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凸二次半定规划一个新的原始对偶路径跟踪算法

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摘要: 本文提出求解凸二次半定规划的一个新的原始对偶路径跟踪算法. 在每次迭代中, 通过求解一个线性方程组产生搜索方向. 在一定条件下证明算法产生的迭代点列落在中心路径的邻域内, 且算法至多经 $\mathcal{O}(n|\log\epsilon|)$次迭代可得到一个$\epsilon$-最优解.

关键词: 凸二次半定规划; 原始对偶路径跟踪算法; 中心路径; 迭代复杂度
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1. 引言

本文考虑如下二次半定规划 (简记为QSDP)

$$\begin{cases}\min & f(X)=\frac{1}{2}\langle\varphi(X),X\rangle+\langle C,X\rangle,\\ \text{s.t.} & \langle A_i,X\rangle=b_i,\ i=1,\cdots,m,\\ & X^{\mathrm{T}}=X\succeq 0,\end{cases}\tag{1.1}$$

其中函数 $\varphi:\mathbb{S}^n\to\mathbb{S}^n$ 为自伴随线性算子, $\mathbb{S}^n$表示 n 阶实对称矩阵空间. $A_i(i=1,\cdots,m),C$ 和 X 都是 $n\times n$ 阶实对称矩阵, $b\in\mathbb{R}^m$. $\langle\cdot\rangle$ 表示矩阵的内积, 即 $\forall A\in\mathbb{R}^{p\times q},B\in\mathbb{R}^{p\times q}$, $\langle A,B\rangle=\mathrm{tr}(A^{\mathrm{T}}B)$. $X\succeq 0$ 和 $X\succ 0$ 分别表示矩阵 X 是对称半正定矩阵和对称正定矩阵.

凸二次半定规划是半定规划[1−4]的推广, 其在证券, 金融, 最优控制等领域中有着广泛的应用, 因此对凸二次半定规划的研究受到学者们的关注, 并已取得一批研究成果(见文[5–10]). 文 [5]提出了一个预估校正算法, 该算法至多经 $\mathcal{O}(\sqrt{n}\log(1/\epsilon))$ 次迭代可得到一个 ϵ 最优解. 文 [8] 提出了一个非精确原始对偶路径跟踪算法. 文 [9] 提出了一个基于参数核函数的原始-对偶内点算法.

受线性半定规划的 HKM方向启发, 本文提出一个基于HKM方向的新原始-对偶路径跟踪算法. 在每次迭代中, 算法通过求解一个线性方程组产生 HKM搜索方向. 在一定条件下证明了算法产生的迭代点列落在中心路径的邻域内, 且算法至多经 $\mathcal{O}(n|\log\epsilon|)$ 次迭代可得到一个 ϵ-最优解.

在一定的假设条件下, 分析了该算法的迭代复杂度.

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QSDP(1.1)的对偶问题(简记为QSDD)为

$$\begin{cases} \max & D(X,y) = -\frac{1}{2}\langle \varphi(X), X\rangle + b^{\mathrm{T}}y, \\ \text{s.t.} & \sum\limits_{i=1}^{m} y_i A_i + Z = C + \varphi(X), \\ & Z \succeq 0, \end{cases} \tag{1.2}$$

其中 $y \in \mathbb{R}^m$, $Z \in \mathbb{S}^n$.

分别记 QSDP(1.1)和 QSDD(1.2)的可行域为 $\mathbb{F}_P$和 $\mathbb{F}_D$, 并记 $\mathbb{F}_P^0$ 和 $\mathbb{F}_D^0$ 分别为 $\mathbb{F}_P$ 和 $\mathbb{F}_D$ 的严格内部, 即

$$\mathbb{F}_P^0 := \{X \in \mathbb{S}^n \mid \langle A_i, X\rangle = b_i, i = 1, \cdots, m,\ X \succ 0\},$$
$$\mathbb{F}_D^0 := \{(X, y, Z) \in \mathbb{S}^n \times \mathbb{R}^m \times \mathbb{S}^n \mid C + \varphi(X) = \sum_{i=1}^{m} y_i A_i + Z,\ Z \succ 0\}.$$

对于任意的可行点 X和 (X,y,Z), 则对偶间隙为

$$\begin{aligned} f(X) - D(X,y) &= \frac{1}{2}\langle \varphi(X), X\rangle + \langle C, X\rangle - (b^{\mathrm{T}}y - \frac{1}{2}\langle \varphi(X), X\rangle) \\ &= \langle C, X\rangle + \langle \varphi(X), X\rangle - b^{\mathrm{T}}y \\ &= \langle C, X\rangle + \langle \varphi(X), X\rangle - \sum_{i=1}^{m}\langle A_i, X\rangle y_i \\ &= \left\langle C + \varphi(X) - \sum_{i=1}^{m} y_i A_i\ ,\ X\right\rangle \\ &= \langle Z, X\rangle . \end{aligned}$$

本文需作如下基本假设:

假设(A_1) 线性算子 $\varphi(X)$ 是对称半正定的, 即满足

$$\varphi(X) = \varphi(X)^{\mathrm{T}},\ \langle \varphi(X), X\rangle \ge 0,\ \ \forall\ X \in \mathbb{R}^{n\times n};$$

假设(A_2) Slater约束规格成立, 即存在 $X \succ 0,\ Z \succ 0,\ y \in \mathbb{R}^m$, 使得 $X \in \mathbb{F}_P^0, (X,y,Z) \in \mathbb{F}_D^0$;

假设(A_3) 矩阵 $A_1, A_2, \cdots, A_m$ 线性无关.

2. HKM 方向

称满足如下方程组

$$\begin{cases} \langle A_i, X\rangle = b_i, \quad i = 1, \cdots, m, & (2.1\text{a}) \\ \sum\limits_{i=1}^{m} y_i A_i + Z = C + \varphi(X), & (2.1\text{b}) \\ XZ = \sigma\mu I & (2.1\text{c}) \end{cases}$$

的点(X,y,Z) 构成的集合为中心路径, 其中 $\sigma \in [0, \frac{1}{2}]$ 为中心参数, $\mu = \frac{\langle X,Z\rangle}{n}$, I 为 n 阶单位矩阵.

设当前迭代点为 (X,y,Z), 且 $X \in \mathbb{F}_P^0$, $(X,y,Z) \in \mathbb{F}_D^0$. 用一步 Newton 法求解方程组(2.1), 得到下面方程组:

$$\begin{cases} \langle A_i, \Delta X\rangle = 0 \quad i = 1, \cdots, m, & (2.2\text{a}) \\ \sum\limits_{i=1}^{m} \Delta y_i A_i + \Delta Z - \varphi(\Delta X) = 0, & (2.2\text{b}) \\ \Delta X Z + X \Delta Z = \sigma\mu I - XZ. & (2.2\text{c}) \end{cases}$$

类似线性半定规划 (见文[2]), HKM 搜索方向 $(\Delta X, \Delta y, \Delta Z)$ 由下列线性方程组确定:

$$\begin{cases} \langle A_i, \Delta X\rangle = 0, \quad i = 1, \cdots, m, & \text{(2.3a)} \\ \sum_{i=1}^{m} \Delta y_i A_i + \Delta Z - \varphi(\Delta X) = 0, & \text{(2.3b)} \\ X^{-\frac{1}{2}}(X\Delta Z + \Delta X Z)X^{\frac{1}{2}} + X^{\frac{1}{2}}(\Delta Z X + Z\Delta X)X^{-\frac{1}{2}} = 2(\sigma\mu I - X^{\frac{1}{2}} Z X^{\frac{1}{2}}). & \text{(2.3c)} \end{cases}$$

下面引理给出了线性方程组 (2.3) 的解 $(\Delta X, \Delta y, \Delta Z)$ 的性质.

引理 2.1　假设(A_1)-(A_3)成立, $X \in \mathbb{F}_P^0$, $(X, y, Z) \in \mathbb{F}_D^0$, $(\Delta X, \Delta y, \Delta Z)$ 是线性方程组 (2.3) 的解, 则下面结论成立:

(i) $\langle \Delta X, \Delta Z\rangle \geq 0$;

(ii) $\langle X, \Delta Z\rangle + \langle Z, \Delta X\rangle = \mathrm{Tr}(\sigma\mu I - X^{\frac{1}{2}} Z X^{\frac{1}{2}})$;

(iii) 对于任意的 $\sigma \in [0, \frac{1}{2}]$, $\mu = \langle X, Z\rangle / n$, $\alpha > 0$, 则

$$\langle X + \alpha\Delta X, Z + \alpha\Delta Z\rangle = (1 - \alpha + \alpha\sigma)\langle X, Z\rangle + \alpha^2 \langle \Delta X, \Delta Z\rangle. \tag{2.4}$$

证　(i) 根据方程(2.3b)和(2.3a)可得

$$\begin{aligned}\langle \Delta X, \Delta Z\rangle = \left\langle \Delta X,\ (\varphi(\Delta X) - \sum_{i=1}^{m} \Delta y_i A_i)\right\rangle &= \langle \Delta X, \varphi(\Delta X)\rangle - \sum_{i=1}^{m} \Delta y_i \langle A_i, \Delta X\rangle \\ &= \langle \Delta X, \varphi(\Delta X)\rangle \geq 0,\end{aligned}$$

其中最后一个不等式由假设(A_1)得到. 于是结论 (i) 成立.

(ii) 由(2.3c)式得

$$\begin{aligned}2\mathrm{Tr}(\sigma\mu I - X^{\frac{1}{2}} Z X^{\frac{1}{2}}) =& \mathrm{Tr}(X^{-\frac{1}{2}}(X\Delta Z + \Delta X Z)X^{\frac{1}{2}}) + \mathrm{Tr}(X^{\frac{1}{2}}(\Delta Z X + Z\Delta X)X^{-\frac{1}{2}}) \\ =& \mathrm{Tr}(X\Delta Z + \Delta X Z) + \mathrm{Tr}(\Delta Z X + Z\Delta X) \\ =& 2\mathrm{Tr}(X\Delta Z + \Delta X Z) \\ =& 2(\langle X, \Delta Z\rangle + \langle \Delta X, Z\rangle),\end{aligned}$$

即结论 (ii) 成立.

(iii) 根据结论 (ii) 以及 $\mu = \langle X, Z\rangle / n$, 得

$$\begin{aligned}\langle X + \alpha\Delta X, Z + \alpha\Delta Z\rangle &= \langle X, Z\rangle + \alpha(\langle X, \Delta Z\rangle + \langle \Delta X, Z\rangle) + \alpha^2 \langle \Delta X, \Delta Z\rangle \\ &= \langle X, Z\rangle + \alpha\mathrm{Tr}(\sigma\mu I - X^{\frac{1}{2}} Z X^{\frac{1}{2}}) + \alpha^2 \langle \Delta X, \Delta Z\rangle \\ &= \langle X, Z\rangle + \alpha n\sigma\mu - \alpha\langle X, Z\rangle + \alpha^2 \langle \Delta X, \Delta Z\rangle \\ &= \langle X, Z\rangle + \alpha\sigma\langle X, Z\rangle - \alpha\langle X, Z\rangle + \alpha^2 \langle \Delta X, \Delta Z\rangle \\ &= (1 - \alpha + \alpha\sigma)\langle X, Z\rangle + \alpha^2 \langle \Delta X, \Delta Z\rangle,\end{aligned}$$

即结论 (iii) 成立.

3. 长步原始对偶路径跟踪算法

本节将给出基于HKM方向 $(\Delta X, \Delta y, \Delta Z)$的长步原始对偶路径跟踪算法的具体步骤, 然后分析算法的性质.

算法 3.1

步 0(初始步)　给定初始点 $X^0 \in \mathbb{F}_P^0$, $(X^0, y^0, Z^0) \in \mathbb{F}_D^0$, 令 $\mu_0 = \frac{\langle X^0, Z^0\rangle}{n}$, $\varepsilon > 0, k := 0$;

步 1　记$(X, y, Z) = (X^k, y^k, Z^k)$, $\mu = \frac{\langle X, Z\rangle}{n}$;

步 2　如果$\langle X, Z\rangle < \epsilon$, 则算法停止; 否则进入步 3;

步 3　求解方程组 (2.3), 得到 $(\Delta X, \Delta y, \Delta Z)$;

步 4 选择合适的步长 $\alpha=\alpha_k\ge 0$, 令 $(X^{k+1},y^{k+1},Z^{k+1})=(X^k,y^k,Z^k)+\alpha(\Delta X,\Delta y,\Delta Z)$, $k:=k+1$, 转回步 1.

本文定义如下中心路径邻域:

$$\mathbb{N}_F(\gamma,\Gamma):=\{(X,y,Z)\in\mathbb{F}_D^0 \mid X\in\mathbb{F}_P^0,\ (1-\gamma)\mu\le\lambda_i(XZ)\le(1+\Gamma)\mu,\ i=1,\cdots,n\},\tag{3.1}$$

其中 $\mu=\langle X,Z\rangle/n, \gamma\in[0,1), 0\le\Gamma<\infty$.

在下面的讨论中, 为简便起见, 略去指标 k, 例: 把 (X^k,y^k,Z^k) 简记为 (X,y,Z).

引理 3.1 假设(A_1)-(A_3)成立, $X\in\mathbb{F}_P^0$, $(X,y,Z)\in\mathbb{F}_D^0$, $(\Delta X,\Delta y,\Delta Z)$是线性方程组(2.3) 的解, 对于任意给定的 $\alpha>0$, 令

$$(X(\alpha),y(\alpha),Z(\alpha))=(X,y,Z)+\alpha(\Delta X,\Delta y,\Delta Z),\tag{3.2}$$

$$\mu(\alpha)=\frac{\langle X(\alpha),Z(\alpha)\rangle}{n},\tag{3.3}$$

$$Q(\alpha)=X^{-\frac12}(X(\alpha)Z(\alpha)-\mu(\alpha)I)X^{\frac12},\tag{3.4}$$

则

$$Q(\alpha)+Q(\alpha)^{\mathrm T}=2(1-\alpha)(X^{\frac12}ZX^{\frac12}-\mu I)-\frac{2\alpha^2}{n}\langle\Delta X,\Delta Z\rangle I+\alpha^2(X^{-\frac12}\Delta X\Delta ZX^{\frac12}+X^{\frac12}\Delta Z\Delta XX^{-\frac12}).\tag{3.5}$$

证 对于任意给定的 $\alpha>0$, 根据引理 2.1(iii)及 (3.3), 有

$$\mu(\alpha)=\frac{\langle X(\alpha),Z(\alpha)\rangle}{n}=\frac{\langle X+\alpha\Delta X,Z+\alpha\Delta Z\rangle}{n}=(1-\alpha+\alpha\sigma)\mu+\frac{\alpha^2}{n}\langle\Delta X,\Delta Z\rangle,\tag{3.6}$$

因此, 有

$$\begin{aligned}&X(\alpha)Z(\alpha)-\mu(\alpha)I\\=&(X+\alpha\Delta X)(Z+\alpha\Delta Z)-\left((1-\alpha+\alpha\sigma)\mu+\frac{\alpha^2}{n}\langle\Delta X,\Delta Z\rangle\right)I\\=&XZ+\alpha X\Delta Z+\alpha\Delta XZ+\alpha^2\Delta X\Delta Z-(1-\alpha)\mu I-\alpha\sigma\mu I-\frac{\alpha^2}{n}\langle\Delta X,\Delta Z\rangle I\\=&(1-\alpha)(XZ-\mu I)+\alpha(XZ-\sigma\mu I)+\alpha(X\Delta Z+\Delta XZ)\\&+\alpha^2\Delta X\Delta Z-\frac{\alpha^2}{n}\langle\Delta X,\Delta Z\rangle I.\end{aligned}\tag{3.7}$$

于是由(3.4)并结合(3.7), 得

$$\begin{aligned}&Q(\alpha)+Q(\alpha)^{\mathrm T}\\=&X^{-\frac12}((1-\alpha)(XZ-\mu I)+\alpha(XZ-\sigma\mu I)+\alpha(X\Delta Z+\Delta XZ)+\alpha^2\Delta X\Delta Z\\&-\frac{\alpha^2}{n}\langle\Delta X,\Delta Z\rangle I)X^{\frac12}+X^{\frac12}((1-\alpha)(ZX-\mu I)+\alpha(ZX-\sigma\mu I)+\alpha(\Delta ZX+Z\Delta X)\\&+\alpha^2\Delta Z\Delta X-\frac{\alpha^2}{n}\langle\Delta X,\Delta Z\rangle I)X^{-\frac12}.\end{aligned}$$

将上式进行化简并结合(2.3c), 得

$$\begin{aligned}&Q(\alpha)+Q(\alpha)^{\mathrm T}\\=&2(1-\alpha)(X^{\frac12}ZX^{\frac12}-\mu I)+2\alpha(X^{\frac12}ZX^{\frac12}-\sigma\mu I)+\alpha(X^{-\frac12}(X\Delta Z+\Delta XZ)X^{\frac12}\\&+X^{\frac12}(\Delta ZX+Z\Delta X)X^{-\frac12})+\alpha^2(X^{-\frac12}\Delta X\Delta ZX^{\frac12}+X^{\frac12}\Delta Z\Delta XX^{-\frac12})-\frac{2\alpha^2}{n}\langle\Delta X,\Delta Z\rangle I\\=&2(1-\alpha)(X^{\frac12}ZX^{\frac12}-\mu I)+2\alpha(X^{\frac12}ZX^{\frac12}-\sigma\mu I)+2\alpha(\sigma\mu I-X^{\frac12}ZX^{\frac12})\\&+\alpha^2(X^{-\frac12}\Delta X\Delta ZX^{\frac12}+X^{\frac12}\Delta Z\Delta XX^{-\frac12})-\frac{2\alpha^2}{n}\langle\Delta X,\Delta Z\rangle I\end{aligned}$$

$$=2(1-\alpha)(X^{\frac{1}{2}}ZX^{\frac{1}{2}}-\mu I)+\alpha^2(X^{-\frac{1}{2}}\Delta X\Delta ZX^{\frac{1}{2}}+X^{\frac{1}{2}}\Delta Z\Delta XX^{-\frac{1}{2}})-\frac{2\alpha^2}{n}\langle\Delta X,\Delta Z\rangle I,$$

即 (3.5) 成立.

引理 3.2　假设(A_1)-(A_3)成立, $(X,y,Z)\in\mathbb{N}_F(\gamma,\Gamma)$, $(\Delta X,\Delta y,\Delta Z)$ 是线性方程组 (2.3) 的解, 则

$$-\gamma\mu(\alpha)\le\frac{1}{2}\lambda_{\min}(Q(\alpha)+Q(\alpha)^{\mathrm{T}})\le\frac{1}{2}\lambda_{\max}(Q(\alpha)+Q(\alpha)^{\mathrm{T}})\le\Gamma\mu(\alpha),\tag{3.8}$$

其中 $\mu(\alpha)$ 的定义见 (3.3), $\alpha\in[0,\bar{\alpha}]$, 且

$$\bar{\alpha}=\min\left\{1,\frac{\sigma\mu\gamma/(2-\gamma)}{\|X^{-\frac{1}{2}}\Delta X\Delta ZX^{\frac{1}{2}}\|_F},\frac{\sigma\mu\Gamma}{\|X^{-\frac{1}{2}}\Delta X\Delta ZX^{\frac{1}{2}}\|_2}\right\}.\tag{3.9}$$

证　由 $(X,y,Z)\in\mathbb{N}_F(\gamma,\Gamma)$ 知

$$\lambda_{\max}(X^{\frac{1}{2}}ZX^{\frac{1}{2}}-\mu I)\le\Gamma\mu,\ 0\le\alpha\le\bar{\alpha}\le 1.\tag{3.10}$$

由 (3.5) 及 $\lambda_{\max}(\cdot)$ 为凸函数知

$$\begin{aligned}
&\frac{1}{2}\lambda_{\max}(Q(\alpha)+Q(\alpha)^{\mathrm{T}})\\
=&\lambda_{\max}\left((1-\alpha)(X^{\frac{1}{2}}ZX^{\frac{1}{2}}-\mu I)-\frac{\alpha^2}{n}\langle\Delta X,\Delta Z\rangle I+\frac{\alpha^2}{2}(X^{-\frac{1}{2}}\Delta X\Delta ZX^{\frac{1}{2}}+X^{\frac{1}{2}}\Delta Z\Delta XX^{-\frac{1}{2}})\right)\\
\le&(1-\alpha)\lambda_{\max}(X^{\frac{1}{2}}ZX^{\frac{1}{2}}-\mu I)+\frac{\alpha}{2}\lambda_{\max}(-\frac{2\alpha}{n}\langle\Delta X,\Delta Z\rangle I)\\
&+\frac{\alpha}{2}\lambda_{\max}\left(\alpha(X^{-\frac{1}{2}}\Delta X\Delta ZX^{\frac{1}{2}}+X^{\frac{1}{2}}\Delta Z\Delta XX^{-\frac{1}{2}})\right)\\
=&(1-\alpha)\lambda_{\max}(X^{\frac{1}{2}}ZX^{\frac{1}{2}}-\mu I)-\frac{\alpha^2}{n}\langle\Delta X,\Delta Z\rangle\lambda_{\max}(I)\\
&+\frac{\alpha^2}{2}\lambda_{\max}(X^{-\frac{1}{2}}\Delta X\Delta ZX^{\frac{1}{2}}+X^{\frac{1}{2}}\Delta Z\Delta XX^{-\frac{1}{2}}).
\end{aligned}$$

结合 (3.10), (3.6), 引理 2.1(1) 及文 [2] 中的引理 3.1 可得

$$\begin{aligned}
&\frac{1}{2}\lambda_{\max}(Q(\alpha)+Q(\alpha)^{\mathrm{T}})\\
\le&(1-\alpha)\Gamma\mu-\frac{\alpha^2}{n}\langle\Delta X,\Delta Z\rangle+\frac{\alpha^2}{2}\parallel X^{-\frac{1}{2}}\Delta X\Delta ZX^{\frac{1}{2}}+X^{\frac{1}{2}}\Delta Z\Delta XX^{-\frac{1}{2}}\parallel_2\\
\le&\Gamma\mu(\alpha)-\alpha\sigma\Gamma\mu-\frac{\alpha^2}{n}\Gamma\langle\Delta X,\Delta Z\rangle-\frac{\alpha^2}{n}\langle\Delta X,\Delta Z\rangle+\alpha^2\parallel X^{-\frac{1}{2}}\Delta X\Delta ZX^{\frac{1}{2}}\parallel_2\\
\le&\Gamma\mu(\alpha)-\alpha\sigma\Gamma\mu+\alpha^2\parallel X^{-\frac{1}{2}}\Delta X\Delta ZX^{\frac{1}{2}}\parallel_2\\
\le&\Gamma\mu(\alpha)-\alpha(\sigma\Gamma\mu-\bar{\alpha}\parallel X^{-\frac{1}{2}}\Delta X\Delta ZX^{\frac{1}{2}}\parallel_2)\\
\le&\Gamma\mu(\alpha).
\end{aligned}$$

最后一个不等式源自(3.9). 由此可知 (3.8) 式右边不等式成立.

同理, 由 $(X,y,Z)\in\mathbb{N}_F(\gamma,\Gamma)$ 知

$$\lambda_{\min}(X^{\frac{1}{2}}ZX^{\frac{1}{2}}-\mu I)\ge-\gamma\mu,\ 0\le\alpha\le\bar{\alpha}\le 1.\tag{3.11}$$

由 (3.5) 及 $\lambda_{\min}(\cdot)$ 是凹函数知

$$\begin{aligned}
\frac{1}{2}\lambda_{\min}(Q(\alpha)+Q(\alpha)^{\mathrm{T}})=&\lambda_{\min}\Big((1-\alpha)(X^{\frac{1}{2}}ZX^{\frac{1}{2}}-\mu I)-\frac{\alpha^2}{n}\langle\Delta X,\Delta Z\rangle I\\
&+\frac{\alpha^2}{2}(X^{-\frac{1}{2}}\Delta X\Delta ZX^{\frac{1}{2}}+X^{\frac{1}{2}}\Delta Z\Delta XX^{-\frac{1}{2}})\Big)\\
\ge&(1-\alpha)\lambda_{\min}(X^{\frac{1}{2}}ZX^{\frac{1}{2}}-\mu I)+\frac{\alpha}{2}\lambda_{\min}(-\frac{2\alpha}{n}\langle\Delta X,\Delta Z\rangle I)
\end{aligned}$$

$$+\frac{\alpha}{2}\lambda_{\min}\left(\alpha(X^{-\frac{1}{2}}\Delta X\Delta ZX^{\frac{1}{2}}+X^{\frac{1}{2}}\Delta Z\Delta XX^{-\frac{1}{2}})\right)$$
$$=(1-\alpha)\lambda_{\min}(X^{\frac{1}{2}}ZX^{\frac{1}{2}}-\mu I)-\frac{\alpha^2}{n}\langle\Delta X,\Delta Z\rangle\lambda_{\min}(I)$$
$$+\frac{\alpha^2}{2}\lambda_{\min}(X^{-\frac{1}{2}}\Delta X\Delta ZX^{\frac{1}{2}}+X^{\frac{1}{2}}\Delta Z\Delta XX^{-\frac{1}{2}}).$$

结合 (3.11), (3.6) 及文 [2] 的引理 3.1 可得

$$\begin{aligned}&\frac{1}{2}\lambda_{\min}(Q(\alpha)+Q(\alpha)^{\mathrm{T}})\\ \geq&-(1-\alpha)\gamma\mu-\frac{\alpha^2}{n}\langle\Delta X,\Delta Z\rangle-\frac{\alpha^2}{2}\parallel X^{-\frac{1}{2}}\Delta X\Delta ZX^{\frac{1}{2}}+X^{\frac{1}{2}}\Delta Z\Delta XX^{-\frac{1}{2}}\parallel_2\\ \geq&-\gamma\mu(\alpha)+\alpha\sigma\gamma\mu+\frac{\alpha^2}{n}\gamma\langle\Delta X,\Delta Z\rangle-\frac{\alpha^2}{n}\langle\Delta X,\Delta Z\rangle-\alpha^2\parallel X^{-\frac{1}{2}}\Delta X\Delta ZX^{\frac{1}{2}}\parallel_2\\ \geq&-\gamma\mu(\alpha)+\alpha\sigma\gamma\mu+\frac{\alpha^2}{n}(\gamma-1)\langle\Delta X,\Delta Z\rangle-\alpha^2\parallel X^{-\frac{1}{2}}\Delta X\Delta ZX^{\frac{1}{2}}\parallel_F.\end{aligned}\tag{3.12}$$

由$\|\cdot\|_F$的定义得

$$\begin{aligned}\parallel X^{-\frac{1}{2}}\Delta X\Delta ZX^{\frac{1}{2}}\parallel_F=&\sqrt{\left\langle X^{-\frac{1}{2}}\Delta X\Delta ZX^{\frac{1}{2}},X^{-\frac{1}{2}}\Delta X\Delta ZX^{\frac{1}{2}}\right\rangle}\\ \geq&\sqrt{\left\langle X^{\frac{1}{2}}\Delta Z\Delta XX^{-\frac{1}{2}},X^{-\frac{1}{2}}\Delta X\Delta ZX^{\frac{1}{2}}\right\rangle}\\ =&\sqrt{\mathrm{Tr}(\Delta X\Delta Z)^2}=\sqrt{\sum_{i=1}^n\lambda_i^2(\Delta X\Delta Z)}\\ \geq&\frac{1}{\sqrt{n}}\left(\sum_{i=1}^n\lambda_i(\Delta X\Delta Z)\right)\geq\frac{1}{n}\langle\Delta X,\Delta Z\rangle.\end{aligned}$$

上述第一个不等式来自 $\langle A,A\rangle\geq\langle A^{\mathrm{T}},A\rangle$, 第二个不等式利用了 Cauchy-Schwarz 不等式. 将上式代入 (3.12), 得

$$\begin{aligned}&\frac{1}{2}\lambda_{\min}(Q(\alpha)+Q(\alpha)^{\mathrm{T}})\\ \geq&-\gamma\mu(\alpha)+\alpha\sigma\gamma\mu+\alpha^2(\gamma-1)\parallel X^{-\frac{1}{2}}\Delta X\Delta ZX^{\frac{1}{2}}\parallel_F-\alpha^2\parallel X^{-\frac{1}{2}}\Delta X\Delta ZX^{\frac{1}{2}}\parallel_F\\ \geq&-\gamma\mu(\alpha)+\alpha\left(\sigma\gamma\mu+\bar{\alpha}(\gamma-2)\parallel X^{-\frac{1}{2}}\Delta X\Delta ZX^{\frac{1}{2}}\parallel_F\right)\\ \geq&-\gamma\mu(\alpha).\end{aligned}$$

最后一个不等式源自(3.9). 因此, (3.8)左边不等式成立.

综上可知本引理成立.

引理 3.3 假设(A_1)-(A_3) 成立, $(X,y,Z)\in\mathbb{N}_F(\gamma,\mathit{\Gamma})$, $(\Delta X,\Delta y,\Delta Z)$是线性方程组(2.3)的解, 则对于任意的 $\alpha\in(0,\hat{\alpha})$, 有 $(X(\alpha),y(\alpha),Z(\alpha))\in\mathbb{N}_F(\gamma,\mathit{\Gamma})$, 其中

$$\hat{\alpha}=\min\left\{1,\frac{1}{\parallel X^{-\frac{1}{2}}\Delta XX^{-\frac{1}{2}}\parallel_2},\frac{\sigma\mu\gamma/(2-\gamma)}{\|X^{-\frac{1}{2}}\Delta X\Delta ZX^{\frac{1}{2}}\|_F},\frac{\sigma\mathit{\Gamma}\mu}{\|X^{-\frac{1}{2}}\Delta X\Delta ZX^{\frac{1}{2}}\|_2}\right\}.\tag{3.13}$$

证 先证明$X(\alpha)\in\mathbb{F}_P^0$. 根据(2.3a)且 $X\in\mathbb{F}_P^0$, 有 $\langle A_i,X(\alpha)\rangle=b_i$, $i=1,\cdots,m$. 由(3.13)及$\alpha<\hat{\alpha}$知$\alpha\parallel X^{-\frac{1}{2}}\Delta XX^{-\frac{1}{2}}\parallel_2<1$,于是$I+\alpha X^{-\frac{1}{2}}\Delta XX^{-\frac{1}{2}}\succ 0$, 进而 $X(\alpha)=X+\alpha\Delta X=X^{\frac{1}{2}}(I+\alpha X^{-\frac{1}{2}}\Delta XX^{-\frac{1}{2}})X^{\frac{1}{2}}\succ 0$, 因此 $X(\alpha)\in\mathbb{F}_P^0$.

令

$$N(\alpha)=X^{-\frac{1}{2}}[X(\alpha)]^{\frac{1}{2}},\ E(\alpha)=[X(\alpha)]^{\frac{1}{2}}Z(\alpha)[X(\alpha)]^{\frac{1}{2}}-\mu(\alpha)I,\tag{3.14}$$

则易知 $N(\alpha)$ 非奇异, 且 $N(\alpha)E(\alpha)N(\alpha)^{-1}=Q(\alpha)$. 根据文[2]的引理 3.3 有

$$\frac{1}{2}\lambda_{\min}(Q(\alpha)+Q(\alpha)^{\mathrm{T}})\leq\lambda_{\min}(E(\alpha))\leq\lambda_{\max}(E(\alpha))\leq\frac{1}{2}\lambda_{\max}(Q(\alpha)+Q(\alpha)^{\mathrm{T}}).\tag{3.15}$$

由(3.9)和(3.13)知$\hat{\alpha} \leq \bar{\alpha}$, 于是结合(3.8)和(3.15), 得

$$-\gamma\mu(\alpha) \leq \lambda_{\min}(E(\alpha)) \leq \lambda_{\max}(E(\alpha)) \leq \Gamma\mu(\alpha).$$

将(3.14)的第二式代入上述左边不等式, 得

$$\lambda_{\min}\left([X(\alpha)]^{\frac{1}{2}}Z(\alpha)[X(\alpha)]^{\frac{1}{2}} - \mu(\alpha)I\right) \geq -\gamma\mu(\alpha),$$

于是, 有

$$\lambda_i\left([X(\alpha)]^{\frac{1}{2}}Z(\alpha)[X(\alpha)]^{\frac{1}{2}} - \mu(\alpha)I\right) \geq -\gamma\mu(\alpha),\ i = 1,\cdots,n.$$

故

$$\lambda_i(X(\alpha)Z(\alpha)) \geq (1-\gamma)\mu(\alpha),\ i = 1,\cdots,n.$$

类似可得

$$\lambda_i(X(\alpha)Z(\alpha)) \leq (1+\Gamma)\mu(\alpha),\ i = 1,\cdots,n.$$

于是

$$(1-\gamma)\mu(\alpha) \leq \lambda_i(X(\alpha)Z(\alpha)) \leq (1+\Gamma)\mu(\alpha),\ i = 1,\cdots,n. \tag{3.16}$$

下面证明 $(X(\alpha), y(\alpha), Z(\alpha)) \in \mathbb{F}_D^0$. 根据(3.16), 设$\gamma \in [0,1)$, 得

$$\lambda_i(X(\alpha)Z(\alpha)) \geq (1-\gamma)\mu(\alpha) > 0,\ i = 1,\cdots,n.$$

由此可知 $X(\alpha)Z(\alpha) \succ 0$, 结合 $X(\alpha) \succ 0$ 即知 $Z(\alpha) \succ 0$. 由 $(X,y,Z) \in \mathbb{F}_D^0$ 及(2.3b)知

$$\begin{aligned}\sum_{i=1}^m y_i(\alpha)A_i + Z(\alpha) &= \sum_{i=1}^m (y_i + \alpha\Delta y_i)A_i + Z + \alpha\Delta Z\\ &= \sum_{i=1}^m y_iA_i + \alpha\sum_{i=1}^m \Delta y_iA_i + Z + \alpha\Delta Z\\ &= C + \varphi(X) + \alpha\varphi(\Delta X)\\ &= C + \varphi(X(\alpha)).\end{aligned}$$

因此, $(X(\alpha), y(\alpha), Z(\alpha)) \in \mathbb{F}_D^0$.

综合以上证明即知 $(X(\alpha), y(\alpha), Z(\alpha)) \in \mathbb{N}_F(\gamma,\Gamma)$.

引理 3.4　假设(A_1)-(A_3)成立, $X \in \mathbb{F}_P^0$, $(X,y,Z) \in \mathbb{F}_D^0$, $(\Delta X, \Delta y, \Delta Z)$ 是线性方程组 (2.3) 的解, 则

$$\| X^{-\frac{1}{2}}\Delta XZ^{\frac{1}{2}} \|_F \leq \sqrt{\mu} \| \sigma D^{-\mathrm{T}} - D \|_F, \tag{3.17}$$

$$\| X^{-\frac{1}{2}}\Delta XX^{-\frac{1}{2}} \|_F \leq \| D^{-1} \|_2 \| \sigma D^{-\mathrm{T}} - D \|_F, \tag{3.18}$$

$$\| Z^{-\frac{1}{2}}\Delta ZZ^{-\frac{1}{2}} \|_F \leq \| D^{-1} \|_2 \| \sigma D^{-\mathrm{T}} - D \|_F, \tag{3.19}$$

$$\| Z^{-\frac{1}{2}}\Delta ZX^{\frac{1}{2}} \|_F \leq \sqrt{\mu} \| D \|_2 \| D^{-1} \|_2 \| \sigma D^{-\mathrm{T}} - D \|_F, \tag{3.20}$$

其中 $D = \frac{1}{\sqrt{\mu}}X^{\frac{1}{2}}Z^{\frac{1}{2}}$.

证　证明中需用到以下两个不等式 (见文[11]):

$$\| AB \|_F \leq \| A \|_2 \| B \|_F,\ \| AB \|_F \leq \| A \|_F \| B \|_2,\ \ \forall\, A, B \in \mathbb{R}^{n\times n}. \tag{3.21}$$

在(2.2c)式两边左乘 $X^{-\frac{1}{2}}$, 右乘$Z^{-\frac{1}{2}}$, 得

$$X^{-\frac{1}{2}}\Delta XZ^{\frac{1}{2}} + X^{\frac{1}{2}}\Delta ZZ^{-\frac{1}{2}} = \sigma\mu X^{-\frac{1}{2}}Z^{-\frac{1}{2}} - X^{\frac{1}{2}}Z^{\frac{1}{2}},$$

从而

$$\| X^{-\frac{1}{2}}\Delta XZ^{\frac{1}{2}} + X^{\frac{1}{2}}\Delta ZZ^{-\frac{1}{2}} \|_F = \| \sigma\mu X^{-\frac{1}{2}}Z^{-\frac{1}{2}} - X^{\frac{1}{2}}Z^{\frac{1}{2}} \|_F = \sqrt{\mu} \| \sigma D^{-\mathrm{T}} - D \|_F\ . \tag{3.22}$$

根据矩阵F-范数的定义和矩阵内积的性质有

$$\| X^{-\frac{1}{2}}\Delta XZ^{\frac{1}{2}} + X^{\frac{1}{2}}\Delta ZZ^{-\frac{1}{2}} \|_F^2 = \left\langle X^{-\frac{1}{2}}\Delta XZ^{\frac{1}{2}}, X^{-\frac{1}{2}}\Delta XZ^{\frac{1}{2}} \right\rangle + 2\left\langle X^{-\frac{1}{2}}\Delta XZ^{\frac{1}{2}}, X^{\frac{1}{2}}\Delta ZZ^{-\frac{1}{2}} \right\rangle$$

$$+\left\langle X^{\frac{1}{2}}\Delta ZZ^{-\frac{1}{2}}, X^{\frac{1}{2}}\Delta ZZ^{-\frac{1}{2}}\right\rangle$$
$$=\parallel X^{-\frac{1}{2}}\Delta XZ^{\frac{1}{2}}\parallel_F^2 +2\left\langle \Delta X, \Delta Z\right\rangle + \parallel X^{\frac{1}{2}}\Delta ZZ^{-\frac{1}{2}}\parallel_F^2$$
$$\geq \parallel X^{-\frac{1}{2}}\Delta XZ^{\frac{1}{2}}\parallel_F^2 + \parallel X^{\frac{1}{2}}\Delta ZZ^{-\frac{1}{2}}\parallel_F^2, \tag{3.23}$$

其中最后一个不等式是利用了引理2.1(1). 于是由(3.22)和(3.23)即得

$$\parallel X^{-\frac{1}{2}}\Delta XZ^{\frac{1}{2}}\parallel_F \leq \sqrt{\mu}\parallel \sigma D^{-\mathrm{T}} - D\parallel_F,$$

且

$$\parallel X^{\frac{1}{2}}\Delta ZZ^{-\frac{1}{2}}\parallel_F \leq \sqrt{\mu}\parallel \sigma D^{-\mathrm{T}} - D\parallel_F, \tag{3.24}$$

即(3.17)成立.

根据(3.21), (3.17)及矩阵D的定义, 得

$$\parallel X^{-\frac{1}{2}}\Delta XX^{-\frac{1}{2}}\parallel_F \leq \parallel X^{-\frac{1}{2}}\Delta XZ^{\frac{1}{2}}\parallel_F \parallel Z^{-\frac{1}{2}}X^{-\frac{1}{2}}\parallel_2 \leq \parallel D^{-1}\parallel_2 \parallel \sigma D^{-\mathrm{T}} - D\parallel_F,$$

即 (3.18) 成立.

根据(3.21), (3.24)及矩阵D的定义, 得

$$\parallel Z^{-\frac{1}{2}}\Delta ZZ^{-\frac{1}{2}}\parallel_F \leq \parallel Z^{-\frac{1}{2}}X^{-\frac{1}{2}}\parallel_2 \parallel X^{\frac{1}{2}}\Delta ZZ^{-\frac{1}{2}}\parallel_F \leq \parallel D^{-1}\parallel_2 \parallel \sigma D^{-\mathrm{T}} - D\parallel_F,$$

即 (3.19) 成立.

由(3.21), (3.19)及矩阵D的定义, 得

$$\parallel Z^{-\frac{1}{2}}\Delta ZX^{\frac{1}{2}}\parallel_F \leq \parallel Z^{-\frac{1}{2}}\Delta ZZ^{-\frac{1}{2}}\parallel_F \parallel Z^{\frac{1}{2}}X^{\frac{1}{2}}\parallel_2 \leq \sqrt{\mu}\parallel D\parallel_2 \parallel D^{-1}\parallel_2 \parallel \sigma D^{-\mathrm{T}} - D\parallel_F,$$

即 (3.20) 成立.

参照文[2]的引理 5.7, 可得如下结论:

引理3.5 设矩阵 W 非奇异且 $\parallel W\parallel_F = \sqrt{n}$, 则对于任意的 $\sigma \in [0, \frac{1}{2}]$, 有

$$\parallel \sigma W^{-\mathrm{T}} - W\parallel_F^2 \leq n(1 - 2\sigma + \sigma^2 \parallel W^{-1}\parallel_2^2).$$

下面定理给出了算法 3.1 的一次迭代后迭代点$(X(\alpha), y(\alpha), Z(\alpha))$的性质.

定理3.1 假设(A_1)-(A_3)成立, $(X, y, Z) \in \mathbb{N}_F(\gamma, \mathit{\Gamma})$, $\mathit{\Gamma} \geq \gamma$, $(\Delta X, \Delta y, \Delta Z)$是线性方程组(2.3)的解. 令

$$\tilde{\alpha} = \frac{\sigma\gamma(1-\gamma)^{\frac{1}{2}}}{n(2-\gamma)(1+\mathit{\Gamma})^{\frac{1}{2}}}\left((1-\sigma)^2 + \frac{\gamma\sigma^2}{1-\gamma}\right)^{-1}, \tag{3.25}$$

则对于任意的 $\alpha \in [0, \tilde{\alpha}]$, 有

$$(X(\alpha), y(\alpha), Z(\alpha)) \in \mathbb{N}_F(\gamma, \mathit{\Gamma}).$$

证 根据引理 3.3, 我们只需证明 $\tilde{\alpha} \leq \hat{\alpha}$, 即证

$$\tilde{\alpha} \leq \min\left\{1, \frac{1}{\parallel X^{-\frac{1}{2}}\Delta XX^{-\frac{1}{2}}\parallel_2}, \frac{\sigma\mu\gamma/(2-\gamma)}{\|X^{-\frac{1}{2}}\Delta X\Delta ZX^{\frac{1}{2}}\|_F}\right\}. \tag{3.26}$$

根据 $D = \frac{1}{\sqrt{\mu}}X^{\frac{1}{2}}Z^{\frac{1}{2}}$ 及 (3.1), 得

$$\parallel D\parallel_2^2 = \lambda_{\max}(D^{\mathrm{T}}D) = \frac{\lambda_{\max}(Z^{\frac{1}{2}}XZ^{\frac{1}{2}})}{\mu} = \frac{\lambda_{\max}(XZ)}{\mu} \leq (1+\mathit{\Gamma}) \tag{3.27}$$

和

$$\parallel D^{-1}\parallel_2^2 = \frac{1}{\lambda_{\min}(D^{\mathrm{T}}D)} = \frac{\mu}{\lambda_{\min}(Z^{\frac{1}{2}}XZ^{\frac{1}{2}})} = \frac{\mu}{\lambda_{\min}(XZ)} \leq \frac{1}{1-\gamma}. \tag{3.28}$$

根据 (3.28) 和引理 3.5, 得

$$\parallel \sigma D^{-\mathrm{T}} - D\parallel_F^2 \leq n(1 - 2\sigma + \frac{\sigma^2}{1-\gamma}) = n\left((1-\sigma)^2 + \frac{\gamma\sigma^2}{1-\gamma}\right). \tag{3.29}$$

根据(3.17), (3.20), (3.27)-(3.29), 有

$$\begin{aligned}\tilde{\alpha}\parallel X^{-\frac{1}{2}}\Delta X\Delta ZX^{\frac{1}{2}}\parallel_F&=\tilde{\alpha}\parallel X^{-\frac{1}{2}}\Delta XZ^{\frac{1}{2}}Z^{-\frac{1}{2}}\Delta ZX^{\frac{1}{2}}\parallel_F\\&\le\tilde{\alpha}\parallel X^{-\frac{1}{2}}\Delta XZ^{\frac{1}{2}}\parallel_F\parallel Z^{-\frac{1}{2}}\Delta ZX^{\frac{1}{2}}\parallel_F\\&\le\tilde{\alpha}\mu\frac{(1+\Gamma)^{\frac{1}{2}}}{(1-\gamma)^{\frac{1}{2}}}\left((1-\sigma)^2+\frac{\gamma\sigma^2}{1-\gamma}\right)n.\end{aligned}$$

将(3.25)代入上式, 即得

$$\tilde{\alpha}\parallel X^{-\frac{1}{2}}\Delta X\Delta ZX^{\frac{1}{2}}\parallel_F\le\sigma\mu\gamma/(2-\gamma).\tag{3.30}$$

根据(3.18), (3.28), (3.29)和(3.25), 可得

$$\begin{aligned}&\tilde{\alpha}\parallel X^{-\frac{1}{2}}\Delta XX^{-\frac{1}{2}}\parallel_2\\&\le\left[\frac{\sigma\gamma(1-\gamma)^{\frac{1}{2}}}{n(1+\Gamma)^{\frac{1}{2}}}\left((1-\sigma)^2+\frac{\gamma\sigma^2}{1-\gamma}\right)^{-1}\right]\frac{1}{(1-\gamma)^{\frac{1}{2}}}\left((1-\sigma)^2+\frac{\gamma\sigma^2}{1-\gamma}\right)^{\frac{1}{2}}n^{\frac{1}{2}}\\&=\frac{\sigma\gamma}{n^{\frac{1}{2}}(1+\Gamma)^{\frac{1}{2}}}\left((1-\sigma)^2+\frac{\gamma\sigma^2}{1-\gamma}\right)^{-\frac{1}{2}}\\&\le\frac{\sigma\gamma}{n^{\frac{1}{2}}(1+\Gamma)^{\frac{1}{2}}}\frac{(1-\gamma)^{\frac{1}{2}}}{\gamma^{\frac{1}{2}}\sigma}<1.\end{aligned}\tag{3.31}$$

最后证明 $\tilde{\alpha}<1$. 由 (3.25) 知

$$\tilde{\alpha}<\frac{\sigma\gamma(1-\gamma)^{\frac{1}{2}}}{n(1+\Gamma)^{\frac{1}{2}}}\left((1-\sigma)^2+\frac{\gamma\sigma^2}{1-\gamma}\right)^{-1}<\frac{\sigma\gamma(1-\gamma)^{\frac{1}{2}}}{n(1+\Gamma)^{\frac{1}{2}}}(\frac{1-\gamma}{\gamma\sigma^2})<1.\tag{3.32}$$

综合(3.30)-(3.32)即知不等式(3.26)成立.

基于定理3.1, 算法 3.1 中步长 α_k 的选取方法如下: 对于任意的 $k\ge 0$, 取

$$\alpha_k=\max\{\alpha\in[0,1]\mid(X^k,y^k,Z^k)+\alpha'(\Delta X^k,\Delta y^k,\Delta Z^k)\in\mathbb{N}_F(\gamma,\Gamma),\ \forall\,\alpha'\in[0,\alpha]\}.\tag{3.33}$$

下面分析证明算法 3.1 的迭代复杂度, 为此需作如下进一步假设:

假设(A_4)　$\langle\Delta X,\Delta Z\rangle$ 满足不等式: $\langle\Delta X,\Delta Z\rangle\le\frac{\sigma}{\alpha}\langle X,Z\rangle$.

定理3.2　假设 (A_1)-(A_4) 成立, 对于任意的 $\epsilon>0$, 算法 3.1 至多经 $\tau^{-1}\log[\epsilon^{-1}\langle X^0,Z^0\rangle]=\mathcal{O}(n\log\frac{\mathrm{Tr}(X^0Z^0)}{\epsilon})$ 次迭代可得到一个 ϵ- 最优解, 其中

$$\tau=\frac{\sigma(1-2\sigma)\gamma(1-\gamma)^{\frac{1}{2}}}{n(2-\gamma)(1+\Gamma)^{\frac{1}{2}}}\left((1-\sigma)^2+\frac{\gamma\sigma^2}{1-\gamma}\right)^{-1}.$$

证　根据(2.4)及假设(A_4), 有

$$\left\langle X^k+\alpha\Delta X,Z^k+\alpha_k\Delta Z\right\rangle\le(1-\alpha_k+2\alpha_k\sigma)\left\langle X^k,Z^k\right\rangle.$$

根据定理 3.1和 (3.33)得 $\alpha_k\ge\tau/(1-2\sigma)$, 代入上式, 即得

$$\langle X^{k+1},Z^{k+1}\rangle\le(1-(1-2\sigma)\alpha_k)\langle X^k,Z^k\rangle\le(1-\tau)\langle X^k,Z^k\rangle.$$

于是有

$$\langle X^k,Z^k\rangle\le(1-\tau)^k\langle X^0,Z^0\rangle.$$

令 $(1-\tau)^k\langle X^0,Z^0\rangle\le\epsilon$, 解得

$$k\ge\tau^{-1}\log[\epsilon^{-1}\left\langle X^0,Z^0\right\rangle],$$

因此结论成立.

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A New Primal-Dual Path-Following Algorithm for Convex Quadratic Semidefinite Programming

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Abstract: In this paper, a new primal-dual path-following algorithm for convex quadratic semidefinite programming is proposed. At each iteration, the search direction of the algorithm is generated by the solution of a system of linear equations. Under some conditions, the iteration points generated by the algorithm lie in the neighborhood of the central path, and an ϵ-optimal solution can be found at most $\mathcal{O}(n|\log\epsilon|)$ iterations.

Key words: Convex quadratic semidefinite programming; Primal-dual path-following algorithm; Center path; Iterative complexity