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Positive Solutions for Fractional Differential Equations with Integral and Infinite-Point Boundary Conditions

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Abstract: We study the positive solutions of a class of fractional differential equations with integral boundary conditions and infinite-point boundary value conditions. By mean of the properties of Green function and the fixed point theorem of cone expansion and compression of norm type, the existence results of the positive solution of the boundary value problem are obtained. Finally, we also give an example to illustrate the applicability of the results.
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1. Introduction

In recent years, the fractional differential equation has been widely used in many fields. Numerous mathematicians has been devoted to the study of it's related properties[1−4]. The solvability analysis of fractional differential equations with boundary conditions has become a hot theme. In this topic, they mainly study the existence or positive solutions of boundary value problems for fractional differential equations[5−10]. In this paper, we consider the integral boundary value condition and the infinite point boundary value condition together, and mainly discuss the existence of positive solutions of fractional differential equations for such boundary value problems.

This article deals with integral and infinite-point boundary value problems of the fractional differential equations

$$\begin{cases} D_{0^+}^{\alpha}u(t)+h(t,u(t))=0,\ t\in[0,1],\\ u^{(i)}(0)=0,\quad i=0,1,2,...,n-2,\\ D_{0^+}^{\beta}u(1)=\sum\limits_{i=1}^{\infty}\beta_i\int_0^{\eta_i}u(s)\mathrm{d}s+\sum\limits_{i=1}^{\infty}\gamma_i u(\eta_i),\end{cases}\tag{1.1}$$

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where D^α represents the standard Riemann-Liouville fractional derivative of order α satisfying $n-1<\alpha\leq n$, $n\geq 3$, $n\in\mathbb{N}^+$, $1\leq\beta\leq\alpha-1$, $0<\eta_1<\eta_2<\cdots<\eta_i<\cdots<1$, $\xi=\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}-\frac{1}{\alpha}\sum\limits_{i=1}^{\infty}\beta_i\eta_i^\alpha-\sum\limits_{i=1}^{\infty}\gamma_i\eta_i^{\alpha-1}>0$, $\beta_i,\gamma_i>0(i=1,2,...)$, $h\in C([0,1]\times[0,+\infty),[0,+\infty))$.

If $u\in C[0,1]$, $u(t)>0(0<t\leq 1)$ and u satisfies (1.1) on $[0,1]$, u is called a positive solution of problem (1.1).

In [11], the author investigated the existence of solutions for fractional differential equations with three-point integral boundary conditions

$$\begin{cases} {}^cD^qx(t)=f(t,x(t)),\ \ 0<t<1, 1<q\leq 2,\\ x(0)=0, x(1)=\alpha\int_0^\eta x(s)\mathrm{d}s, \quad 0<\eta<1,\end{cases}$$

where ${}^cD^q$ represents the Caputo fractional derivative of order q, $\alpha\in\mathbb{R}$, $\alpha\neq\eta^2$, $f:[0,1]\times X\to X$ is continuous, $(X,\|\cdot\|)$ is a Banach space.

In [12], the authors considered the existence and multiplicity of positive solutions for the following infinite-point boundary value problem of fractional differential equations

$$\begin{cases} D_{0+}^\alpha u(t)+q(t)f(t,u(t))=0,\ \ 0<t<1,\\ u(0)=0, u(1)=\sum\limits_{i=1}^{\infty}\alpha_iu(\xi_i),\end{cases}$$

where D_{0+}^α denotes the standard Riemann-Liouville fractional derivative satisfying $1<\alpha\leq 2$, $\xi_i\in(0,1)$, $\alpha_i\in[0,+\infty)$ with $\sum\limits_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1}<1$, $q(t)\in C([0,1],[0,+\infty))$, $f(t,u)\in C([0,1]\times[0,+\infty),[0,+\infty))$.

Recently, in [13], the authors investigated the existence results for fractional differential equations with integral and multi-point boundary conditions

$$\begin{cases} D^\sigma x(t)+f(t,x(t))=0, \quad t\in[0,1],\\ x^{(i)}(0)=0, \quad i=0,1,2,...,n-2,\\ x(1)=\sum\limits_{i=1}^{m-2}\beta_i\int_0^{\eta_i}x(s)\mathrm{d}s+\sum\limits_{i=1}^{m-2}\gamma_ix(\eta_i),\end{cases}$$

where D^σ is the standard Riemann-Liouville fractional derivative of order σ satisfying $n-1<\sigma\leq n$, $n\geq 3$, $n\in\mathbb{N}^+$, $0<\eta_1<\eta_2<\cdots<\eta_{m-2}<1$, $\beta_i,\gamma_i>0(1\leq i\leq m-2)$, m is an integer satisfying $m\geq 3$, $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a given continuous function.

Inspired by the above literature, we obtain the existence of positive solutions for fractional differential equations by means of the fixed point theorem of the cone. Compared with the existing literature, this paper has the following two features. First, integral and infinite-point boundary value conditions are considered together in this paper. Second, the aim to this paper is to investigate the existence of positive solutions for boundary value problem (1.1).

In the rest of this paper, the following arrangements have been made. Section 2 is aimed to recall certain basic definitions and lemmas. Section 3 is devoted to gain the main results by virtue of the Guo-Krasnoselskii's fixed point theorem. Section 4 is intended to illustrate the conclusion with an example.

2. Preliminaries

Definition 2.1[13] The Riemann-Liouville fractional integral of order $\alpha>0$ of a continuous function $h:(0,+\infty)\to\mathbb{R}$ is defined as

$$I_{0+}^\alpha h(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}h(s)\mathrm{d}s,$$

provided that the right-hand side is pointwise defined on $(0,+\infty)$, where $\Gamma(\alpha)$ is the gamma function.

Definition 2.2[13] The Riemann-Liouville fractional derivative of order $\alpha > 0$ for a function $f:(0,+\infty)\to\mathbb{R}$ is defined by

$$D_{0+}^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_0^t(t-s)^{n-\alpha-1}f(s)\mathrm{d}s,\quad n-1\le\alpha<n,\ n\in\mathbb{N}.$$

Lemma 2.1[14] Let $\alpha>0$. Assume that $u\in C(0,1)\bigcap L(0,1)$ with a fractional derivative of order α that belongs to $C(0,1)\bigcap L(0,1)$:

$$I_{0+}^{\alpha}D_{0+}^{\alpha}u(t)=u(t)+C_1t^{\alpha-1}+C_2t^{\alpha-2}+\cdots+C_nt^{\alpha-n},$$

where $C_i\in\mathbb{R},\ i=1,2,\cdots,n,\ n-1<\alpha\le n$.

Lemma 2.2 Let $y\in L^1[0,1]$, then the solution of the problem

$$\begin{cases} D_{0+}^{\alpha}u(t)+y(t)=0,\ t\in[0,1],\\ u^{(i)}(0)=0,\ i=0,1,2,...,n-2,\\ D_{0+}^{\beta}u(1)=\sum\limits_{i=1}^{\infty}\beta_i\int_0^{\eta_i}u(s)\mathrm{d}s+\sum\limits_{i=1}^{\infty}\gamma_iu(\eta_i)\end{cases}\tag{2.1}$$

can be expressed by

$$u(t)=\int_0^1G(t,s)y(s)\mathrm{d}s,$$

where

$$G(t,s)=\frac{1}{p(0)\Gamma(\alpha)}\begin{cases}t^{\alpha-1}p(s)(1-s)^{\alpha-\beta-1}-p(0)(t-s)^{\alpha-1}, & 0\le s\le t\le 1,\\ t^{\alpha-1}p(s)(1-s)^{\alpha-\beta-1}, & 0\le t\le s\le 1.\end{cases}\tag{2.2}$$

$$p(s)=\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}-\frac{1}{\alpha}\sum_{s\le\eta_i}\beta_i(\frac{\eta_i-s}{1-s})^{\alpha}(1-s)^{\beta+1}-\sum_{s\le\eta_i}\gamma_i(\frac{\eta_i-s}{1-s})^{\alpha-1}(1-s)^{\beta}.\tag{2.3}$$

Proof By Lemma 2.1, we get

$$u(t)=-I_{0+}^{\alpha}y(t)+C_1t^{\alpha-1}+C_2t^{\alpha-2}+\cdots+C_nt^{\alpha-n}.$$

In view of the boundary conditions $u^i(0)=0,\ i=0,1,2,\cdots n-2$, the parameters $C_2=C_3=\cdots=C_n=0$ are concluded and

$$u(t)=-I_{0+}^{\alpha}y(t)+C_1t^{\alpha-1}.$$

By the boundary value condition

$$D_{0+}^{\beta}u(1)=\sum_{i=1}^{\infty}\beta_i\int_0^{\eta_i}u(s)\mathrm{d}s+\sum_{i=1}^{\infty}\gamma_iu(\eta_i),$$

we have

$$\begin{aligned}D_{0+}^{\beta}u(1)&=-\frac{1}{\Gamma(\alpha-\beta)}\int_0^1(1-s)^{\alpha-\beta-1}y(s)\mathrm{d}s+C_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}\\&=\sum_{i=1}^{\infty}\beta_i\int_0^{\eta_i}u(s)\mathrm{d}s+\sum_{i=1}^{\infty}\gamma_iu(\eta_i)\\&=\sum_{i=1}^{\infty}\beta_i[-I^{\alpha+1}y(\eta_i)+\frac{C_1\eta_i^{\alpha}}{\alpha}]+\sum_{i=1}^{\infty}\gamma_i[-I^{\alpha}y(\eta_i)+C_1\eta_i^{\alpha-1}],\end{aligned}$$

that is to say,

$$C_1\{\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}-\frac{1}{\alpha}\sum_{i=1}^{\infty}\beta_i\eta_i^{\alpha}-\sum_{i=1}^{\infty}\gamma_i\eta_i^{\alpha-1}\}$$

$$
\begin{aligned}
=&\frac{1}{\Gamma(\alpha-\beta)}\int_0^1(1-s)^{\alpha-\beta-1}y(s)\mathrm{d}s-\sum_{i=1}^{\infty}\frac{\beta_i}{\Gamma(\alpha+1)}\int_0^{\eta_i}(\eta_i-s)^{\alpha}y(s)\mathrm{d}s\\
&-\sum_{i=1}^{\infty}\frac{\gamma_i}{\Gamma(\alpha)}\int_0^{\eta_i}(\eta_i-s)^{\alpha-1}y(s)\mathrm{d}s.
\end{aligned}
$$

So,

$$
C_1=\frac{1}{\xi}\{\int_0^1\frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}y(s)\mathrm{d}s-\sum_{i=1}^{\infty}\beta_i\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha}}{\Gamma(\alpha+1)}y(s)\mathrm{d}s-\sum_{i=1}^{\infty}\gamma_i\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s\}.
$$

Hence, the solution is

$$
\begin{aligned}
u(t)=&-\int_0^t\frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}y(s)\mathrm{d}s+\int_0^1\frac{(1-s)^{\alpha-\beta-1}p(s)}{\Gamma(\alpha)p(0)}y(s)\mathrm{d}s\cdot t^{\alpha-1}\\
=&-\int_0^t\frac{p(0)(t-s)^{\alpha-1}}{\Gamma(\alpha)p(0)}y(s)\mathrm{d}s+\int_0^t\frac{(1-s)^{\alpha-\beta-1}p(s)t^{\alpha-1}}{\Gamma(\alpha)p(0)}y(s)\mathrm{d}s\\
&+\int_t^1\frac{(1-s)^{\alpha-\beta-1}p(s)t^{\alpha-1}}{\Gamma(\alpha)p(0)}y(s)\mathrm{d}s\\
=&\int_0^t\frac{p(s)(1-s)^{\alpha-\beta-1}t^{\alpha-1}-p(0)(t-s)^{\alpha-1}}{\Gamma(\alpha)p(0)}y(s)\mathrm{d}s+\int_t^1\frac{p(s)(1-s)^{\alpha-\beta-1}t^{\alpha-1}}{\Gamma(\alpha)p(0)}y(s)\mathrm{d}s\\
=&\int_0^1 G(t,s)y(s)\mathrm{d}s.
\end{aligned}
$$

This completes the proof.

Lemma 2.3 Suppose that $p(0)>0$, and then the function $p(s)>0$, $s\in[0,1]$ and $p(s)$ is nondecreasing on $[0,1]$.

Proof Using hypothesis made by (1.1), we have

$$
p(0)=\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}-\frac{1}{\alpha}\sum_{i=1}^{\infty}\beta_i\eta_i^{\alpha}-\sum_{i=1}^{\infty}\gamma_i\eta_i^{\alpha-1}>0,
$$

$$
\begin{aligned}
p'(s)=&\frac{1}{\alpha}\sum_{s\le\eta_i}\alpha\beta_i(\eta_i-s)^{\alpha-1}(1-s)^{-\alpha}(1-s)^{\beta+1}-\frac{1}{\alpha}\sum_{s\le\eta_i}\beta_i\alpha(\eta_i-s)^{\alpha}(1-s)^{-\alpha-1}(1-s)^{\beta+1}\\
&+\frac{1}{\alpha}\sum_{s\le\eta_i}\beta_i(\beta+1)(\eta_i-s)^{\alpha}(1-s)^{-\alpha}(1-s)^{\beta}\\
&+\sum_{s\le\eta_i}\gamma_i(1-\alpha)(\eta_i-s)^{\alpha-1}(1-s)^{-\alpha}(1-s)^{\beta}\\
&+\sum_{s\le\eta_i}\gamma_i(\alpha-1)(\eta_i-s)^{\alpha-2}(1-s)^{-\alpha+1}(1-s)^{\beta}\\
&+\sum_{s\le\eta_i}\beta\gamma_i(\eta_i-s)^{\alpha-1}(1-s)^{-\alpha+1}(1-s)^{\beta-1}\\
=&\sum_{s\le\eta_i}\beta_i(\eta_i-s)^{\alpha-1}(1-s)^{-\alpha+\beta}[(1-s)-(\eta_i-s)+\frac{\beta+1}{\alpha}(\eta_i-s)]\\
&+\sum_{s\le\eta_i}\gamma_i(\eta_i-s)^{\alpha-2}(1-s)^{-\alpha+\beta}[(\alpha-1)(1-s)-(\alpha-1)(\eta_i-s)+\beta(\eta_i-s)]>0.
\end{aligned}
$$

So $p(s)$ is nondecreasing on $[0,1]$. This completes the proof.

Lemma 2.4 The function $G(t,s)$ defined by (2.2) satisfies the following properties:

1) $G(t,s)\ge 0, \frac{\partial}{\partial t}G(t,s)\ge 0, 0\le t,s\le 1$;

2) $\max\limits_{t\in[0,1]} G(t,s) = G(1,s), 0 \le s \le 1;$

3) $G(t,s) \ge t^{\alpha-1}G(1,s), 0 \le t, s \le 1;$

4) $\max\limits_{\frac{1}{4}\le t\le \frac{3}{4}} G(t,s) \ge (\frac{1}{4})^{\alpha-1}G(1,s), 0 \le s \le 1.$

Proof 1) For $0 \le s \le t \le 1$, by Lemma 2.3, we can see that

$$\begin{aligned}G(t,s) =&\frac{1}{p(0)\Gamma(\alpha)}[t^{\alpha-1}p(s)(1-s)^{\alpha-\beta-1} - p(0)(t-s)^{\alpha-1}]\\ =&\frac{t^{\alpha-1}}{p(0)\Gamma(\alpha)}[p(s)(1-s)^{\alpha-\beta-1} - p(0)(1-\frac{s}{t})^{\alpha-1}]\\ \ge&\frac{p(s)t^{\alpha-1}}{p(0)\Gamma(\alpha)}[(1-s)^{\alpha-\beta-1} - (1-\frac{s}{t})^{\alpha-1}]\\ >&\frac{p(s)t^{\alpha-1}}{p(0)\Gamma(\alpha)}[(1-s)^{\alpha-1} - (1-\frac{s}{t})^{\alpha-1}] \ge 0.\end{aligned}$$

It is clear that for $0 \le t \le s \le 1$, $G(t,s) \ge 0$.

By simple computation, we have

$$\frac{\partial}{\partial t}G(t,s) = \frac{1}{p(0)\Gamma(\alpha)}\begin{cases}(\alpha-1)p(s)(1-s)^{\alpha-\beta-1}t^{\alpha-2} - (\alpha-1)p(0)(t-s)^{\alpha-2}, & 0 \le s \le t \le 1,\\ (\alpha-1)p(s)(1-s)^{\alpha-\beta-1}t^{\alpha-2}, & 0 \le t \le s \le 1.\end{cases}$$

Apparently, $\frac{\partial}{\partial t}G(t,s)$ is continuous on $[0,1]\times[0,1]$. By Lemma 2.3, we only need to consider on $0 \le s \le t \le 1$. Here,

$$\begin{aligned}\frac{\partial}{\partial t}G(t,s) =&\frac{1}{p(0)\Gamma(\alpha)}[(\alpha-1)p(s)(1-s)^{\alpha-\beta-1}t^{\alpha-2} - (\alpha-1)p(0)(t-s)^{\alpha-2}]\\ \ge&\frac{(\alpha-1)p(0)t^{\alpha-2}}{p(0)\Gamma(\alpha)}[(1-s)^{\alpha-\beta-1} - (1-\frac{s}{t})^{\alpha-2}]\\ \ge&\frac{(\alpha-1)}{\Gamma(\alpha)}t^{\alpha-2}[(1-s)^{\alpha-2} - (1-\frac{s}{t})^{\alpha-2}] \ge 0.\end{aligned}$$

2) By 1), we get that $G(t,s)$ is nondecreasing. So,

$$\max_{t\in[0,1]} G(t,s) = G(1,s) = \frac{1}{p(0)\Gamma(\alpha)}[p(s)(1-s)^{\alpha-\beta-1} - p(0)(1-s)^{\alpha-1}], 0 \le s \le 1.$$

3) For $0 \le s \le t \le 1$, we can see that

$$\begin{aligned}G(t,s) =&\frac{1}{p(0)\Gamma(\alpha)}[t^{\alpha-1}p(s)(1-s)^{\alpha-\beta-1} - p(0)(t-s)^{\alpha-1}]\\ =&\frac{1}{p(0)\Gamma(\alpha)}t^{\alpha-1}[p(s)(1-s)^{\alpha-\beta-1} - p(0)(1-\frac{s}{t})^{\alpha-1}]\\ \ge&\frac{1}{p(0)\Gamma(\alpha)}t^{\alpha-1}[p(s)(1-s)^{\alpha-\beta-1} - p(0)(1-s)^{\alpha-1}]\\ =&t^{\alpha-1}G(1,s).\end{aligned}$$

For $0 \le t \le s \le 1$, we can notice that

$$G(t,s) = \frac{1}{p(0)\Gamma(\alpha)}t^{\alpha-1}p(s)(1-s)^{\alpha-\beta-1} \ge t^{\alpha-1}G(1,s).$$

4) By 3), it is easy to see that $\max\limits_{\frac{1}{4}\le t\le\frac{3}{4}} G(t,s) \ge (\frac{1}{4})^{\alpha-1}G(1,s)$, $0 \le s \le 1$.

Let $E = C[0,1]$ be the Banach space equipped with the norm $\| u \| = \sup\limits_{0\le t\le 1} |u(t)|$ and define a cone P by

$$P = \{u \in E | u(t) \ge t^{\alpha-1} \| u \|, t \in [0,1]\}.$$

Define the operator T as follows:

$$(Tu)(t)=\int_0^1 G(t,s)h(s,u(s))\mathrm{d}s, 0\le t\le 1. \tag{2.4}$$

Clearly, $T:P\to C[0,1]$. By Lemma 2.2, we get that the fixed point of T is the solution of problem (1.1).

Lemma 2.5 $T:P\to P$ is completely continuous.

Proof First, $\forall u\in P$, it follows from the definition of T and the non-negativity of $G(t,s)$ and h that, $(Tu)(t)\ge 0(0\le t\le 1)$. By Lemma 2.4 and (2.4), we have that

$$(Tu)(t)=\int_0^1 G(t,s)h(s,u(s))\mathrm{d}s\le\int_0^1 G(1,s)h(s,u(s))\mathrm{d}s,\quad \forall t\in[0,1],$$

$$(Tu)(t)=\int_0^1 G(t,s)h(s,u(s))\mathrm{d}s\ge t^{\alpha-1}\int_0^1 G(1,s)h(s,u(s))\mathrm{d}s,\quad \forall t\in[0,1].$$

From the above two inequalities, we can get

$$(Tu)(t)\ge t^{\alpha-1}\parallel Tu\parallel,\quad \forall t\in[0,1].$$

So, $Tu\in P$. That means $T:P\to P$.

From the continuity of h, we know that T is continuous on P. Next, we are in a position to show that T is a compact mapping. $\forall D\subset P$, D is bounded. That is to say, $\exists M>0,\forall u\in D$, we have $\parallel u\parallel\le M$. Denote $L=\max\limits_{0\le t\le 1,0\le u\le M}\mid h(t,u)\mid +1$. For any $u\in D$, we get that

$$\mid Tu(t)\mid\le\int_0^1 G(1,s)h(s,u(s))\mathrm{d}s\le L\int_0^1 G(1,s)\mathrm{d}s.$$

So, $T(D)$ is bounded. For $t_1,t_2\in[0,1],t_1<t_2$, we have

$$\begin{aligned}
&\mid Tu(t_2)-Tu(t_1)\mid\\
=&\mid\int_0^{t_1}(G(t_2,s)-G(t_1,s))h(s,u(s))\mathrm{d}s+\int_{t_1}^{t_2}(G(t_2,s)-G(t_1,s))h(s,u(s))\mathrm{d}s\\
&+\int_{t_2}^1(G(t_2,s)-G(t_1,s))h(s,u(s))\mathrm{d}s\mid\\
\le&\frac{L}{p(0)\Gamma(\alpha)}\int_0^{t_1}\mid p(s)(1-s)^{\alpha-\beta-1}t_2^{\alpha-1}-p(0)(t_2-s)^{\alpha-1}-p(s)(1-s)^{\alpha-\beta-1}t_1^{\alpha-1}\\
&+p(0)(t_1-s)^{\alpha-1}\mid\mathrm{d}s+\frac{L}{p(0)\Gamma(\alpha)}\int_{t_1}^{t_2}\mid p(s)(1-s)^{\alpha-\beta-1}-p(0)(t_2-s)^{\alpha-1}\\
&-p(s)(1-s)^{\alpha-\beta-1}t_1^{\alpha-1}\mid\mathrm{d}s+\frac{L}{p(0)\Gamma(\alpha)}\int_{t_2}^1\mid p(s)(1-s)^{\alpha-\beta-1}(t_2^{\alpha-1}-t_1^{\alpha-1})\mid\mathrm{d}s\\
\le&\frac{L}{p(0)\Gamma(\alpha)}\int_0^{t_1}\mid p(s)(1-s)^{\alpha-\beta-1}(t_2^{\alpha-1}-t_1^{\alpha-1})\mid\mathrm{d}s+\frac{L}{\Gamma(\alpha)}\int_0^{t_1}\mid(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\mid\mathrm{d}s\\
&+\frac{L}{p(0)\Gamma(\alpha)}\int_{t_1}^{t_2}\mid p(s)(1-s)^{\alpha-\beta-1}(t_2^{\alpha-1}-t_1^{\alpha-1})\mid\mathrm{d}s+\frac{L}{p(0)\Gamma(\alpha)}\int_{t_1}^{t_2}\mid p(0)(t_2-s)^{\alpha-1}\mid\mathrm{d}s\\
&+\frac{L}{p(0)\Gamma(\alpha)}\int_{t_2}^1\mid p(s)(1-s)^{\alpha-\beta-1}(t_2^{\alpha-1}-t_1^{\alpha-1})\mid\mathrm{d}s\\
\le&\frac{L}{p(0)\Gamma(\alpha-\beta)}\int_0^{t_1}(1-s)^{\alpha-\beta-1}(t_2^{\alpha-1}-t_1^{\alpha-1})\mathrm{d}s+\frac{L}{\Gamma(\alpha)}\int_0^{t_1}((t_2-s)^{\alpha-1}\\
&-(t_1-s)^{\alpha-1})\mathrm{d}s+\frac{L}{p(0)\Gamma(\alpha-\beta)}\int_{t_1}^{t_2}(1-s)^{\alpha-\beta-1}(t_2^{\alpha-1}-t_1^{\alpha-1})\mathrm{d}s
\end{aligned}$$

$$+\frac{L}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\mathrm{d}s+\frac{L}{p(0)\Gamma(\alpha-\beta)}\int_{t_2}^{1}(1-s)^{\alpha-\beta-1}(t_2^{\alpha-1}-t_1^{\alpha-1})\mathrm{d}s$$
$$=\frac{L}{p(0)\Gamma(\alpha-\beta+1)}({t_2}^{\alpha-1}-{t_1}^{\alpha-1})+\frac{L}{\Gamma(\alpha+1)}(t_2^{\alpha}-t_1^{\alpha}-(t_2-t_1)^{\alpha})+\frac{L}{\Gamma(\alpha+1)}(t_2-t_1)^{\alpha}.$$

So, $T(D)$ is equicontinuous. By the Arzela-Ascoli theorem, we can conclude that the operator $T:P\to P$ is completely continuous.

Theorem 2.1 (Guo-Krasnoselskii's Fixed Point Theorem)[15] Let E be a Banach space, $P\subseteq E$ a cone, and Ω_1,Ω_2 two bounded open balls of E centered at the origin with $\overline{\Omega_1}\subset\Omega_2$. Suppose that:$A:P\bigcap(\overline{\Omega_2}\backslash\Omega_1)\longrightarrow P$ is a completely continuous operator such that either

1) $\|Ax\|\le\|x\|$, $x\in P\bigcap\partial\Omega_1$, and $\|Ax\|\ge\|x\|$, $x\in P\bigcap\partial\Omega_2$ or

2) $\|Ax\|\ge\|x\|$, $x\in P\bigcap\partial\Omega_1$, and $\|Ax\|\le\|x\|$, $x\in P\bigcap\partial\Omega_2$

holds. Then A has at least one fixed point in $P\bigcap(\overline{\Omega}_2\backslash\Omega_1)$.

3. Main Results

Definition 3.1 Let

$$h^0=\lim_{u\to0^+}\max_{0\le t\le1}\frac{h(t,u)}{u},h_\infty=\lim_{u\to+\infty}\min_{0\le t\le1}\frac{h(t,u)}{u}.$$

Theorem 3.1 If $h^0=0$,$h_\infty=+\infty$, then the boundary value problem (1.1) has at least a positive solution u^*.

Proof By the definition of h^0, we have that $\exists r_1>0$, for $0<u\le r_1$, then

$$\max_{0\le t\le1}\frac{h(t,u)}{u}\le(\int_0^1G(1,s)\mathrm{d}s)^{-1}.$$

That is to say, for $0\le u\le r_1$, $0\le t\le1$, we get that

$$h(t,u)\le(\int_0^1G(1,s)\mathrm{d}s)^{-1}u.$$

Denote $\Omega_1=\{u\in P:\parallel u\parallel<r_1\}$. If $u\in\partial\Omega_1$, we know that $\parallel u\parallel=r_1$. So, for $0\le s\le1$, we have

$$0\le u(s)\le r_1,\ \ h(s,u(s))\le(\int_0^1G(1,s)\mathrm{d}s)^{-1}u(s).$$

By Lemma 2.4, we have that

$$\begin{aligned}\parallel Tu\parallel&=\max_{0\le t\le1}\mid(Tu)(t)\mid=\max_{0\le t\le1}(Tu)(t)=\max_{0\le t\le1}\int_0^1G(t,s)h(s,u(s))\mathrm{d}s\\&\le\int_0^1G(1,s)h(s,u(s))\mathrm{d}s\le(\int_0^1G(1,s)\mathrm{d}s)^{-1}(\int_0^1G(1,s)u(s)\mathrm{d}s)\\&\le(\int_0^1G(1,s)\mathrm{d}s)^{-1}(\int_0^1G(1,s)\mathrm{d}s)\cdot r_1=\parallel u\parallel.\end{aligned}$$

By the definition of h_∞, we get that $\exists\overline{r_2}>0$, for $u\ge\overline{r_2}$, we have

$$\min_{0\le t\le1}\frac{h(t,u)}{u}>16^{\alpha-1}(\int_{\frac14}^{\frac34}G(1,s)\mathrm{d}s)^{-1}.$$

That is to say, for $u\ge\overline{r_2}$, $0\le t\le1$, we get that

$$h(t,u)>16^{\alpha-1}(\int_{\frac14}^{\frac34}G(1,s)\mathrm{d}s)^{-1}u.$$

Let $r_2>\max\{r_1,4^{\alpha-1}\overline{r_2}\}$, $\Omega_2=\{u\in P:\parallel u\parallel<r_2\}$. If $u\in\partial\Omega_2$, then $\parallel u\parallel=r_2$. For $\frac14\le s\le\frac34$, we have

$$u(s)\ge s^{\alpha-1}\parallel u\parallel\ge(\frac14)^{\alpha-1}\parallel u\parallel=(\frac14)^{\alpha-1}r_2\ge\overline{r_2},$$

$$h(s,u(s)) > 16^{\alpha-1}(\int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)\mathrm{d}s)^{-1}u(s).$$

By Lemma 2.4, we obtain

$$\begin{aligned}\| Tu \| &= \max_{0\le t\le 1} | (Tu)(t) | = \max_{0\le t\le 1} \int_0^1 G(t,s)h(s,u(s))\mathrm{d}s \\ &\ge \max_{0\le t\le 1} \int_{\frac{1}{4}}^{\frac{3}{4}} G(t,s)h(s,u(s))\mathrm{d}s \\ &> (\frac{1}{4})^{\alpha-1} \int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)16^{\alpha-1}[\int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)\mathrm{d}s]^{-1}\mathrm{d}s \\ &\ge 16^{\alpha-1}[\int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)\mathrm{d}s]^{-1}(\frac{1}{4})^{\alpha-1}\int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)\mathrm{d}s\cdot(\frac{1}{4})^{\alpha-1}r_2 \\ &= r_2 = \| u \| .\end{aligned}$$

It follows from Theorem 2.1 that T has at least one fixed point u^* in $P\bigcap(\overline{\Omega_2}\backslash\Omega_1)$. So the boundary value problem (1.1) has at least one positive solution u^*. This completes the proof.

4. Example

In this section, we will present an example to illustrate our main results.

Example 4.1 Consider the following boundary value problem

$$\begin{cases} D_{0^+}^{\frac{7}{2}}u(t) + \frac{u^3(t)}{5\sqrt[4]{u+1}\sqrt[3]{1+t}} = 0, \quad 0\le t\le 1, \\ u(0) = u'(0) = u''(0) = 0, \\ D_{0^+}^{1}u(1) = \sum\limits_{i=1}^{\infty} \frac{1}{2^i}\int_0^{1-\frac{1}{i+1}} u(s)\mathrm{d}s + \sum\limits_{i=1}^{\infty} \frac{1}{3^i}u(1-\frac{1}{i+1}). \end{cases} \tag{4.1}$$

Here, $\alpha = \frac{7}{2}$, $\beta_i = \frac{1}{2^i}$, $\gamma_i = \frac{1}{3^i}$, $\eta_i = 1-\frac{1}{i+1}$, $h(t,u) = \frac{u^3}{5\sqrt[4]{u+1}}\frac{1}{\sqrt[3]{1+t}}$, $\beta = 1$. Clearly, $\beta_i, \gamma_i > 0$, $0<\eta_1<\eta_2<\cdots<\eta_i<\cdots<1$, $i = 1,2,\cdots$. And

$$\begin{aligned}\xi &= \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} - \frac{1}{\alpha}\sum_{i=1}^{\infty}\beta_i{\eta_i}^{\alpha} - \sum_{i=1}^{\infty}\gamma_i{\eta_i}^{\alpha-1} \\ &= \frac{5}{2} - \frac{2}{7}\sum_{i=1}^{\infty}\frac{1}{2^i}(1-\frac{1}{i+1})^{\frac{7}{2}} - \sum_{i=1}^{\infty}\frac{1}{3^i}(1-\frac{1}{i+1})^{\frac{5}{2}} \\ &\ge \frac{5}{2} - \frac{2}{7}(\frac{1}{2}\cdot 1^{\frac{7}{2}} + \frac{1}{2^2}\cdot 1^{\frac{7}{2}} + \cdots + \frac{1}{2^n}\cdot 1^{\frac{7}{2}} + \cdots) - (\frac{1}{3}\cdot 1^{\frac{5}{2}} + \frac{1}{3^2}\cdot 1^{\frac{5}{2}} + \cdots + \frac{1}{3^n}\cdot 1^{\frac{5}{2}} + \cdots) \\ &= \frac{12}{7} > 0.\end{aligned}$$

Noticing

$$\frac{h(t,u)}{u} = \frac{u^2}{5\sqrt[4]{u+1}}\frac{1}{\sqrt[3]{1+t}},$$

we have

$$h^0 = \lim_{u\to 0^+}\max_{0\le t\le 1}\frac{h(t,u)}{u} = \lim_{u\to 0^+}\frac{u^2}{5\sqrt[4]{u+1}}\cdot 1 = 0,$$

$$h_\infty = \lim_{u\to +\infty}\min_{0\le t\le 1}\frac{h(t,u)}{u} = \lim_{u\to +\infty}\frac{u^2}{5\sqrt[4]{u+1}}\cdot\frac{1}{\sqrt[3]{2}} = +\infty.$$

Therefore the assumptions of Theorem 3.1 are satisfied. Thus Theorem 3.1 ensures that the problem (4.1) has at least one positive solution.

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带有积分与无穷点边值条件的分数阶微分方程的正解

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摘要: 研究一类带有积分边值条件和无穷点边值条件的分数阶微分方程的正解问题. 借助Green函数有关的性质, 并利用锥上不动点定理, 获得该问题正解的存在性结果. 最后给出一个例子说明所得结果的应用性.

关键词: 积分边值条件; 分数阶微分方程; 不动点定理; 正解