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具有惯性项和阻尼项的Cahn-Hilliard方程的整体吸引子

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摘要: 本文研究具有惯性项和阻尼项的亚三次非线性Cahn-Hilliard方程的初边值问题. 在非线性弱正则的条件下, 我们建立弱解的适定性, 而不考虑非线性项的一阶导数的下界条件. 接着利用弱解的渐近紧和能量解的严格Lyapunov函数的存在性, 证明在空间$(H^2(\Omega)\cap H_0^1(\Omega))\times L^2(\Omega)$上存在整体吸引子.

关键词: Cahn-Hilliard方程; 阻尼项; 惯性项; 整体吸引子
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1. 引言

考虑下面具有惯性项和阻尼项的Cahn-Hilliard方程

$$u_{tt}+u_t+\Delta^2 u-\Delta u-\Delta f(u)-\Delta(\beta u_t)=h(x),\ (t,x)\in(0,\infty)\times\Omega, \tag{1.1}$$

$$u=\Delta u=0,\ (t,x)\in(0,\infty)\times\partial\Omega, \tag{1.2}$$

$$u(0,x)=u_0(x),\ u_t(0,x)=u_1(x),\ x\in\Omega, \tag{1.3}$$

其中$\Omega\in\mathbb{R}^2$是有界正则域, $h\in L^2(\Omega)$, 粘性系数$\beta\geq 0$且非线性函数f满足下列条件

$$f\in C^2(\mathbb{R}),\ |f''(x)|\leq C(1+|x|^p),\ \forall x\in\mathbb{R},\ p<1, \tag{1.4}$$

$$\liminf_{|s|\to\infty}\frac{f(s)}{s}>-\lambda_1,\ \lambda_1=\inf_{v\in H_0^1, v\neq 0}\frac{\|\nabla v\|^2_{L^2(\Omega)}}{\|v\|^2_{L^2(\Omega)}}, \tag{1.5}$$

$$\limsup_{|t|+|x|\to 0}|t|^q|x|^{-\alpha}|f''(\frac{1}{t}+x)-f''(\frac{1}{t})|<\infty,\ 0\leq q<\alpha\leq 1. \tag{1.6}$$

20世纪60年代末, Cahn和Hilliard提出了经典的Cahn-Hilliard方程, 即$u_t-\Delta(-\Delta u+f(u))=0$. Cahn-Hilliard方程是用来模拟合金等二元材料的相对分离过程的. 为了将经典理论推广到强非平衡分解的情形, 在文[2-3]中, 作者通过在方程中加入惯性项εu_{tt}, 提出了Cahn-Hilliard方程的双曲松弛. Khanmamedov和Yayla在文[1]中讨论在相应的初边值条件下弱解的适定性, 且证明了整体吸引子的存在性. 带有阻尼项的方程的吸引子在文[4-5, 12]中也给出了严格的论证. 近年来, 很多研究者对Cahn-Hilliard方程产生了浓厚的兴趣, 但在特殊边界条件下对带有阻尼项的双曲Cahn-Hilliard方程的研究仍无果. 本文主要研究了在亚三次非线性二维情形下, 带有阻尼项的双曲Cahn-Hilliard方程的解的适定性, 并讨论了方程吸引子的存在性. 另外由于特殊项$-\Delta(\beta u_t)$的出现及分数阶空间嵌入定理的处理使得工作更为复杂.

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2. 基本假设及主要结论

定义2.1 函数$u\in C([0,T];H^2(\Omega)\cap H_0^1(\Omega))\cap C^1([0,T];L^2(\Omega))$且$u(0)=u_0,\ u_t(0)=u_1$, 若对每一个$\varphi\in(H^2(\Omega)\cap H_0^1(\Omega))((.,.)$表示$L^2(\Omega)$上的内积), 下列式子在$[0,T)\times\Omega$上成立,

$$\frac{\mathrm{d}}{\mathrm{d}t}(u_t,\varphi)+(u_t,\varphi)+(\Delta u,\Delta\varphi)-(\Delta u,\varphi)-(\Delta f(u),\varphi)-(\Delta(\beta u_t),\varphi)=(h,\varphi),$$

则u是方程(1.1)-(1.3)在$[0,T)\times\Omega$上的弱解. 若函数$u\in C([0,T];H^4(\Omega)\cap H_0^1(\Omega))\cap C^1([0,T];H^2(\Omega)\cap H_0^1(\Omega))\cap C^2([0,T];L^2(\Omega))$且上述条件成立, 则$u$为方程(1.1)-(1.3)的强解.

注2.1[1] 对任意的$f\in C^3(\mathbb{R})$, 且具有增长条件

$$|f'''|\le C(1+|x|^p),\ \forall x\in\mathbb{R},\ p<1,$$

该条件为(1.6)中当$\alpha=1$时的情况. 因此(1.6)比文[6]中关于f的三阶导数的条件要弱.

注2.2 若$u(t,x)$是方程(1.1)-(1.3)的弱解, 我们有

$$\frac{\mathrm{d}}{\mathrm{d}t}(u_t,\varphi)+(u_t,\varphi)+(\Delta u,\Delta\varphi)-(\Delta u,\varphi)-(\Delta f(u),\varphi)-(\Delta(\beta u_t),\varphi)=(h,\varphi),$$

对每一个$\varphi\in C_0^\infty(\Omega)$, 在区域$D'((0,T)\times\Omega)$中有

$$u_{tt}+u_t+\Delta^2u-\Delta u-\Delta f(u)-\Delta(\beta u_t)=h(x). \tag{2.1}$$

由(2.1)式得$u_{tt}\in H^{-1}(0,\tau;L^2(\Omega))$, $\Delta^2u\in H^{-1}(0,\tau;L^2(\Omega))$, $\tau\in(0,T)$.

在文[7]中定义两个算子γ_0和γ_1, 满足以下性质

$$\gamma_0(v)=v\Big|_{\partial\Omega},\ \gamma_1(v)=\frac{\partial}{\partial\nu}v\Big|_{\partial\Omega},$$

$\gamma_0(v)$是$H_\Delta(\Omega):=\{u\in L^2(\Omega):\triangle u\in L^2(\Omega)\}$到$H^{-1/2}(\partial\Omega)$的线性连续算子, $\gamma_1(v)$是$H_\Delta(\Omega):=\{u\in L^2(\Omega):\triangle u\in L^2(\Omega)\}$到$H^{-3/2}(\partial\Omega)$的线性连续算子. 则由$\Delta u\in C([0,T];L^2(\Omega))$和$\Delta^2u\in H^{-1}(0,\tau;L^2(\Omega))$可以得出

$$\Delta u\in H^{-1}(0,\tau;H^{-1/2}(\partial\Omega)),\ \frac{\partial}{\partial\nu}\Delta u\in H^{-1}(0,\tau;H^{-3/2}(\partial\Omega)),$$

其中$\tau\in(0,T)$, $\frac{\partial}{\partial\nu}$表示$\partial\Omega$ 上的外法向导数. 因此, (2.1)式在$H^{-1}(0,\tau;L^2(\Omega))$ 上成立, 用$\varphi\in H^2(\Omega)\cap H_0^1(\Omega)$检验这个方程, 在$H^{-1}(0,\tau)$中得到下式

$$\frac{\mathrm{d}}{\mathrm{d}t}(u_t,\varphi)+(u_t,\varphi)+(\Delta u,\Delta\varphi)-\langle\langle\Delta u,\frac{\partial}{\partial\nu}\varphi\rangle\rangle-(\Delta u,\varphi)-(\Delta f(u),\varphi)-(\Delta(\beta u_t),\varphi)=(h,\varphi).$$

在空间$H^{-1}(0,\tau)$中, $\langle\langle\Delta u,\frac{\partial}{\partial\nu}\varphi\rangle\rangle=0$, 其中$\varphi\in H^2(\Omega)\cap H_0^1(\Omega)$, $\tau\in(0,T)$, $\langle\langle.,.\rangle\rangle$是$H^{1/2}(\partial\Omega)$和$H^{-1/2}(\partial\Omega)$的二元形式. 关于上述等式, 在$H^{-1}(0,\tau)$中, 对$\forall\psi\in H^{1/2}(\partial\Omega)$, 有$\langle\langle\Delta u,\psi\rangle\rangle=0$, 对$\forall\tau\in(0,T)$, 在$H^{-1}(0,\tau;H^{-1/2}(\partial\Omega))$上, 有$\Delta u=0$.

定理2.1 假设条件(1.4)-(1.6)成立, 对每个初始条件$(u_0,u_1)\in(H^2(\Omega)\cap H_0^1(\Omega))\times L^2(\Omega))$, 方程(1.1)-(1.3)有唯一的弱解$u\in C([0,\infty);H^2(\Omega)\cap H_0^1(\Omega))\cap C^1([0,\infty);L^2(\Omega))$, 使得

$$\|u(t)\|_{H^2(\Omega)}+\|u_t(t)\|_{L^2(\Omega)}\le c_1(\|(u_0,u_1)\|_{H^2(\Omega)\times L^2(\Omega)}),\ \forall t\ge0,$$

其中c_1是$\mathbb{R}^+\to\mathbb{R}^+$上的一个非减函数.

定义2.2[1] 设$\{V(t)\}_{t\ge0}$是度量空间(X,d)上的半群, 如果$\mathcal{A}$满足:

1)$\mathcal{A}$是紧集,

2)$\mathcal{A}$是不变的, 即$V(t)\mathcal{A}=\mathcal{A},\forall t\ge0$,

3)$\lim\limits_{t\to\infty}\sup\limits_{v\in B}\inf\limits_{\omega\in\mathcal{A}}d(V(t)v,\omega)=0,\forall B\subset X$, B为有界集.

那么, 紧集$\mathcal{A}\subset X$称为半群$\{V(t)\}_{t\ge0}$的整体吸引子.

定理2.2 若条件(1.4)-(1.6)成立, 则方程(1.1)-(1.3)生成的半群$\{S(t)\}_{t\ge0}$在空间$(H^2(\Omega)\cap H_0^1(\Omega))\times L^2(\Omega)$中存在整体吸引子$\mathcal{A}$.

3. 弱解的适定性

引理3.1[1] 设$g \in C(\mathbb{R})$, 且

$$|g(x)| \le M(1+|x|^p),\ \forall x \in \mathbb{R}, \tag{3.1}$$

$$|g(\frac{1}{t}+x)-g(\frac{1}{t})| \le M|x|^{\alpha}|t|^{-q},\ \forall(|t|+|x|)<\frac{1}{r}, \tag{3.2}$$

其中$p<1,\ 0 \le q<\alpha \le 1,\ M \ge 1$且$r \ge 1$.

接下来对$\forall \varepsilon \in (0,\frac{1}{2r})$, 都存在$g_{\varepsilon} \in C^{\infty}(\mathbb{R})$, 使得

$$|g_{\varepsilon(x)}| \le \hat{c_1}(1+|x|^p),\ \forall x \in \mathbb{R}, \tag{3.3}$$

$$|g(x)-g_{\varepsilon}(x)| \le \max_{\substack{y,z \in [-\hat{c_2},\hat{c_2}] \\ |y-z| \le \varepsilon}} |g(y)-g(z)|+\hat{c_2}\varepsilon^{\alpha}|x|^q,\ \forall x \in \mathbb{R}, \tag{3.4}$$

$$|g_{\varepsilon}'(x)| \le \hat{c_3}\varepsilon^{-1}(1+\varepsilon^{\alpha}|x|^q),\ \forall x \in \mathbb{R}, \tag{3.5}$$

其中$\hat{c_i}(i=1,2,3)$为与M有关的常数, r与ε, g, x有关.

引理3.2[1] 设$f \in C^2(\mathbb{R})$, 且

$$|f^{''}(\frac{1}{t}+x)-f^{''}(\frac{1}{t})| \le M|x|^{\alpha}|t|^{-q},\ \forall(|t|+|x|)<\frac{1}{r},$$

其中$\alpha>0, q \ge 0, M \ge 1$且$r \ge 1$. 则有以下不等式成立

$$\|f-f_{\varepsilon}\|_{C^2[-k,k]} \le \sum_{i=0}^{2} \max_{\substack{y,z \in [-k-1,k+1] \\ |y-z| \le \varepsilon}} |f^i(y)-f^i(z)|,\ \forall k>0,\ \forall \varepsilon \in (0,1),$$

$$|f_{\varepsilon}^{''}(\frac{1}{t}+x)-f_{\varepsilon}^{''}(\frac{1}{t})| \le 2^q M|x|^{\alpha}|t|^{-q}, \forall(|t|+|x|) \le \frac{1}{3r},\ \forall \varepsilon \in (0,1),$$

其中$f_{\varepsilon}(x)=(f*\rho_{\varepsilon})(x)$.

定理2.1的证明 首先证明弱解的存在性, 即证假设条件(1.4)-(1.6)成立, 对每个初始条件$(u_0,u_1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega)$, 方程(1.1)-(1.3)有弱解$u \in C([0,\infty);H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0,\infty);L^2(\Omega))$, 使得

$$\|u(t)\|_{H^2(\Omega)}+\|u_t(t)\|_{L^2(\Omega)} \le c_1(\|(u_0,u_1)\|_{H^2(\Omega) \times L^2(\Omega)}),\ \forall t \ge 0,$$

其中c_1是$\mathbb{R}^+ \to \mathbb{R}^+$上的一个非减函数.

证 步1 设$(u_0,u_1) \in \mathcal{H}_1 := \{(u,v) \in (H^4(\Omega) \cap H_0^1(\Omega)) \times (H^2(\Omega) \cap H_0^1(\Omega)) : \Delta u\Big|_{\partial\Omega}=0\}$, 函数$f$不仅满足条件(1.4)-(1.6), 还满足局部Lipschitz条件. 我们可以将问题(1.1)-(1.3) 归结为$\mathcal{H} := (H^2(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega)$中的柯西问题:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}(u(t),u_t(t))=A((u(t),u_t(t)))+\mathcal{F}((u(t),u_t(t))) \\ (u(0),u_t(0))=(u_0,u_1), \end{cases} \tag{3.6}$$

$D(A)=\mathcal{H}_1$, $A(u,v)=(v,-\Delta^2 u-v+\Delta u+\Delta(\beta v))$, $\mathcal{F}=(0,\Delta f(u)+h)$. 因为算子$A$ 在$\mathcal{H}$中是极大耗散的, 由文[8]中Lumer-Phillips定理, 它生成一个线性连续半群$\{e^{tA}\}_{t \ge 0}$. 易得算子$\mathcal{F}:\mathcal{H} \to \mathcal{H}$ 是Lipschitz连续的. 故根据文[9]中的半群理论, 方程组(3.6)有强解$(u,u_t) \in C([0,T_{\max});\mathcal{H}_1) \cap C^1([0,T_{\max});\mathcal{H})$.

算子$-\Delta : H^2(\Omega) \cap H_0^1(\Omega) \subset L^2(\Omega) \to L^2(\Omega)$是自伴随正定的, 因此$u(t,x)$是$[0,T_{\max}) \times \Omega$中(1.1)-(1.3)的强解, 将(1.1)式乘以$(-\Delta)^{-1}u_t(t)$, 并在$(s,t) \times \Omega$ 上积分, 其中$f(u)|_{\partial\Omega}=f(0)$, 得到下列等式

$$\frac{1}{2}\|(-\Delta)^{-\frac{1}{2}}u_t(t)\|_{L^2(\Omega)}^2+\frac{1}{2}\|(-\Delta)^{\frac{1}{2}}u_t(t)\|_{L^2(\Omega)}^2+\int_s^t \|(-\Delta)^{-\frac{1}{2}}u_t(\sigma)\|_{L^2(\Omega)}^2 \mathrm{d}\sigma$$

$$+\int_s^t \beta\|(u_t(\sigma))\|^2 \mathrm{d}\sigma+\int_s^t \int_\Omega u(\sigma)u_t(\sigma)\mathrm{d}x\mathrm{d}\sigma+(F(u(t))-f(0)u(t),1)-(h,(-\Delta)^{-1}u(t))$$
$$=\frac{1}{2}\|(-\Delta)^{-\frac{1}{2}}u_t(s)\|^2_{L^2(\Omega)}+\frac{1}{2}\|(-\Delta)^{\frac{1}{2}}u(s)\|^2_{L^2(\Omega)}+(F(u(s))-f(0)u(s),1)$$
$$-(h,(-\Delta)^{-1}u(s)),\ 0\le s\le t<T_{\max}, \tag{3.7}$$

其中$F(z)=\int_0^z f(t)\mathrm{d}t$. 在上式中结合(1.4)-(1.5)式我们得到

$$\begin{cases} \|u_t(t)\|_{H^{-1}(\Omega)}+\|u(t)\|_{H^1(\Omega)}\le Q(\|(u_0,u_1)\|_{\mathcal{H}}), \\ \displaystyle\int_0^t \|u_t(s)\|^2_{H^{-1}(\Omega)}\mathrm{d}s\le Q(\|(u_0,u_1)\|_{\mathcal{H}}), \qquad \forall t\in[0,T_{\max}) \\ \displaystyle\int_\Omega\int_0^t \beta|u_t(x,s)|^2\mathrm{d}x\mathrm{d}s\le Q(\|(u_0,u_1\|_{\mathcal{H}}), \end{cases} \tag{3.8}$$

其中$Q:\mathbb{R}^+\to\mathbb{R}^+$是非减函数. 接下来将(1.1)式乘以$2u_t$并在$\Omega$上进行积分, 得到下列等式

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|u_t(t)\|^2_{L^2(\Omega)}+\|\Delta u(t)\|^2_{L^2(\Omega)}+\|\nabla u(t)\|^2_{L^2(\Omega)}$$
$$+(f'(u(t)),|\nabla u(t)|^2)\Big)+2\|u_t(t)\|^2_{L^2(\Omega)}+2\beta\|\nabla u_t(t)\|^2_{L^2(\Omega)}\mathrm{d}x$$
$$=(f''(u(t))u_t(t),|\nabla u(t)|^2)+2(h,u_t(t)),\ \forall t\in[0,T_{\max}). \tag{3.9}$$

下面将(1.1)式乘以δu并在Ω上进行积分, 得到下列等式

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\delta(u_t(t),u(t))+\frac{1}{2}\beta\delta\|\nabla u(t)\|^2_{L^2(\Omega)}\Big)+\delta\|\Delta u(t)\|^2_{L^2(\Omega)}-\delta\|u_t(t)\|^2_{L^2(\Omega)}$$
$$+\delta\|\nabla u(t)\|^2_{L^2(\Omega)}+\delta(u_t(t),u(t))+\delta(f'(u(t)),|\nabla u(t)|^2)$$
$$=\delta(h,u_t(t)),\ \forall t\in[0,T_{\max}). \tag{3.10}$$

将(3.9)式与(3.10)式相加得

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|u_t(t)\|^2_{L^2(\Omega)}+\|\Delta u(t)\|^2_{L^2(\Omega)}+\|\nabla u(t)\|^2_{L^2(\Omega)}+\delta(u_t(t),u(t))+(f'(u(t)),|\nabla u(t)|^2)$$
$$+\frac{1}{2}\beta\delta\|\nabla u(t)\|^2_{L^2(\Omega)}\Big)+2\|u_t(t)\|^2_{L^2(\Omega)}+\delta\|\Delta u(t)\|^2_{L^2(\Omega)}+\delta\|\nabla u(t)\|^2_{L^2(\Omega)}$$
$$+\delta(u_t(t),u(t))+\delta(f'(u(t)),|\nabla u(t)|^2)-\delta\|u_t(t)\|^2_{L^2(\Omega)}+2\beta\|\nabla u_t(t)\|^2_{L^2(\Omega)}\mathrm{d}x$$
$$=(f''(u(t))u_t(t),|\nabla u(t)|^2)+2(h,u_t(t))+\delta(h,u_t(t)),\ \forall t\in[0,T_{\max}). \tag{3.11}$$

由(1.4)式, $(3.8)_1$式及$H^{1/2}(\Omega)\hookrightarrow L^2(\Omega)$得

$$\Big|(f'(u(t)),|\nabla u(t)|^2)\Big|\le\|f'(u(t))\|_{L^2(\Omega)}\|\nabla u(t)\|^2_{L^4(\Omega)}$$
$$\le c_1\|\nabla u(t)\|^2_{H^{1/2}(\Omega)}\le c_2\|u(t)\|_{H^2(\Omega)}\|u(t)\|_{H^1(\Omega)}$$
$$\le c_3\|u(t)\|_{H^2(\Omega)}. \tag{3.12}$$

结合(3.11)式, 并在(3.10)式中令δ足够小, 我们得到

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\varepsilon(u(t))\Big)+c_4E(u(t))\le(f''(u(t))u_t(t),|\nabla u(t)|^2)+c_5,\ \forall t\in[0,T_{\max}). \tag{3.13}$$

其中$E(u(t))=\|u_t(t)\|^2_{L^2(\Omega)}+\|\Delta u(t)\|^2_{L^2(\Omega)}$, $\varepsilon(u(t))=E(u(t))+(f'(u(t)),|\nabla u(t)|^2)+\delta(u_t(t),u(t))+\|\nabla u(t)\|^2_{L^2(\Omega)}+\frac{1}{2}\beta\delta\|\nabla u(t)\|^2_{L^2(\Omega)}$. 为了估计(3.12)式右边, 利用引理3.1, 令$g(z):=f''(z)$, 根据(1.4)式和(1.6)式, 函数$g(z)$满足引理3.1的条件, 则有以下式子成立

$$\Big|(f''(u(t))u_t(t),|\nabla u(t)|^2)\Big|=\Big|(g(u(t))u_t(t),|\nabla u(t)|^2)\Big|$$
$$\le\Big|((g(u(t))-g_\varepsilon(u(t)))u_t(t),|\nabla u(t)|^2)\Big|$$

$$+\Big|(g_\varepsilon(u(t))u_t(t),|\nabla u(t)|^2)\Big|, \tag{3.14}$$

其中$g_\varepsilon(z)=(g*\rho_\varepsilon)(z)$, 由(3.4)式, $(3.8)_1$式得

$$\begin{aligned}\Big|((g(u(t))-g_\varepsilon(u(t)))u_t(t),|\nabla u(t)|^2)\Big| \le & \max_{\substack{y,z\in[-\hat{c_2},\hat{c_2}]\\|y-z|\le\varepsilon}}|g(y)-g(z)|\|u_t(t)\|_{L^2(\Omega)}\|\nabla u(t)\|^2_{L^4(\Omega)}\\&+\hat{c_2}\varepsilon^\alpha(|u(t)|^q|u_t(t)|,|\nabla u(t)|^2)\\ \le & c_6\max_{\substack{y,z\in[-\hat{c_2},\hat{c_2}]\\|y-z|\le\varepsilon}}|g(y)-g(z)|\|u_t(t)\|_{L^2(\Omega)}\|u(t)\|^2_{H^2(\Omega)}\\&+c_7\varepsilon^\alpha\|u(t)\|^q_{L^\infty(\Omega)}\|u_t(t)\|_{L^2(\Omega)}\|u(t)\|_{H^2(\Omega)}.\end{aligned} \tag{3.15}$$

由(3.3)式, (3.5)式和嵌入定理$W^{1,\frac{2}{2-\sigma}}(\Omega)\hookrightarrow H^\sigma(\Omega)(0<\sigma<1)$得

$$\begin{aligned}\Big|(g_\varepsilon(u(t))|\nabla u(t)|^2,u_t(t))\Big| \le & \Big|((g_\varepsilon(u(t))-g_\varepsilon(0))|\nabla u(t)|^2,u_t(t))\Big|+|g_\varepsilon(0)|\Big|(|\nabla u(t)|^2,u_t(t))\Big|\\ \le & \Big\|(g_\varepsilon(u(t))-g_\varepsilon(0))|\nabla u(t)|^2\Big\|_{W_0^{1,\frac{2}{2-\sigma}}(\Omega)}\|u_t(t)\|_{H^{-\sigma}(\Omega)}\\&+|g_\varepsilon(0)|\Big\||\nabla u(t)|^2\Big\|_{W^{1,\frac{2}{2-\sigma/2}}(\Omega)}\|u_t(t)\|_{H^{-\frac{\sigma}{2}}(\Omega)}\\ \le & c_8\Big[(1+\|u(t)\|^p_{L^\infty(\Omega)})\|u(t)\|_{H^2(\Omega)}\|\nabla u(t)\|_{L^{\frac{2}{1-\sigma}}(\Omega)}\\&+\varepsilon^{-1}(1+\varepsilon^\alpha\|u(t)\|^q_{L^\infty(\Omega)})\|\nabla u(t)\|^3_{L^{\frac{6}{2-\sigma}}(\Omega)}\Big]\|u_t(t)\|_{H^{-\sigma}(\Omega)}\\&+c_8(\|u(t)\|_{H^2(\Omega)}\|\nabla u(t)\|_{L^{\frac{4}{2-\sigma}}(\Omega)}+\|\nabla u(t)\|^2_{L^{\frac{8}{4-\sigma}}(\Omega)})\|u_t(t)\|_{H^{-\frac{\sigma}{2}}(\Omega)}\\ \le & c_9\Big[(1+\|u(t)\|^p_{L^\infty(\Omega)})\|u(t)\|_{H^2(\Omega)}\|\nabla u(t)\|_{H^\sigma(\Omega)}\\&+\varepsilon^{-1}(1+\varepsilon^\alpha\|u(t)\|^q_{L^\infty(\Omega)})\|\nabla u(t)\|^3_{H^{\frac{1+\sigma}{3}}(\Omega)}\Big]\|u_t(t)\|_{H^{-\sigma}(\Omega)}\\&+c_8(\|u(t)\|_{H^2(\Omega)}\|\nabla u(t)\|_{H^{\frac{\sigma}{2}}(\Omega)}+\|\nabla u(t)\|^2_{H^{\frac{\sigma}{4}}(\Omega)})\|u_t(t)\|_{H^{-\frac{\sigma}{2}}(\Omega)}.\end{aligned} \tag{3.16}$$

由内插不等式

$$\begin{cases}\|\nabla u(t)\|_{H^\sigma(\Omega)}\le c_{10}\|u(t)\|^\sigma_{H^2(\Omega)}\|u(t)\|^{1-\sigma}_{H^1(\Omega)},\\ \|\nabla u(t)\|^2_{H^{\frac{\sigma}{2}}(\Omega)}\le c_{10}\|\nabla u(t)\|_{H^\sigma(\Omega)}\|\nabla u(t)\|_{L^2(\Omega)},\\ \|\nabla u(t)\|^3_{H^{\frac{1+\sigma}{3}}(\Omega)}\le c_{10}\|u(t)\|^{1+\sigma}_{H^2(\Omega)}\|u(t)\|^{2-\sigma}_{H^1(\Omega)},\\ \|u_t(t)\|_{H^{-\sigma}(\Omega)}\le c_{10}\|u_t(t)\|^{1-\sigma}_{L^2(\Omega)}\|u_t(t)\|^\sigma_{H^{-1}(\Omega)},\end{cases}$$

$(3.8)_1$式及(3.16)式得

$$\begin{aligned}\Big|((g_\varepsilon(u(t)))|\nabla u(t)|^2,u_t(t))\Big| \le & c_{11}\Big[\Big(1+\|u(t)\|^p_{L^\infty(\Omega)}\Big)\Big(\|u(t)\|^{1+\sigma}_{H^2(\Omega)}+1\Big)\\&+\varepsilon^{-1}\Big(1+\varepsilon^\alpha\|u(t)\|^q_{L^\infty(\Omega)}\Big)\|u(t)\|^{1+\sigma}_{H^2(\Omega)}\Big]\|u_t(t)\|^{1-\sigma}_{L^2(\Omega)}\|u_t(t)\|^\sigma_{H^{-1}(\Omega)}\\&+c_{11}(\|u(t)\|^{1+\frac{\sigma}{2}}_{H^2(\Omega)}+1)\|u_t(t)\|^{1-\frac{\sigma}{2}}_{L^2(\Omega)}\|u_t(t)\|^{\frac{\sigma}{2}}_{H^{-1}(\Omega)}\\ \le & c_{12}\Big(1+\|u(t)\|^p_{L^\infty(\Omega)}\Big)\Big(E(u(t))\|u_t(t)\|^\sigma_{H^{-1}(\Omega)}+1\Big)\\&+c_{12}\varepsilon^{-1}\Big(1+\varepsilon^\alpha\|u(t)\|^q_{L^\infty(\Omega)}\Big)E(u(t))\|u_t(t)\|^\sigma_{H^{-1}(\Omega)}\\&+c_{12}(E(u(t))+1)\|u_t(t)\|^{\frac{\sigma}{2}}_{H^{-1}(\Omega)},\end{aligned} \tag{3.17}$$

其中$\sigma\in(0,1)$. 由(3.14)式, (3.15)式, (3.17)式得

$$\begin{aligned}\left|(f''(u(t))u_t(t),|\nabla u(t)|^2)\right| \le& c_6\max_{y,z\in[-\hat{c_2},\hat{c_2}]|y-z|\le\varepsilon}|(g(y)-g(z)|E(u(t))+c_7\varepsilon^{\alpha}\|u(t)\|^q_{L^\infty(\Omega)}E(u(t))\\&+c_{12}\Big(1+\|u(t)\|^p_{L^\infty(\Omega)}\Big)\|u_t(t)\|^\sigma_{H^{-1}(\Omega)}E(u(t))\\&+c_{12}\varepsilon^{-1}\Big(1+\varepsilon^\alpha\|u(t)\|^q_{L^\infty(\Omega)}\Big)\|u_t(t)\|^\sigma_{H^{-1}(\Omega)}E(u(t))\\&+c_{12}\|u_t(t)\|^{\frac{\sigma}{2}}_{H^{-1}(\Omega)}E(u(t))+c_{13}\Big(1+\|u(t)\|_{H^2(\Omega)}\Big).\end{aligned}$$

由(3.13)式, 当ε足够小时, 得到

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\varepsilon(u(t))+\tilde{c_1}E(u(t))\le&\tilde{c_2}\varepsilon^\alpha\|u(t)\|^q_{L^\infty(\Omega)}E(u(t))\\&+\tilde{c_2}\Big(1+\|u(t)\|^p_{L^\infty(\Omega)}\Big)\|u_t(t)\|^\sigma_{H^{-1}(\Omega)}E(u(t))\\&+\tilde{c_2}\varepsilon^{-1}\Big(1+\varepsilon^\alpha\|u(t)\|^q_{L^\infty(\Omega)}\Big)\|u_t(t)\|^\sigma_{H^{-1}(\Omega)}E(u(t))\\&+\tilde{c_2},\ \forall t\in[0,T_{\max}).\end{aligned}$$

因此令$\varepsilon=\Big(\frac{\delta}{1+\|u\|^q_{L^\infty((0,\tau)\times\Omega)}}\Big)^{\frac{1}{\alpha}}$, 当$\delta$足够小时, 得到

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\varepsilon(u(t))+\tilde{c_3}E(u(t))\le&\tilde{c_4}\Big(1+\|u\|^p_{L^\infty((0,\tau)\times\Omega)}\Big)\|u_t(t)\|^\sigma_{H^{-1}(\Omega)}E(u(t))\\&+\tilde{c_4}\Big(1+\|u\|^{\frac{q}{\alpha}}_{L^\infty((0,\tau)\times\Omega)}\Big)\|u_t(t)\|^\sigma_{H^{-1}(\Omega)}E(u(t))\\&+\tilde{c_2},\ 0\le t\le\tau<T_{\max}.\end{aligned}\tag{3.18}$$

再由(3.12)式可以得到

$$\hat{\theta}_1E(u(t))-M\le\varepsilon(u(t))\le\hat{\theta}_2E(u(t))+M,\tag{3.19}$$

其中$0<\hat{\theta}_1<\hat{\theta}_2$, $M>0$, 由不等式(3.18)式得

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\varepsilon(u(t))+\tilde{c_5}\varepsilon(u(t))\le&\tilde{c_6}\Big(1+\|u\|^r_{L^\infty((0,\tau)\times\Omega)}\Big)\|u_t(t)\|^\sigma_{H^{-1}(\Omega)}\varepsilon(u(t))\\&+\tilde{c_7}\Big(1+\|u\|^r_{L^\infty((0,\tau)\times\Omega)}\Big),\ 0\le t\le\tau<T_{\max},\end{aligned}$$

其中$r=\max\{p,\frac{q}{\alpha}\}$. 令$\eta(t):=\tilde{c_5}-\tilde{c_6}(1+\|u\|^r_{L^\infty((0,\tau)\times\Omega)})\|u_t(t)\|^\sigma_{H^{-1}(\Omega)}$ 可得

$$\frac{\mathrm{d}}{\mathrm{d}t}\varepsilon(u(t))+\eta(t)\varepsilon(u(t))\le\tilde{c_7}(1+\|u\|^r_{L^\infty((0,\tau)\times\Omega)})$$

则

$$\begin{aligned}\varepsilon(u(t))\le&\mathrm{e}^{-\int_0^t\eta(s)\mathrm{d}s}\varepsilon(u(0))\\&+\tilde{c_7}\Big(1+\|u\|^r_{L^\infty((0,\tau)\times\Omega)}\Big)\int_0^t\mathrm{e}^{-\int_\lambda^t\eta(s)\mathrm{d}s}\mathrm{d}\lambda,\ 0\le t\le\tau<T_{\max}.\end{aligned}\tag{3.20}$$

因为$r\in(0,1)$, 当$\sigma\in(r,1)$时, 由$(3.8)_2$式得

$$\begin{aligned}-\int_\lambda^t\eta(s)\mathrm{d}s=&-\tilde{c_5}(t-\lambda)+\tilde{c_8}(t-\lambda)^{\frac{2-\sigma}{2}}+\tilde{c_8}(t-\lambda)^{\frac{2-\sigma}{2}}\|u\|^r_{L^\infty((0,\tau)\times\Omega)}\\\le&-\frac{1}{2}\tilde{c_5}(t-\lambda)+\delta\|u\|^2_{L^\infty((0,T)\times\Omega)}+\tilde{c_\delta},\ 0\le\lambda\le t\le\tau<T_{\max},\end{aligned}\tag{3.21}$$

其中$\delta>0$. 为了估计$\|u\|_{L^\infty((0,T)\times\Omega)}$, 由文[10]中的Brezis-Gallouet不等式,

$$\|v\|_{L^\infty(\Omega)}\le k\|v\|_{H^1(\Omega)}\log^{\frac{1}{2}}(1+R)+k\|v\|_{H^2(\Omega)}(1+R)^{-1},\ \forall v\in H^2(\Omega),\ R>0,$$

其中$k>0$. 令$R=\|v\|_{H^2(\Omega)}$, 得

$$\|v\|_{L^\infty(\Omega)}\le k\|v\|_{H^1(\Omega)}\log^{\frac{1}{2}}\Big(1+\|v\|_{H^2(\Omega)}\Big)+k,\ \forall v\in H^2(\Omega).$$

由上述不等式和$(3.8)_1$式得

$$\|u\|_{L^\infty((0,\tau)\times\Omega)} \le \tilde{c_9}\Big[\log^{\frac{1}{2}}\Big(1+\|u\|_{L^\infty(0,\tau;H^2(\Omega))}\Big)+1\Big]. \tag{3.22}$$

结合(3.20)式, (3.21)式, (3.22)式可得

$$\begin{aligned}\varepsilon(u(t)) \le&\bar{c_1}\mathrm{e}^{-\bar{c_2}t}\|u\|_{L^\infty(0,\tau;H^2(\Omega))}\varepsilon(u(0))\\&+\bar{c_1}\|u\|_{L^\infty(0,\tau;H^2(\Omega))}\Big(1+\log^{\frac{1}{2}}\Big(1+\|u\|_{L^\infty(0,\tau;H^2(\Omega))}\Big)\Big),\ 0\le t\le\tau<T_{\max},\end{aligned} \tag{3.23}$$

其中常数$\bar{c_i}(i=1,2)$与$\|(u_0,u_1)\|_{\mathcal{H}}$有关, 与$t$和$\tau$无关.

由(3.19)式, (3.23)式可以得出

$$\|u\|_{L^\infty(0,\tau;H^2(\Omega))} \le \bar{c_3},\ 0\le\tau<T_{\max},$$

$\bar{c_3}$与$\|(u_0,u_1)\|_{\mathcal{H}_\infty}$有关, 与$\tau$无关. 最后估计(3.23)式得

$$E(u(t)) \le \bar{c_4},\ \forall t\in[0,T_{\max}).$$

因此, 由文[9], 我们得出当$T_{\max}=\infty$时上式也成立.

步2　令$(u_0,u_1)\in\mathcal{H}$, 函数f满足条件(1.4)-(1.6), 因$\mathcal{H}_1$在$\mathcal{H}$中稠密, 则存在一个序列$\{(u_{0n},u_{1n})\}_{n=1}^{n=\infty}\subset\mathcal{H}_1$使得$(u_{0n},u_{1n})$在$\mathcal{H}$中强收敛于$(u_0,u_1)$.

令$f_n(z):=(f*\rho_{\frac{1}{n}})(z)$, 由引理3.1和(3.3)式可得$f_n$对每个$n$都满足引理3.1, 因此可将步1中的过程应用于该问题

$$\begin{cases}u_{ntt}+u_{nt}+\Delta^2u_n-\Delta u_n-\Delta f_n(u_n)-\Delta(\beta u_{nt})=h(x),\ (t,x)\in(0,\infty)\times\Omega,\\u_n=\Delta u_n=0,\ (t,x)\in(0,\infty)\times\partial\Omega,\\u_n(0,x)=u_{0n}(x),\ u_{nt}(0,x)=u_{1n}(x),\ x\in\Omega.\end{cases} \tag{3.24}$$

注意到在引理3.1中常数$\hat{c_i}(i=1,2,3)$与g无关, 故

$$E(u_n(t)) \le M,\ \forall t\ge0, \tag{3.25}$$

其中M与$\|(u_0,u_1)\|_{\mathcal{H}_1}$有关, 与$n$无关. 由(3.25)得序列$(u_n,u_{nt})$在$L^\infty(0,\infty;\mathcal{H})$中弱$*$ 收敛.

令下式成立

$$\begin{cases}u_n\to u\text{在}L^\infty\Big(0,\infty;H^2(\Omega)\cap H_0^1(\Omega)\Big)\text{上弱}*\text{收敛},\\u_{nt}\to u_t\text{在}L^\infty\Big(0,\infty;L^2(\Omega)\Big)\text{上弱}*\text{收敛},\end{cases}$$

其中$u\in L^\infty(0,\infty;H^2(\Omega)\cap H_0^1(\Omega))\cap W^{1,\infty}(0,\infty;L^2(\Omega))$. 由于文[11]中的推论4, 可得

$$u_n\to u\text{在}C\Big([0,T];H^{2-\varepsilon}(\Omega)\Big)\text{上强收敛},\ \forall\varepsilon>0.$$

因此对每个$T\ge0$有

$$u_n\to u\text{在}C\Big([0,T]\times\bar{\Omega}\Big)\text{上强收敛},\ \forall\varepsilon>0. \tag{3.26}$$

接下来用$2(u_n-u_m)_t$乘以下列方程并在$(0,t)\times\Omega$上积分, 得

$$(u_n-u_m)_{tt}+(u_n-u_m)_t+\Delta^2(u_n-u_m)-\Delta(u_n-u_m)-\Delta(f_n(u_n)-f_m(u_m))-\Delta(\beta(u_n-u_m)_t)=0.$$

结合(3.25)式得

$$\begin{aligned}E(u_n(t)-u_m(t)) \le&E(u_n(0)-u_m(0))+c\int_0^t E(u_n(s)-u_m(s))\mathrm{d}s\\&+cT\|f_n'(u_n)-f_m'(u_m)\|_{C([0,T]\times\bar{\Omega})}\\&+cT\|f_n''(u_n)-f_m''(u_m)\|_{C([0,T]\times\bar{\Omega})},\ \forall t\in[0,T].\end{aligned}$$

由Gronwall's引理, (3.26)式和引理3.2得

$$\lim_{\substack{n\to\infty\\m\to\infty}}\max_{0\le t\le T}E(u_n(t)-u_m(t))=0,\ \forall T\ge0,$$

这里

$$\begin{cases} u_n \to u\text{在}C\Big([0,T];H^2(\Omega)\cap H_0^1(\Omega)\Big)\text{上强收敛}, \\ u_{nt} \to u_t\text{在}C\Big([0,T];L^2(\Omega)\Big)\text{上强收敛}. \end{cases} \tag{3.27}$$

再由(3.24)式得

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}(u_{nt},\varphi)+(u_{nt},\varphi)+(\Delta u_n,\Delta\varphi)-(\Delta u_n,\varphi)-(\Delta f(u_n),\varphi) \\ \qquad -(\Delta(\beta u_{nt}),\varphi)=(h,\varphi),\ \forall\varphi\in H^2(\Omega)\cap H_0^1(\Omega), \\ u_n(0)=u_{0n},\ u_{nt}(0)=u_{1n}. \end{cases} \tag{3.28}$$

最后, 由(3.25)式, (3.27)式, (3.28)式, 结论得证.

下证弱解的唯一性, 即若$u(t,x)$和$v(t,x)$是$[0,T)\times\Omega$上方程(1.1)-(1.3)的弱解, 初始条件为$(u_0,u_1),(v_0,v_1)$, 有下列式子成立

$$\begin{aligned} &\|u-v\|_{C([0,t];H^1(\Omega))}+\|u_t-v_t\|_{C([0,t];H^{-1}(\Omega))} \\ \le& c(t,r)[\|u_0-v_0\|_{H^1(\Omega)}+\|u_1-v_1\|_{H^{-1}(\Omega)}],\ \forall t\in[0,T), \end{aligned} \tag{3.29}$$

其中$r=\max\{\|(u,u_t)\|_{C([0,T];\mathcal{H})},\|(v,v_t)\|_{C([0,T];\mathcal{H})}\}$, 对于每一个变量, c都是$\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$上的非减函数.

步3 令$(u_0,u_1)=(v_0,v_1)$. 设$\omega=u-v$, 这里有$\omega\in C([0,T);H^2(\Omega)\cap H_0^1(\Omega))\cap C^1([0,T);L^2(\Omega))$和

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}(\omega_t,\psi)+(\omega_t,\psi)+(\Delta\omega,\Delta\psi)-(\Delta\omega,\psi)-(\Delta(f(u)-f(v)),\psi) \\ \qquad -(\Delta(\beta\omega_t),\psi)=0,\ \forall\psi\in H^2(\Omega)\cap H_0^1(\Omega), \\ \omega(0)=\omega_t(0)=0. \end{cases} \tag{3.30}$$

由$(3.30)_1$, 对每一个$t\in(0,T)$, 有$(\omega_t,\psi)\in W^{1,\infty}(0,t)$. 因此, 令$\varphi(t)=\int_0^t\omega(s)\mathrm{d}s$, 将$(3.30)_1$式在$(0,t)$上积分, 在空间$C[0,T)$上有

$$\begin{aligned} &(\varphi_{tt},\psi)+(\varphi_t,\psi)+(\Delta\varphi,\Delta\psi)-(\Delta\varphi,\psi)-(\Delta\int_0^t(f(u(s,x))-f(v(s,x)))\mathrm{d}s,\psi) \\ &-(\Delta(\beta\varphi_t),\psi)=0,\ \forall\psi\in H^2(\Omega)\cap H_0^1(\Omega). \end{aligned}$$

下面由注2.2得到

$$\begin{cases} \varphi_{tt}+\varphi_t+\Delta^2\varphi-\Delta\varphi-\Delta\displaystyle\int_0^t(f(u(s,x))-f(v(s,x)))\mathrm{d}s-\Delta(\beta\varphi_t)=0,\ \text{于}C([0,T);L^2(\Omega)), \\ \varphi=0,\ \text{于}C([0,T);H^{3/2}(\partial\Omega)), \\ \Delta\varphi=0,\ \text{于}C([0,T);H^{-1/2}(\partial\Omega)), \\ \varphi(0)=\varphi_t(0)=0,\ \text{于}L^2(\Omega). \end{cases} \tag{3.31}$$

接下来, 用$(-\Delta)^{-1}\varphi_t$乘以$(3.31)_1$式, 并在$(0,t)\times\Omega$上积分, 得到

$$\begin{aligned} &\int_0^t\|\varphi_t(s)\|_{H^{-1}(\Omega)}^2\mathrm{d}s+\int_0^t\int_\Omega\varphi(s)\varphi_t(s)\mathrm{d}x\mathrm{d}s+\|\varphi_t(t)\|_{H^{-1}(\Omega)}^2+\|\varphi(t)\|_{H_0^1(\Omega)}^2 \\ \le&\Big(\int_0^t\|f(u(s))-f(v(s))\|_{H^{-1}(\Omega)}\mathrm{d}s\Big)\|\varphi(t)\|_{H_0^1(\Omega)} \\ &+\int_0^t\|f(u(s))-f(v(s))\|_{H^{-1}(\Omega)}\|\varphi(s)\|_{H_0^1(\Omega)}\mathrm{d}s,\ \forall t\in[0,T). \end{aligned}$$

由$(3.8)_2$式得

$$\|\varphi_t(t)\|^2_{H^{-1}(\Omega)}+\|\varphi(t)\|^2_{H^1_0(\Omega)}\le\Big(\int_0^t\|f(u(s))-f(v(s))\|_{H^{-1}(\Omega)}\mathrm{d}s\Big)\|\varphi(t)\|_{H^1_0(\Omega)}+\int_0^t\|f(u(s))-f(v(s))\|_{H^{-1}(\Omega)}\|\varphi(s)\|_{H^1_0(\Omega)}\mathrm{d}s,\ \forall t\in[0,T).\tag{3.32}$$

对任意的$s\in[0,T)$, 应用估计$\|f(u(s))-f(v(s))\|_{H^{-1}(\Omega)}\le c\|\omega(s)\|_{H^{-1}(\Omega)}$, 得(3.32)为

$$\|\varphi_t(t)\|^2_{H^{-1}(\Omega)}+\|\varphi(t)\|^2_{H^1(\Omega)}\le\hat{c}\int_0^t\Big[\|\varphi_t(s)\|^2_{H^{-1}(\Omega)}+\|\varphi(s)\|^2_{H^1(\Omega)}\Big]\mathrm{d}s.$$

由Gronwall's引理得

$$\varphi_t(t)=0,\ \forall t\in[0,T).$$

最后得出$u(t)=v(t)$, $\forall t\in[0,T)$.

步4　在步3中, 我们证明了弱解的唯一性. 由步1和步2, 每个弱解都可以用强解逼近. 因此完成了这个证明就足以证明(3.29)对于强解的有效性. 设$u(t,x)$, $v(t,x)$是方程(1.1)-(1.3)的强解. 将下列方程乘以$(-\Delta)^{-1}(u-v)_t$, 并在$(0,t)\times\Omega$上积分

$$(u-v)_{tt}+(u-v)_t+\Delta^2(u-v)-\Delta(u-v)-\Delta(f(u)-f(v))-\Delta(\beta(u-v)_t)=0.$$

最后应用Gronwall's引理, (3.29)式成立.

注3.1　根据定理2.1可知, 由公式$S(t)(u_0,u_1)=(u(t),u_t(t))$所定义的算子族$S(t):\mathcal{H}\to\mathcal{H}$, 满足半群的性质并且是$\mathcal{H}$上的弱连续半群.

4. 整体吸引子的存在性

引理4.1　设条件(1.4)-(1.6)成立, B是$\mathcal{H}$的有界子集. 则对每一个$\{S(t_k)\varphi_k\}_{k=1}^\infty$, 在$\mathcal{H}$中都有收敛的子序列, 其中$\{\varphi_k\}_{k=1}^\infty\subset B$, $t_k\to\infty$.

证　由定理2.1, 序列$\{S(.)\varphi_k\}_{k=1}^\infty$在$L^\infty(0,\infty;\mathcal{H})$中有界. 则对每一个$T>0$, 存在一个子序列$\{k_m\}_{m=1}^\infty$, 使得$t_{km}\ge T$, 而且有

$$\begin{cases}u_m\to u\text{在}L^\infty(0,\infty;H^2(\Omega)\cap H^1_0(\Omega))\text{上弱}*\text{收敛},\\ u_{mt}\to u_t\text{在}L^\infty(0,\infty;L^2(\Omega))\text{上弱}*\text{收敛},\end{cases}\tag{4.1}$$

这里$u\in L^\infty(0,\infty;H^2(\Omega)\cap H^1_0(\Omega))\cap W^{1,\infty}(0,\infty;L^2(\Omega))$, $(u_m(t),u_{mt}(t))=S(t+t_{km}-T)\varphi_{km}$.

接下来, 将方程

$$(u_n-u_m)_{tt}+(u_n-u_m)_t+\Delta^2(u_n-u_m)-\Delta(u_n-u_m)-\Delta(f(u_n)-f(u_m))-\Delta(\beta(u_n-u_m)_t)=0$$

乘以$2(u_n-u_m)_t+\delta(u_n-u_m)$, 并在$(0,T)\times\Omega$ 上积分, 当δ足够小时, 我们得到下列式子

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|u_{nt}(t)-u_{mt}(t)\|^2_{L^2(\Omega)}+\delta(u_{nt}(t)-u_{mt}(t),u_n(t)-u_m(t))+\|\Delta(u_n(t)-u_m(t))\|^2_{L^2(\Omega)}\\&+(f'(u_n(t)-u_m(t)),|\nabla(u_n(t)-u_m(t))|^2)+c\|\nabla(u_n(t)-u_m(t))\|^2_{L^2(\Omega)}\Big)\\&+\Big(\|u_{nt}(t)-u_{mt}(t)\|^2_{L^2(\Omega)}+\delta(u_{nt}(t)-u_{mt}(t),u_n(t)-u_m(t))+\delta\|\Delta(u_n(t)-u_m(t))\|^2_{L^2(\Omega)}\\&+\delta(f'(u_n(t)-u_m(t)),|\nabla(u_n(t)-u_m(t))|^2)+\delta\|\nabla(u_n(t)-u_m(t))\|^2_{L^2(\Omega)}\Big)\\&+2\beta\|\nabla(u_{nt}(t)-u_{mt}(t))\|^2_{L^2(\Omega)}\\\le\,&c_1\|u_n(t)-u_m(t)\|_{H^1(\Omega)}+\|f'(u_n(t))-f'(u_m(t))\|_{L^\infty(\Omega)}+\|f''(u_n(t))-f''(u_m(t))\|_{L^\infty(\Omega)},\end{aligned}$$

则有

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(u_n(t),u_m(t))+c_1\Phi(u_n(t),u_m(t))\le c_2H(u_n(t),u_m(t)),\ \forall t\ge0,$$

其中

$$\begin{aligned}\Phi(u_n(t),u_m(t)) =&E(u_n(t),u_m(t))+\delta(u_{nt}(t)-u_{mt}(t),u_n(t)-u_m(t))\\&+(f^{'}(u(t)),|\nabla(u_n(t)-u_m(t))|^2)+c\|\nabla(u_n(t)-u_m(t))\|^2_{L^2(\Omega)},\\ H(u_n(t),u_m(t)) =&\|u_n(t)-u_m(t)\|_{H^1(\Omega)}+\|f^{'}(u_n(t))-f^{'}(u(t))\|_{L^\infty(\Omega)}\\&+\|f^{'}(u_m(t))-f^{'}(u(t))\|_{L^\infty(\Omega)}+\|f^{''}(u_n(t))-f^{''}(u_m(t))\|_{L^\infty(\Omega)}.\end{aligned}$$

由Gronwall's引理得

$$\Phi(u_n(T),u_m(T))\le \mathrm{e}^{-c_1T}\Phi(u_n(0),u_m(0))+c_2\int_0^T \mathrm{e}^{-c_1(T-s)}H(u_n(s),u_m(s))\mathrm{d}s. \tag{4.2}$$

因此, 由(4.1)式, (4.2)式和紧嵌入定理得

$$\liminf_{n\to\infty}\liminf_{m\to\infty}\Phi(u_n(T),u_m(T))\le c_3\mathrm{e}^{-c_1T},$$

则有

$$\liminf_{n\to\infty}\liminf_{m\to\infty}\|S(t_{k_n})\varphi_{kn}-S(t_{k_m})\varphi_{km}\|_{\mathcal{H}}\le c_3\mathrm{e}^{-c_1T},$$

由上述不等式得

$$\liminf_{n\to\infty}\liminf_{m\to\infty}\|S(t_n)\varphi_n-S(t_m)\varphi_m\|_{\mathcal{H}}\le c_3\mathrm{e}^{-c_1T},$$

当$T\to\infty$时, 有

$$\liminf_{n\to\infty}\liminf_{m\to\infty}\|S(t_n)\varphi_n-S(t_m)\varphi_m\|_{\mathcal{H}}=0.$$

上式的证明可见文[12]中引理3.4, 结论得证.

定理2.2的证明 即证若条件(1.4)-(1.6)成立, 半群$\{S(t)\}_{t\ge0}$有整体吸引子$\mathcal{A}$, 其中$\mathcal{A}\in\mathcal{H}$, 且$\mathcal{A}=\mathcal{M}^u(\mathcal{N})$, $\mathcal{M}^u(\mathcal{N})$是由不动点集$\mathcal{N}$产生的不稳定流形.

设B是$\mathcal{H}$的有界子集. 由引理(4.1)得

$$\omega(B)=\bigcap_{\tau\ge0}\overline{\bigcup_{t\ge\tau}S(t)B}$$

是一个非空紧集合, 它对于$S(T)$是不变的并且吸引B. 设$\theta\in\omega(B)$, 由$\omega(B)$的不变性, 则存在一个完整的轨迹$\{(u(t),u_t(t)),\ t\in\mathbb{R}\}\subset\omega(B)$, 使得$(u(0),u_t(0))=\theta$. 接下来定义Lyapunov函数$L:\mathcal{H}\to\mathbb{R}$,

$$\begin{aligned}L((\varphi_0,\varphi_1)):=&\frac12\|(-\Delta)^{-\frac12}\varphi_1\|^2_{L^2(\Omega)}+\frac12\|(-\Delta)^{\frac12}\varphi_0\|^2_{L^2(\Omega)}\\&+(F(\varphi_0)-f(0)\varphi_0,1)-(h,(-\Delta)^{-1}\varphi_0).\end{aligned}$$

由(3.7)式, 函数$L(u(t),u_t(t))$可以推广到弱解, 从而它是$\mathbb{R}$上的一个减函数. 并考虑$L(u(t),u_t(t))$是有界的, 得到

$$\lim_{t\to\infty}L(u(t),u_t(t))=l. \tag{4.3}$$

接下来, 定义$\alpha(\theta):=\{\varphi\in\omega(B)$, 存在一个序列$\{(u(t_k),u_t(t_k))_{k=1}^\infty\}$, 使得$t_k\searrow-\infty,(u(t_k),u_t(t_k))$在$\mathcal{H}$中强收敛于$\varphi\}$. 这里$\alpha(\theta)$是紧的且是$\omega(B)$的不变子集. 由(4.3)式得

$$L(\varphi)=l,\ \forall\varphi\in\alpha(\theta);$$

再由$\alpha(\theta)$的不变性得

$$L(S(t)\varphi)=l,\ \forall\varphi\in\alpha(\theta),\ \forall t\ge0,$$

这个不等式和(3.7)式都用到了$\varphi\in\mathcal{N}$, 这里$\mathcal{N}$是$\{S(t)\}_{t\ge0}$的一组稳定点. 故有$\alpha(\theta)\subset\mathcal{N}$, 且

$$\omega(B)\subset\mathcal{M}^u(\mathcal{N}). \tag{4.4}$$

利用$\{S(t)\}_{t\ge0}$的弱连续性和渐近紧性以及$\mathcal{N}$在$\mathcal{H}$中有界, 很容易得出$\mathcal{M}^u(\mathcal{N})$ 是不变的, 紧的. 因此由(4.4)式得出$\mathcal{A}:=\mathcal{M}^u(\mathcal{N})$是空间$(H^2(\Omega)\cap H_0^1(\Omega))\times L^2(\Omega)$上的整体吸引子.

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Global Attractor for the Cahn-Hilliard Equation with Inertial Term and Damping Term

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Abstract: In this paper, we consider the initial boundary value problem of sub-cubic nonlinearity Cahn-Hilliard equation with inertial term and damping term. Under mild regularity conditions on the nonlinearity, we prove the uniform boundedness of the solutions without considering lower bound condition on the first derivative of the nonlinear term. Then, we first establish the asymptoticwe compactness of the weak solutions. Next, taking advantage of the presence of strict Lyapunov function for the energy solutions, we prove the existence of the global attractor in $(H^2(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega)$.

Key words: Cahn-Hilliard equation; Damping term; Inertial term; Global attractor