\le 2\|f\|_{p(\cdot)}\le 1$. Using Hölder's inequality for the fixed exponent $p^-_{I_n}$, $\|f\|_{p^-_{I_n}}\le 1$, we find that

$$\begin{aligned}\left(\frac{1}{\mu(I_n(x))}\Big|\int_{I_n(x)}f(t)\mathrm{d}t\Big|\right)^{p(x)}\le&\left(\frac{1}{\mu(I_n(x))}\int_{I_n(x)}|f(t)|^{p^-_{I_n(x)}}\mathrm{d}t\right)^{\frac{p(x)}{p^-_{I_n(x)}}}\\ =&2^{\frac{np(x)}{p^-_{I_n(x)}}}\left(\int_{I_n(x)}|f(t)|^{p^-_{I_n(x)}}\mathrm{d}t\right)^{\frac{p(x)}{p^-_{I_n(x)}}}\\ \le&2^{-n\frac{p^-_{I_n(x)}-p(x)}{p^-_{I_n(x)}}}\left(\frac{1}{\mu(I_n(x))}\int_{I_n(x)}|f(t)|^{p^-_{I_n(x)}}\mathrm{d}t\right)\\ \le&c\left(\frac{1}{\mu(I_n(x))}\int_{I_n(x)}(|f(t)|^{p(x)}+1)\mathrm{d}t\right)\\ \le&c(M(f^{p(x)})(x)+1).\end{aligned} \tag{3.4}$$

Then for every $x\in[0,1)$ we have

$$(Mf(x))^{p(x)}\le c(M(f^{p(x)})(x)+1).$$

Theorem 3.1 Suppose that p is log-Hölder continuous with $1 < p^- \le p^+ < \infty$. Then for any martingale $f \in L_{p(\cdot)}$ we have

$$\| \sup_n |f_n| \|_{p(\cdot)} \le c \|f\|_{p(\cdot)}.$$

Proof We assume that $\|f\|_{p(\cdot)} \le 1/2$ by homogeneity. Since $[0,1) = \cup_{k=0}^{2^n-1} I_n^k$, we have

$$f_n = \sum_{k=0}^{2^n-1} \left(\frac{1}{\mu(I_n^k)} \int_{I_n^k} f(t)\mathrm{d}t \right) \chi_{I_n^k}.$$

Since $I_n^k = I_n(2^{-k})$ and the translation invariance $\frac{1}{\mu(I_n^k)} \int_{I_n^k} f(t)\mathrm{d}t = \frac{1}{\mu(I_n(x))} \int_{I_n(x)} f(s)\mathrm{d}s$, we have

$$\begin{aligned}
\int_{[0,1)} M(f)^{p(x)}\mathrm{d}t \le& \int_{[0,1)} \sup_n \left\{ \sum_{k=0}^{2^n-1} \left(\frac{1}{\mu(I_n^k)} \int_{I_n^k} |f(t)|\mathrm{d}t \right) \chi_{I_n^k} \right\}^{p(x)} \mathrm{d}t \\
\le& \int_{[0,1)} \sup_n \left\{ \sum_{k=0}^{2^n-1} \left(\frac{1}{\mu(I_n(x))} \int_{I_n(x)} |f(t)|\mathrm{d}t \right)^{\frac{p(x)}{p^-}} \chi_{I_n^k} \right\}^{p^-} \mathrm{d}t \\
\le& \int_{[0,1)} \sup_n \left\{ \sum_{k=0}^{2^n-1} \left(c\frac{1}{\mu(I_n(x))} \int_{I_n(x)} |f(t)|^{\frac{p(x)}{p^-}} + 1\mathrm{d}t \right) \chi_{I_n^k} \right\}^{p^-} \mathrm{d}t \\
=& c^{p^-} \|E_n(f^{\frac{p(x)}{p^-}} + 1)\|_{p^-}^{p^-} \le c\|f^{\frac{p(x)}{p^-}} + 1\|_{p^-}^{p^-} \le c.
\end{aligned}$$

4. Dyadic Derivative

First we introduce the Walsh system. Every point $x \in [0,1)$ can be written in the following way:

$$x = \sum_{k=0}^{\infty} \frac{x_k}{2^{k+1}}, 0 \le x_k < 2, x_k \in \mathbb{N}.$$

In case there are two different forms, we choose the one for which $\lim_{k\to\infty} x_k = 0$.

For $x, y \in [0,1)$ we define

$$x \oplus y = \sum_{k=0}^{\infty} \frac{|x_k - y_k|}{2^{k+1}} := d(x,y),$$

which is also called dyadic distance.

The functions

$$r_n(x) := \exp(\pi x_n \sqrt{-1}) \quad (n \in \mathbb{N})$$

are called Rademacher functions.

The product system generated by these functions is the Walsh system:

$$\omega_n(x) := \prod_{k=0}^{\infty} r_k(x)^{n_k},$$

where $n = \sum_{k=0}^{\infty} n_k 2^k, 0 \le n_k < 2$ and $n_k \in \mathbb{N}$.

Recall that the Walsh-Dirichlet kernels

$$D_n := \sum_{k=0}^{n-1} \omega_k$$

satisfy

$$D_{2^n}(x) = \begin{cases} 0, & \text{if } x \in [2^{-n}, 1)\ , \\ 2^n, & \text{if } x \in [0, 2^{-n}). \end{cases} \tag{4.1}$$

Moreover, for any measurable function f, the sequence $\{f * D_{2^n} := f_n\}$ is a martingale sequence.

The Walsh-Fejér kernels are defined with

$$K_n := \frac{1}{n}\sum_{k=1}^{n} D_k$$

and can be estimated by

$$|K_n(x)| \le \sum_{j=0}^{N-1} 2^{j-N} \sum_{i=j}^{N-1} (D_{2^i}(x) + D_{2^i}(x \oplus 2^{-j-1})),$$

where $x \in [0,1), n, N \in \mathbb{N}, 2^{N-1} \le n < 2^N$.

Butzer and Wagner[13] introduced the concept of the dyadic derivative as follows. For each function f defined on $[0,1)$, set

$$(d_n f)(x) := \sum_{j=0}^{n-1} 2^{j-1}(f(x) - f(x \oplus 2^{-j-1})) \quad (x \in [0,1)).$$

Then f is said to be dyadically differentiable at $x \in [0,1)$ if $(d_n f)(x)$ converges as $n \to \infty$. It is verified that every Walsh function is dyadically differentiable and $\lim\limits_{n\to\infty}(d_n\omega_k)(x) = k\omega_k(x)$ for all $x \in [0,1)$ and $k \in \mathbb{N}$. Let W be the function whose Walsh-Fourier coefficients satisfy

$$\hat{W}(k) := \int_0^1 W\omega_k \mathrm{d}\mu = \begin{cases} 1, & \text{if } k = 0\ , \\ 1/k, & \text{if } k \in \mathbb{N}, k \ne 0. \end{cases} \tag{4.2}$$

Set

$$W_K := \sum_{n=2^K}^{\infty} \frac{\omega_n}{n}$$

and let us estimate $|d_n W_K|$.

Lemma 4.1[14] We have for all $n, K \in \mathbb{N}$,

$$|d_n W_K| \le c \sum_{i=1}^{5} F_{K,n}^i,$$

where

$$F_{K,n}^1 = \frac{1}{2^{K-n} \vee 1} \sum_{j=0}^{n-1} \sum_{i=j+1}^{n-1} (n-i) 2^{j-n} D_{2^i}(x \oplus 2^{-j-1}),$$

$$F_{K,n}^2 = \frac{1}{2^{K-n} \vee 1} \sum_{i=0}^{n-1} (n-i) 2^{i-n} D_{2^i}(x),$$

$$F_{K,n}^3 = \sum_{j=0}^{n-1} 2^j \sum_{i=n}^{\infty} (n-i) 2^{-i} \frac{D_{2^i}(x \oplus 2^{-j-1})}{2^{K-i} \vee 1},$$

$$F_{K,n}^4 = \sum_{k=0}^{\infty} 2^{-k} \frac{D_{2^{n+k}}(x)}{2^{K-k-n} \vee 1},$$

$$F_{K,n}^5 = D_{2^K}(x)\chi_{\{n>K\}}.$$

The dyadic integral of $f \in L_1$ is introduced by

$$\mathbf{I}f(x) := (f * W)(x) := \int_0^1 f(t) W(x \oplus t) \mathrm{d}\mu.$$

Notice that $W \in L_2 \subset L_1$, so $\mathbf{I}$ is well-defined on L_1.

We consider the maximal dyadic derivative operator

$$\mathbf{I}^* f := \sup_{n\in\mathbb{N}} |d_n(\mathbf{I}f)|.$$

Definition 4.1 A pair (a, B) of measurable function a and $B \in A(F_n)$ is called a $p(\cdot)$-atom if

1) $E_n(a) = 0$,

2) $\|M(a)\|_\infty \le \|\chi_B\|_{p(\cdot)}^{-1}$,

3) $\{a \ne 0\} \subset B$.

Theorem 4.1 Suppose that p is log-Hölder continuous and $1/2 < p(\cdot) \le 1$. For any $f = (f_n) \in H_{p(\cdot)}$, there exist $(a^B, B)_{B\in A}$ of $p(\cdot)$-atoms and $(u_B)_{B\in A}$ of nonnegative real numbers such that

$$f_n = \sum_{B\in A} u_B E_n a^B, \text{a.e.},$$

and

$$\inf\left\| \sum_{B\in A} \left(\frac{u_B \chi_B}{\|\chi_B\|_{p(\cdot)}} \right) \right\|_{p(\cdot)} \le c\|f\|_{H_{p(\cdot)}}.$$

Proof We apply Proposition 2.1 to the process $(|f_n|)_{n\ge0}$ and $\lambda = 2^k, k \in \mathbb{Z}$. We obtain the stopping time τ_k satisfying $\tau_k \ge \tau_l, k \ge l, \lim_{k\to\infty} \tau_k = \infty$ a.e., $\lim_{k\to\infty} f_n^{\tau_k} = f_n$ a.e. and

$$\lim_{k\to-\infty} |f_n^{\tau_k}| \le \lim_{k\to-\infty} M_{\tau_k}(f_n) \le \lim_{k\to-\infty} 2^k = 0 \quad \text{a.e.}$$

Consequently, f_n can be written as

$$f_n = \sum_{k\in\mathbb{Z}} \sum_{l=0}^{n-1} \chi_{\{\tau_k = l\}} (f_n^{\tau_{k+1}} - f_n^{\tau_k}). \tag{4.3}$$

Since $\{\tau_k = l\} \in \mathcal{F}_l$, we have a family of $B_{i,k,l} \in A(\mathcal{F}_l)$ such that

$$\{\tau_k = l\} = \cup_{i\in\mathbb{N}} B_{i,k,l}.$$

Write $u_{B_{i,k,l}} = 3 \cdot 2^{k+1} \|\chi_{B_{i,k,l}}\|_{p(\cdot)}$. When $u_{B_{i,k,l}} \ne 0$, define

$$a_n^{B_{i,k,l}} = \frac{f_n^{\tau_{k+1}} - f_n^{\tau_k}}{u_{B_{i,k,l}}} \chi_{B_{i,k,l}}.$$

Since $M(f_n^{\tau_k}) \le 2^k, k \in \mathbb{Z}$, the definition of $u_{B_{i,k,l}}$ yields

$$M(a_n^{B_{i,k,l}}) \le \frac{M(f_n^{\tau_{k+1}}) + M(f_n^{\tau_k})}{u_{B_{i,k,l}}} \le \|\chi_{B_{i,k,l}}\|_{p(\cdot)}^{-1}. \tag{4.4}$$

Thus $a^{B_{i,k,l}} = (a_n^{B_{i,k,l}})_{n\ge0}$ is a L_2-bounded martingale. Consequently, there exists a $a^{B_{i,k,l}} \in L_2$ such that $E_n(a^{B_{i,k,l}}) = a_n^{B_{i,k,l}}$. Since $\tau_{k+1} > \tau_k = l$, we have

$$E_l(a^{B_{i,k,l}}) = \frac{f_n^{\min(l,\tau_{k+1})} - f_n^{\min(l,\tau_k)}}{u_{B_{i,k,l}}} = 0 \quad \text{and} \quad \{a^{B_{i,k,l}} \ne 0\} \subset B_{i,k,l}.$$

Therefore $(a^{B_{i,k,l}}, B_{i,k,l})$ is really a $p(\cdot)$-atom.

For any $k \in \mathbb{Z}$, define $D_k = \{\tau_k < \infty\}$. Since τ_k is non-decreasing for each $k \in \mathbb{Z}$, we have $D_k \supset D_{k+1}$. For any given $x \in [0, 1)$, there is $k_0 \in \mathbb{Z}$ such that $x \in D_{k_0} \backslash D_{k_0+1}$. Then

$$\sum_{k\in\mathbb{Z}} 3 \cdot 2^k \chi_{D_k}(x) = \sum_{k=-\infty}^{k_0} 3 \cdot 2^k \chi_{D_k}(x) = 3 \sum_{k=-\infty}^{k_0} 2^k = 3 \cdot 2^{k_0+1} = 2 \sum_{k\in\mathbb{Z}} (3 \cdot 2^k) \chi_{D_k \backslash D_{k+1}}(x). \tag{4.5}$$

Therefore, for any $Q \in A$ we have

$$\begin{aligned}\left\|\sum_{B_{i,k,l}\subset Q}\left(\frac{\chi_{B_{i,k,l}}u_{B_{i,k,l}}}{\|\chi_{B_{i,k,l}}\|_{p(\cdot)}}\right)\right\|_{p(\cdot)} &\le c\left\|\sum_{k\in\mathbb{Z}}(3\cdot 2^k)\chi_{D_k}(x)\right\|_{p(\cdot)} \le c\left\|\sum_{k\in\mathbb{Z}}(3\cdot 2^k)\chi_{D_k\setminus D_{k+1}}\right\|_{p(\cdot)}\\ &\le c\inf\left\{\lambda>0:\int_0^1\left(\frac{\sum_{k\in\mathbb{Z}}(3\cdot 2^k)\chi_{D_k\setminus D_{k+1}}}{\lambda}\right)^{p(\cdot)}\mathrm{d}\mu\le 1\right\}\\ &\le c\inf\left\{\lambda>0:\sum_{k\in\mathbb{Z}}\int_{D_k\setminus D_{k+1}}\left(\frac{(3\cdot 2^k)}{\lambda}\right)^{p(\cdot)}\mathrm{d}\mu\le 1\right\}\\ &\le c\inf\{\lambda>0:\int_\Omega\left(\frac{M(f)}{\lambda}\right)^{p(\cdot)}\mathrm{d}\mu\le 1\}\\ &\le c\|f\|_{H_{p(\cdot)}}. \qquad (4.6)\end{aligned}$$

Thus we complete the proof.

Theorem 4.2 Suppose that the operator T is sublinear and for each $p_0 < p(\cdot) \le 1$, there exists a constant $c > 0$ such that

$$\int_{[0,1)\setminus B}|Ta|^{p(\cdot)}\mathrm{d}\mu \le c \qquad (4.7)$$

for every $p(\cdot)$-atom (a, B). If T is bounded from L_∞ to L_∞, then

$$\|Tf\|_{p(\cdot)} \le c\|f\|_{H_{p(\cdot)}} \quad (f \in H_{p(\cdot)})$$

Proof Supposing (a, B) is a $p(\cdot)$-atom and L_∞ boundedness of T, we obtain

$$\begin{aligned}\int_0^1|Ta|^{p(\cdot)}\mathrm{d}\mu &= \int_B|Ta|^{p(\cdot)}\mathrm{d}\mu + \int_{[0,1)\setminus B}|Ta|^{p(\cdot)}\mathrm{d}\mu\\ &\le c\|T\|_\infty\int_B|a|^{p(\cdot)}\mathrm{d}\mu + c \le c\|T\|_\infty\|a\|_\infty^{p_B^+}\mu(B)+c\\ &\le c\|T\|_\infty(\|\chi_B\|_{p(\cdot)}^{-1})^{p_B^+}\mu(B)+c \le c\|T\|_\infty(\mu(B)^{\frac{-1}{p_B^+}})^{p_B^+}\mu(B)+c\\ &= c\|T\|_\infty\mu(B)^{-1}\mu(B)+c\le c.\end{aligned}$$

Thus for every $f \in H_{p(\cdot)}$, we have

$$\begin{aligned}\|Tf\|_{p(\cdot)} &\le \left\|\sum_{B\in A}u_B T(E_n a^B)\right\|_{p(\cdot)} \le \sum_{B\in A}u_B\|T(E_n a^B)\|_{p(\cdot)}\\ &\le c\sum_{B\in A}u_B \le c\sum_{B\in A}\left\|\frac{u_B\chi_B}{\|\chi_B\|_{p(\cdot)}}\right\|_{p(\cdot)} \le c\|f\|_{H_{p(\cdot)}}. \qquad (4.8)\end{aligned}$$

Theorem 4.3 Suppose that p is log-Hölder continuous and $1/2 < p^- \le p^+ \le 1$. Then for any $f \in H_{p(\cdot)}$ we have

$$\|\mathbf{I}^*f\|_{p(\cdot)} \le c\|f\|_{H_{p(\cdot)}}.$$

Proof By Theorem 4.2, the proof of Theorem 4.3 will be completed if we show that the operator $\mathbf{I}^*$ satisfies (4.7) and is bounded from L_∞ to L_∞.

Obviously,

$$d_n(\mathbf{I}f(x)) = d_n(\int_0^1 f(t)W(x\oplus t)\mathrm{d}\mu) = \int_0^1 f(t)(d_nW(x\oplus t))\mathrm{d}\mu \qquad (4.9)$$

and

$$\|d_n(\mathbf{I}f(x))\|_\infty \le \|f\|_\infty\int_0^1|d_nW(x\oplus t)|\mathrm{d}\mu$$

Since $\|D_{2^n}\|_1 = 1$, we can show that $\|F^i_{0,n}\|_1 \le c$ for $i = 1, 2, \cdots 5$. From this it follows that $\|d_n W\|_1 \le c$ for all $n \in \mathbb{N}$, which verifies that $\mathbf{I}^*$ is bounded on L_∞.

If $a \equiv 1$, then the left hand side of (4.9) is zero. Let $a \ne 1$ be an arbitrary $p(\cdot)$-atom with support B and $\mu(B) = 2^{-\tau}$. Without loss of generality, we may suppose that $B = [0, 2^{-\tau})$.

For $k < 2^\tau$, ω_k is constant on B and so

$$\mathbf{I}a(x) = \int_0^1 a(t) W_\tau(x \oplus t)\mathrm{d}\mu.$$

Moreover

$$|d_n(\mathbf{I}a)(x)| \le \int_0^1 |a(t) W_\tau(x \oplus t)|\mathrm{d}\mu \le |a| * (d_n W_\tau)(x).$$

To verify (4.7) we have to investigate the integral of $(\sup_n |a| * F^i_{\tau,n}(x))^{p(\cdot)}$ over $[0,1)\backslash B$ for all $i = 1, 2, \cdots, 5$. Since $x \in [0,1)\backslash B$, we have

$$\int_B D_{2^i}(x \oplus t \oplus 2^{-j-1})\mathrm{d}\mu = 2^{i-\tau}\chi_{[2^{-j-1}, 2^{-j-1}\oplus 2^{-i})}(x) \quad \text{if} \quad j < i \le \tau - 1,$$

$$\int_B D_{2^i}(x \oplus t)\mathrm{d}\mu = 2^{i-\tau}\chi_{[2^{-\tau}, 2^{-i})}(x) \quad \text{if} \quad i \in \mathbb{N},$$

$$\int_B D_{2^i}(x \oplus t \oplus 2^{-j-1})\mathrm{d}\mu = \chi_{[2^{-j-1}, 2^{-j-1}\oplus 2^{-\tau})}(x) \quad \text{if} \quad i \ge \tau.$$

By the definition of an atom,

$$\begin{aligned}\sup_{n \le \tau} |a| * F^1_{\tau,n}(x) &= \frac{1}{2^{\tau-n}} \sum_{j=0}^{n-1} \sum_{i=j+1}^{n-1} (n-i) 2^{j-n} \int_B |a(t)| D_{2^i}(x \oplus t \oplus 2^{-j-1})\mathrm{d}\mu \\ &\le \frac{\|a\|_\infty}{2^{\tau-n}} \sum_{j=0}^{n-1} \sum_{i=j+1}^{n-1} (n-i) 2^{j-n} \int_B D_{2^i}(x \oplus t \oplus 2^{-j-1})\mathrm{d}\mu \\ &\le \frac{\|a\|_\infty}{2^{2\tau}} \sum_{j=0}^{\tau-1} \sum_{i=j+1}^{\tau-1} (\tau - i) 2^{i+j} \chi_{[2^{-j-1}, 2^{-j-1}\oplus 2^{-i})}(x).\end{aligned}$$

Integrating over $[0,1)\backslash B$, we obtain

$$\begin{aligned}&\int_{[0,1)\backslash B} (\sup_{n\le\tau} |a| * F^1_{\tau,n}(x))^{p(\cdot)}\mathrm{d}\mu \\ &\le \frac{\|a\|_\infty^{p^+}}{2^{2\tau p^-}} \sum_{j=0}^{\tau-1} 2^{jp^-} \sum_{i=j+1}^{\tau-1} (\tau-i)^{p^-} 2^{ip^-} \|\chi_{[2^{-j-1}, 2^{-j-1}\oplus 2^{-i})}\|_{p(\cdot)} \\ &\le \frac{\|a\|_\infty^{p^+}}{2^{2\tau p^-}} \sum_{j=0}^{\tau-1} 2^{jp^-} \sum_{i=j+1}^{\tau-1} (\tau-i)^{p^-} 2^{i(p^- - 1)} \le \frac{\|a\|_\infty^{p^+}}{2^{2\tau p^-}} \sum_{j=0}^{\tau-1} 2^{2jp^- - j} (\tau - j)^{(2p^- + 1)/2} \\ &\le \frac{\|\chi_B\|_{p(\cdot)}^{-p^+}}{2^{2\tau p^-}} \sum_{j=0}^{\tau-1} 2^{2jp^- - 1}(\tau - j)^{(2p^- + 1)/2} \le \sum_{j=0}^{\tau-1} 2^{2jp^- - j - 2\tau p^- + \tau p^+} (\tau - j)^{(2p^+ + 1)/2} \\ &\le \sum_{j=0}^{\tau-1} 2^{2jp^- - j - 2\tau p^- + \tau} (\tau - j)^{(2p^+ + 1)/2} = \sum_{j=0}^{\tau-1} 2^{(j-\tau)(2p^- - 1)} (\tau - j)^{(2p^+ + 1)/2} \le c. \end{aligned} \tag{4.10}$$

Let $n > \tau$. If $j > \tau$ and $x \notin B$, then $x \oplus 2^{-j-1} \notin B$. Consequently, for $x \notin B$, $i \ge j > \tau$ we have

$$\int_B |a(t)| D_{2^i}(x \oplus t)\mathrm{d}\mu = \int_B |a(t)| D_{2^i}(x + t \oplus 2^{-j-1})\mathrm{d}\mu = 0.$$

Thus

$$\begin{aligned}\sup_{n>\tau}|a|*F^1_{\tau,n}\le&\sup_{n>\tau}\sum_{j=0}^{\tau-1}\sum_{i=j+1}^{\tau-1}(n-i)2^{j-n}\int_B|a(t)|D_{2^i}(x+t\oplus 2^{-j-1})\mathrm{d}\mu\\&+\sup_{n>\tau}\sum_{j=0}^{\tau-1}\sum_{i=\tau}^{n-1}(n-i)2^{j-n}\int_B|a(t)|D_{2^i}(x+t\oplus 2^{-j-1})\mathrm{d}\mu\\\le&\|a\|_\infty\sup_{n>\tau}\sum_{j=0}^{\tau-1}\sum_{i=j+1}^{\tau-1}(n-i)2^{j-n}2^{i-\tau}\chi_{[2^{-j-1},2^{-j-1}\oplus 2^{i-})}\\&+\|a\|_\infty\sup_{n>\tau}\sum_{j=0}^{\tau-1}\sum_{i=\tau}^{n-1}(n-i)2^{i-n}2^{j-i}\chi_{[2^{-j-1},2^{-j-1}\oplus 2^{-\tau})}\\\le&\|a\|_\infty\sup_{n>\tau}\sum_{j=0}^{\tau-1}\sum_{i=j+1}^{\tau-1}(\tau-i)2^{j-\tau}2^{i-\tau}\chi_{[2^{-j-1},2^{-j-1}\oplus 2^{i-})}\\&+\frac{1}{2}\|a\|_\infty\sup_{n>\tau}\sum_{j=0}^{\tau-1}\sum_{i=\tau}^{\infty}2^{j-i}\chi_{[2^{-j-1},2^{-j-1}\oplus 2^{-\tau})}.\end{aligned}\tag{4.11}$$

We can see as above that

$$\begin{aligned}&\int_{[0,1)\backslash B}(\sup_{n>\tau}|a|*F^1_{\tau,n}(x))^{p(\cdot)}\mathrm{d}\mu\\\le&c\|a\|_\infty^{p^+}2^{-2\tau p^-}\sum_{j=0}^{\tau-1}2^{jp^-}\sum_{i=j+1}^{\tau-1}(\tau-i)^{p^-}2^{i(p^--1)}+\|a\|_\infty^{p^+}\sum_{j=0}^{\tau-1}2^{jp^-}\sum_{i=\tau}^{\infty}2^{-ip^--\tau}\\\le&c2^{\tau p^+}2^{-2\tau p^-}\sum_{j=0}^{\tau-1}2^{jp^-}\sum_{i=j+1}^{\tau-1}(\tau-i)^{p^-}2^{i(p^--1)}+2^{\tau p^+}\sum_{j=0}^{\tau-1}2^{jp^-}\sum_{i=\tau}^{\infty}2^{-ip^--\tau}\\\le&c.\end{aligned}\tag{4.12}$$

By (4.11) and (4.12) we have

$$\int_{[0,1)\backslash B}(\sup_{n\in\mathbb{N}}|a|*F^1_{\tau,n}(x))^{p(\cdot)}\mathrm{d}\mu\le c.$$

Similarly, we can also estimate the integral $(\sup_n|a|*F^i_{\tau,n}(x))^{p(\cdot)}$ over $[0,1)\backslash B$ for all $i=2,\cdots,5$. We omit it. By Theorem 4.2, we complete the proof.

Corollary 4.1 Suppose that p is log-Hölder continuous and $1/2<p^-\le p^+\le 1$. If $f\in H_{p(\cdot)}$ is of mean zero then

$$\lim_{n\to\infty}d_n(\mathbf{I}f)=f\quad\text{a.e.}$$

References:

[1] AOYAMA H. Lebesgue spaces with variable exponent on a probability space[J]. Hiroshima Math., 2009, 39: 207-216.

[2] CRUZ-URIBE D, FIORENZA A, MARTRLL J M, et al. The boundedness of classical operators on variable L_p spaces[J]. Ann. Acad. Sci. Fenn. Math., 2006, 31: 239-264.

[3] CRUZ-URIBE D, FIORENZA A, NEUGEBAUER C J. The maximal function on variable $L_{p(\cdot)}$ spaces[J]. Ann. Acad. Sci. Fenn. Math., 2003, 28: 223-238; 2004, 29: 247-249.

[4] CRUZ-URIBE D, FIORENZA A, NEUGEBAUER C J. Weighted norm inequalities for the maximal operator on variable Lebesgue spaces[J]. J. Math. Anal. Appl., 2012, 394: 744-760.

[5] DIENING L. Maximal function on generalized Lebesgue spaces $L_{p(\cdot)}$[J]. Math. Inequal. Appl., 2004, 7: 245-253.
[6] FAN X, ZHAO D. On the spaces $L_{p(x)}$ and $W_{m,p(x)}$[J]. J. Math. Anal. Appl., 2011, 263: 424-446.
[7] HAO Z W. Atomic decomposition of predictable martingale Hardy spaces with variable exponents[J]. Czechoslovak Mathematical Journal, 2015, 65: 1033-1045.
[8] HUDZIK H, KOWALEWSKI W. On some global and local geometry properties of Musielak-Orlicz spaces[J]. Publ Math Debrecen, 2005, 67: 41-64.
[9] KOVACIK O, RAKOSNIK J. On spaces $L_p(x)$ and $W_{k,p}(x)$[J]. Czechoslovak Math. J., 1991, 41: 592-618.
[10] LONG R L. Martingale Spaces and Inequalities[M]. Beijing: Peking Univ. Press, 1993.
[11] MUSIELAK J. Orlicz Spaces and Modular Spaces[M]. Berlin: Springer-Verlag, 1983.
[12] NAKAI E, SADASUE G. Maximal function on generalized martingale Lebesgue spaces with variable exponent[J]. Statis. Probab. Letters, 2013, 83: 2167-2171.
[13] BUTZER P L, WAGNER H J. Walsh series and the concept of a derivative[J]. Applic. Anal., 1973, 3: 29-46.
[14] WEISZ F. Martingale Hardy spaces and the dyadic derivative[J]. Analysis Mathematica, 1998, 24: 59-77.

变指数二进鞅空间上二进求导极大算子有界性研究

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摘要：本文研究变指数二进鞅空间理论. 借助于对数Hölder连续的等价刻画, 得到Doob不等式.借助于变指数鞅空间的原子分解理论, 证明二进求导极大算子的有界性, 上述结果推广了经典情形结论.

关键词：鞅; 变指数; 二进导数; 原子分解