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Comparison Principle of Very Weak Solutions for Nonhomogeneous Elliptic Equations

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Abstract: Under some suitable assumptions, a comparison principle of very weak solutions for quasi-linear elliptic equation $-\text{div}A(x,\nabla u) = f(x,u)$ is given by using McShane extension theorem to construct the Lipschitz continuous test function, and the corresponding results of some homogeneous equations are generalized.
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1. Introduction

In this paper , we study a comparison principle of very weak solutions to nonhomogeneous elliptic equation

$$-\text{div}A(x,\nabla u) = f(x,u). \tag{1.1}$$

In recent years, there exist many results of weak solutions for quasilinear elliptic equations. Gilbarg and Trudinger[1] established a comparison principle of classical solutions for second-order quasilinear elliptic equation. Tolksdorf[2] generalized the results of [1] and obtained a comparison principle of weak solutions for $\sum_{j=1}^{n} A_j(x,u,\nabla u) - A(x,u,\nabla u) = 0$. Damascelli[3] studied a comparison principle of weak solutions(in $W^{1,\infty}(\Omega)$) for $-\text{div}A(x,\nabla u) = g(x,u)$ by taking an appropriate test function. The right hand side of the equation(i.e.lower order term) in [1-3] satisfied non-increasing for u. The definition of very weak solutions for A-harmonic equation were given[4], Iwaniec et al.[5] obtained the existence and local integrability of very weak solutions to the A-harmonic equation by using Hodge decomposition method. By Hodge decomposition method to construct a proper test function, the uniqueness of the very weak solutions of $-\text{div}A(x,\nabla u) = f(x,u)$ is obtained under the condition of weak boundary value in [6]. More references about Hodge decomposition see [7-9].

Lewis[10] and ZHONG[11] studied the existence and uniqueness of very weak solutions for $D^m A(x, D^m u) = 0$ and $-\text{div}A(x,\nabla u) = \mu$ in Grand Sobolev space by using the method of maximal function to construct Lipschitz-type continuous test function. SHI[12]

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studied a comparison principle of very weak solutions to $-\mathrm{div}A(x,\nabla u)=0$ by the similar method in [10-11]. ZHU[13] introduced a comparison principle of very weak solutions for $-\mathrm{div}A(x,u,\nabla u)=f(x)+\mathrm{div}(|\nabla u^{p-2}|\nabla u)$ by constructing a suitable test function, and combining Hardy-Littlewood maximum function et al.

However, the comparison principle of very weak solutions to equation (1.1) has not been studied yet. Inspired by [6,11-12], we study a comparison principle of equation (1.1) by using McShane extension theorem to contract the Lipschitz continuous test function. Compared with the right hand side in [13], our $f(x,u)$ in (1.1) is more general, the reason is that the right hand side of [13] is actually $f(x)$ which is independent of u. In particular, owing to the appearance of $f(x,u)$ in the proof, we apply Theorem 2.7 in [14] (i.e. Lemma 2.3 in this paper), Sobolev embedding theorem, Hölder's and Young's inequalities in order to estimate the integral term of $f(x,u)$. In this paper, we assume $f(x,u)$ to be non-increasing for u, so that our result also holds in classical solution and weak solution cases.

Let $\Omega\subset\mathbb{R}^n(n\geq 2)$ be a bounded open subset, $1<p<\infty$, the Carathéodory function $A(x,\xi):\Omega\times\mathbb{R}^{nN}\to\mathbb{R}^n$, $f(x,\xi):\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$, and $f(x,\xi)$ is non-increasing for ξ, $A(x,\xi)$ and $f(x,\xi)$ satisfies:

$$\langle A(x,\xi_1)-A(x,\xi_2),\xi_1-\xi_2\rangle\geq\alpha(|\xi_1|+|\xi_2)^{p-2}|\xi_1-\xi_2|^2,\tag{1.2}$$

$$|A(x,\xi_1)-A(x,\xi_2)|\leq\beta(|\xi_1|+|\xi_2|)^{p-2}|\xi_1-\xi_2|,\tag{1.3}$$

$$|f(x,\zeta_1)-f(x,\zeta_2)|\leq|\zeta_1-\zeta_2|^{p-1},\tag{1.4}$$

where $x\in\Omega$, $\xi_i(i=1,2)\in\mathbb{R}^{nN}$, $\zeta_i(i=1,2)\in\mathbb{R}^n$, $0<\alpha<\beta<\infty$.

Remark 1 We know that $f(x,u)$ is non-increasing for u, this condition and above (1.4) are not contradict, for example, when $u>0$,

$$f(x,u)=\begin{cases}u^p, & 0<u<1\\ u^{1-p}, & u\geq 1.\end{cases}$$

Definition1.1 A function $u(x)\in W^{1,r}(\Omega)$, $\max\{1,p-1\}<r<p$ is called a very weak solution to elliptic equation (1.1) if

$$\int_\Omega\langle A(x,\nabla u),\nabla\phi\rangle\mathrm{d}x=\int_\Omega\langle f(x,u),\phi\rangle\mathrm{d}x\tag{1.5}$$

for all $\phi\in C_0^\infty(\Omega)$.

2. Preliminary Lemmas and Main Result

In order to prove the main result, we need the following lemmas.

Lemma 2.1[9] Let $h\in W^{1,q}(\Omega)$, $1<q<\infty$. Then there exists a set $N\subset\mathbb{R}^n$, $|N|=0$, such that

$$|h(x)-h(y)|\leq c|x-y|(M(|Dh|)(x)+M(|Dh|)(y))$$

for every $x,y\in\mathbb{R}^n\backslash N$, where $c=c(n)>0$, $Mh(x)$ is a Hardy-littlewood maximal function of $h(x)$.

Let $h(x)\in W_0^{1,r}(\Omega)$,$1<r<\infty$, extending $h(x)=0$ to $\mathbb{R}^n$, so that $h(x)=0$ in $C\Omega$. For a real number $\lambda>0$, we denote

$$\begin{aligned}F_\lambda&=\{x\in\Omega\backslash N:M(|Dh|)(x)\leq\lambda,|h(x)|\leq\lambda d(x,\partial\Omega)\}\\&=\{x\in\Omega\backslash N:g(x)\leq\lambda\},\end{aligned}$$

where $g(x)=\max\{M(|Dh|)(x),|h(x)|d^{-1}(x,\Omega)\}$.

It is easy to prove that $h|_{F_\lambda \cup C\Omega}$ is $C\lambda$-Lipschitz continuity, where the constant $C = C(n) \geq 1$. Applying McShane extension theorem, we get the following lemma:

Lemma 2.2[11] Let $h(x) \in W_0^{1,r}(\Omega)$, for any given $\lambda > 0$, F_λ was defined by Lemma 2.1, then there exists Lipschitz continuous function h_λ satisfying:

(i) $h_\lambda(x) = h(x)$ for every $x \in F_\lambda$;

(ii) $|Dh_\lambda(x)| \leq C(n)\lambda$ for every $x \in \mathbb{R}^n$;

(iii) $h_\lambda(x) = 0$ where $x \in C\Omega$;

(iv) $\|Dh_\lambda(x)\|_\infty = \|Dh(x)\|_\infty$ for a.e. $x \in F_\lambda$.

Lemma 2.3[14] Every function u: $U \to \mathbb{R}$ of class $C^{0,1}(U)$ belongs to $W^{1,\infty}_{\text{loc}}(U)$ (where U is the open set in $\mathbb{R}^n$).

Our main result is the following comparison principle:

Theorem 2.1 Assume that the equation (1.1) satisfies (1.2), (1.3) and (1.4). There exists a constant $0 < \varepsilon_0 = \varepsilon_0(n, p, \alpha, \beta, \gamma) < 1$ such that solutions $u_1, u_2 \in W^{1,r}(\Omega)$, where $r > p - \varepsilon_0$, if $u_1(x) \geq u_2(x)$ on $\partial\Omega$, then $u_1(x) \geq u_2(x)$ on Ω.

3. Proof of Theorem

In the following proof, all the constants C may change from line to line.

Proof Let u_1, $u_2 \in W^{1,r}(\Omega)$ are solutions of equation (1.1), and $u_1 \geq u_2$ on $\partial\Omega$. We consider $v(x) = \min\{0, u_1 - u_2\}$ and know that $v(x) \in W_0^{1,r}(\Omega)$. From Lemma 2.1 and Lemma 2.2, there exists v_λ as Lipschitz continuous extension of function $v(x)$ on $F_\lambda \cup C\Omega$, and v_λ can be used as a test function in Definition 1.1 because it satisfied (i)-(iv) from Lemma 2.2. Next we have

$$\int_\Omega \langle A(x, \nabla u), \nabla v_\lambda \rangle \mathrm{d}x = \int_\Omega \langle f(x, u), v_\lambda \rangle \mathrm{d}x.$$

Due to u_1, u_2 are solutions of equation (1.1), we obtained

$$\int_\Omega \langle A(x, \nabla u_1), \nabla v_\lambda \rangle \mathrm{d}x = \int_\Omega \langle f(x, u_1), v_\lambda \rangle \mathrm{d}x,$$

$$\int_\Omega \langle A(x, \nabla u_2), \nabla v_\lambda \rangle \mathrm{d}x = \int_\Omega \langle f(x, u_2), v_\lambda \rangle \mathrm{d}x.$$

The above two formulas are subtracted,

$$\int_\Omega \langle A(x, \nabla u_1) - A(x, \nabla u_2), \nabla v_\lambda \rangle \mathrm{d}x = \int_\Omega \langle f(x, u_1) - f(x, u_2), v_\lambda \rangle \mathrm{d}x.$$

Note that $\Omega = \Omega_1 \bigcup \Omega_2 = \{x \in \Omega : u_1(x) \geq u_2(x)\} \bigcup \{x \in \Omega : u_1(x) < u_2(x)\}$. Theorem 2.1 is clearly in Ω_1. In order to get our result, we only need to prove Ω_2 is empty, that is $u_1(x) \geq u_2(x)$ on Ω_2. There have

$$\int_{\Omega_2} \langle A(x, \nabla u_1) - A(x, \nabla u_2), \nabla v_\lambda \rangle \mathrm{d}x = \int_{\Omega_2} \langle f(x, u_1) - f(x, u_2), v_\lambda \rangle \mathrm{d}x. \tag{3.1}$$

Owing to Lemma 2.1, there have $v_\lambda = v$ on F_λ, so

$$\begin{aligned} &\int_{F_\lambda} \langle A(x, \nabla u_1) - A(x, \nabla u_2), \nabla u_1 - \nabla u_2 \rangle \mathrm{d}x \\ =& -\int_{\Omega_2 \backslash F_\lambda} \langle A(x, \nabla u_1) - A(x, \nabla u_2), \nabla v_\lambda \rangle \mathrm{d}x + \int_{F_\lambda} \langle f(x, u_1) - f(x, u_2), u_1 - u_2 \rangle \mathrm{d}x \\ &+ \int_{\Omega_2 \backslash F_\lambda} \langle f(x, u_1) - f(x, u_2), v_\lambda \rangle \mathrm{d}x \end{aligned}$$

$$\leq |I_1| + |I_2| + |I_3|. \tag{3.2}$$

We combined (1.2) to get the following estimate,

$$\begin{aligned} &\int_{F_\lambda} \langle A(x,\nabla u_1) - A(x,\nabla u_2), \nabla u_1 - \nabla u_2 \rangle \mathrm{d}x \\ \geq\ & \alpha \int_{F_\lambda} (|\nabla u_1| + |\nabla u_2|)^{p-2} |\nabla u_1 - \nabla u_2|^2 \mathrm{d}x. \end{aligned} \tag{3.3}$$

According to the condition (1.3) and (1.4), Lemma 2.2 and the Sobolev embedding theorem, we deduce that

$$|I_1| \leq C\beta\lambda \int_{\Omega_2 \setminus F_\lambda} (|\nabla u_1| + |\nabla u_2|)^{p-2} |\nabla u_1 - \nabla u_2| \mathrm{d}x,$$

$$|I_2| \leq \int_{F_\lambda} |u_1 - u_2|^p \mathrm{d}x \leq C \int_{F_\lambda} |\nabla u_1 - \nabla u_2|^p \mathrm{d}x.$$

Next, by using Hölder's and Young's inequalities, the Sobolev embedding theorem, Lemma 2.2 and Lemma 2.3, we have

$$\begin{aligned} |I_3| \leq & \int_{\Omega_2 \setminus F_\lambda} |u_1 - u_2|^{p-1} |v_\lambda| \mathrm{d}x \\ \leq & \left(\int_{\Omega_2 \setminus F_\lambda} |u_1 - u_2|^p \mathrm{d}x \right)^{\frac{p-1}{p}} \left(\int_{\Omega_2 \setminus F_\lambda} |v_\lambda|^p \mathrm{d}x \right)^{\frac{1}{p}} \\ \leq & C(\varepsilon') \int_{\Omega_2 \setminus F_\lambda} |u_1 - u_2|^p \mathrm{d}x + \varepsilon' \int_{\Omega_2 \setminus F_\lambda} |v_\lambda|^p \mathrm{d}x \\ \leq & C(\varepsilon') C \int_{\Omega_2 \setminus F_\lambda} |\nabla u_1 - \nabla u_2|^p \mathrm{d}x + \varepsilon' C \int_{\Omega_2 \setminus F_\lambda} |\nabla v_\lambda|^p \mathrm{d}x \\ \leq & C(\varepsilon') C \int_{\Omega_2 \setminus F_\lambda} |\nabla u_1 - \nabla u_2|^p \mathrm{d}x + \varepsilon' C \lambda^{p-1} \int_{\Omega_2 \setminus F_\lambda} \lambda \mathrm{d}x, \end{aligned}$$

where from the third line to the fourth line, we apply Lemma 2.3, since $v_\lambda \in C^{0,1}(\Omega)$, $v_\lambda \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$, that is $v_\lambda \in W^{1,p}_{\mathrm{loc}}(\Omega), 1 < p \leq +\infty$.

Combing (3.3) and the estimates of $|I_1|, |I_2|, |I_3|$, there have

$$\begin{aligned} & \alpha \int_{F_\lambda} (|\nabla u_1| + |\nabla u_2|)^{p-2} |\nabla u_1 - \nabla u_2|^2 \mathrm{d}x \\ \leq & C\beta\lambda \int_{\Omega_2 \setminus F_\lambda} (|\nabla u_1| + |\nabla u_2|)^{p-2} |\nabla u_1 - \nabla u_2| \mathrm{d}x \\ & + C \int_{F_\lambda} |\nabla u_1 - \nabla u_2|^p \mathrm{d}x + C(\varepsilon') C \int_{\Omega_2 \setminus F_\lambda} |\nabla u_1 - \nabla u_2|^p \mathrm{d}x + \varepsilon' C \lambda^{p-1} \int_{\Omega_2 \setminus F_\lambda} \lambda \mathrm{d}x. \end{aligned} \tag{3.4}$$

Multiplying both sides of the above inequality by $\lambda^{-1-\varepsilon} (0 < \varepsilon < 1)$ and integrating λ on $(0, +\infty)$, we deduce that

$$\begin{aligned} & \frac{\alpha}{\varepsilon} \int_{\Omega_2} (|\nabla u_1| + |\nabla u_2|)^{p-2} |\nabla u_1 - \nabla u_2|^2 g(x)^{-\varepsilon} \mathrm{d}x \\ \leq & \frac{C\beta}{1-\varepsilon} \int_{\Omega_2} (|\nabla u_1| + |\nabla u_2|)^{p-2} |\nabla u_1 - \nabla u_2| g(x)^{1-\varepsilon} \mathrm{d}x \\ & + \frac{C}{\varepsilon} \int_{\Omega_2} |\nabla u_1 - \nabla u_2|^p g(x)^{-\varepsilon} \mathrm{d}x + \frac{C(\varepsilon') C}{-\varepsilon} \int_{\Omega_2} |\nabla u_1 - \nabla u_2|^p g(x)^{-\varepsilon} \mathrm{d}x \\ & + \frac{\varepsilon' C \lambda^{p-1}}{(1-\varepsilon)} \int_{\Omega_2} g(x)^{1-\varepsilon} \mathrm{d}x \\ \leq & \frac{C\beta}{1-\varepsilon} \int_{\Omega_2} (|\nabla u_1| + |\nabla u_2|)^{p-2} |\nabla u_1 - \nabla u_2| g(x)^{1-\varepsilon} \mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&+\frac{C(1-C(\varepsilon'))}{\varepsilon}\int_{\Omega_2}|\nabla u_1-\nabla u_2|^p g(x)^{-\varepsilon}\mathrm{d}x+\frac{\varepsilon' C\lambda^{p-1}}{(1-\varepsilon)}\int_{\Omega_2}g(x)^{1-\varepsilon}\mathrm{d}x\\
\leq&\frac{C\beta}{1-\varepsilon}\int_{\Omega_2}(|\nabla u_1|+|\nabla u_2|)^{p-2}|\nabla u_1-\nabla u_2|g(x)^{1-\varepsilon}\mathrm{d}x+\frac{\varepsilon' C}{(1-\varepsilon)}\int_{\Omega_2}|g(x)|^{1-\varepsilon}\mathrm{d}x\\
\leq&|I_4|+|I_5|.
\end{aligned}\tag{3.5}
$$

Remark 2 In (3.5), we need point out, since $\varepsilon' < 1$ and $C(\varepsilon') > 1$ are coefficients of Young's inequality, C is a positive constant in Sobolev embedding theorem, then $\frac{C(1-C(\varepsilon'))}{\varepsilon} < 0$ in the sixth line of (3.5). Meanwhile, λ^{p-1} can be merged into C because $\lambda > 0$.

Similar to [12], recalling the definition of $g(x)$, combing our test function, we deduce that

$$
\begin{aligned}
|I_4|=&\frac{C\beta}{1-\varepsilon}\int_{\Omega_2}(|\nabla u_1|+|\nabla u_2|)^{p-2}|\nabla u_1-\nabla u_2|g(x)^{1-\varepsilon}\mathrm{d}x\\
\leq&\frac{C\beta}{1-\varepsilon}\int_{\Omega_2}(|\nabla u_1|+|\nabla u_2|)^{p-1}g(x)^{1-\varepsilon}\mathrm{d}x\\
\leq&\frac{C\beta}{1-\varepsilon}\left(\int_{\Omega_2}(|\nabla u_1|+|\nabla u_2|)^{p-\varepsilon}\mathrm{d}x\right)^{\frac{p-1}{p-\varepsilon}}\left(\int_{\Omega_2}g(x)^{p-\varepsilon}\mathrm{d}x\right)^{\frac{1-\varepsilon}{p-\varepsilon}}\\
\leq&\frac{C\beta}{1-\varepsilon}\left(\int_{\Omega_2}(|\nabla u_1|+|\nabla u_2|)^{p-\varepsilon}\mathrm{d}x\right)^{\frac{p-1}{p-\varepsilon}}\left(\int_{\Omega_2}|\nabla u_1-\nabla u_2|^{p-\varepsilon}\mathrm{d}x\right)^{\frac{1-\varepsilon}{p-\varepsilon}}.
\end{aligned}\tag{3.6}
$$

By using Hölder's and Young's inequalities, we obtain

$$
\begin{aligned}
|I_5|=&\frac{\varepsilon' C}{(1-\varepsilon)}\int_{\Omega_2}|g(x)|^{1-\varepsilon}\mathrm{d}x\\
\leq&\frac{\varepsilon' C}{(1-\varepsilon)}\left(\varepsilon''\int_{\Omega_2}|g(x)|^{p-\varepsilon}\mathrm{d}x+C(\varepsilon'')|\Omega_2|\right)\\
\leq&\frac{\varepsilon' C}{(1-\varepsilon)}\left(\varepsilon''\int_{\Omega_2}|\nabla u_1-\nabla u_2|^{p-\varepsilon}\mathrm{d}x+C(\varepsilon'')|\Omega_2|\right),
\end{aligned}\tag{3.7}
$$

where ε'' and $C(\varepsilon'')$ stand for the small and large constants respectively from Young's inequality.

Similar to [12], We distinguish the proof into two cases:

Case 1 $p \geq 2$. In this case, by using Hölder's inequality we have

$$
\begin{aligned}
&\int_{\Omega_2}|\nabla u_1-\nabla u_2|^{p-\varepsilon}\mathrm{d}x\\
\leq&C\left[\int_{\Omega_2}|\nabla u_1-\nabla u_2|^p g(x)^{-\varepsilon}\mathrm{d}x\right]^{\frac{p-\varepsilon}{p}}\left[\int_{\Omega_2}g(x)^{p-\varepsilon}\mathrm{d}x\right]^{\frac{\varepsilon}{p}}\\
\leq&C\left[\int_{\Omega_2}|\nabla u_1-\nabla u_2|^p g(x)^{-\varepsilon}\mathrm{d}x\right]^{\frac{p-\varepsilon}{p}}\left[\int_{\Omega_2}|\nabla u_1-\nabla u_2|^{p-\varepsilon}\mathrm{d}x\right]^{\frac{\varepsilon}{p}}.
\end{aligned}\tag{3.8}
$$

This is easy to implies

$$
\begin{aligned}
&\int_{\Omega_2}|\nabla u_1-\nabla u_2|^{p-\varepsilon}\mathrm{d}x\\
\leq&C\int_{\Omega_2}|\nabla u_1-\nabla u_2|^p g(x)^{-\varepsilon}\mathrm{d}x\\
\leq&C\int_{\Omega_2}(|\nabla u_1|+|\nabla u_2|)^{p-2}|\nabla u_1-\nabla u_2|^2 g(x)^{-\varepsilon}\mathrm{d}x.
\end{aligned}\tag{3.9}
$$

Combining (3.5), (3.6), (3.7),(3.8)and (3.9), we have

$$\begin{aligned}
&\int_{\Omega_2} |\nabla u_1 - \nabla u_2|^{p-\varepsilon} \mathrm{d}x \\
\le & \frac{C\varepsilon}{1-\varepsilon}\left[\int_{\Omega_2} (|\nabla u_1| + |\nabla u_2|)^{p-\varepsilon} \mathrm{d}x\right]^{\frac{p-1}{p-\varepsilon}} \left[\int_{\Omega_2} |\nabla u_1 - \nabla u_2|^{p-\varepsilon} \mathrm{d}x\right]^{\frac{1-\varepsilon}{p-\varepsilon}} \\
&+ \frac{\varepsilon\varepsilon' C}{(1-\varepsilon)}\left(\varepsilon'' \int_{\Omega_2} |\nabla u_1 - \nabla u_2|^{p-\varepsilon} \mathrm{d}x + C(\varepsilon'')|\Omega_2|\right),
\end{aligned} \tag{3.10}$$

where the above inequality holds for all $0 < \varepsilon \le \varepsilon_0 < 1$, and C doesn't depend on ε. We let $\varepsilon \to 0$, then we get $u_1 = u_2$ a.e. in Ω_2 and Theorem 2.1 were proved.

Case 2 $1 < p < 2$. This implies

$$\begin{aligned}
&\int_{\Omega_2} |\nabla u_1 - \nabla u_2|^{p-\varepsilon} \mathrm{d}x \\
=& \int_{\Omega_2} \left((|\nabla u_1| + |\nabla u_2|)^{p-2} |\nabla u_1 - \nabla u_2|^2 g(x)^{-\varepsilon}\right)^{\frac{p-\varepsilon}{2}} \\
&\times (|\nabla u_1| + |\nabla u_2|)^{(p-\varepsilon)\frac{2-p}{2}} g(x)^{(p-\varepsilon)\frac{\varepsilon}{2}} \mathrm{d}x \\
\le & \left(\int_{\Omega_2} (|\nabla u_1| + |\nabla u_2|)^{p-2} |\nabla u_1 - \nabla u_2|^2 g(x)^{-\varepsilon} \mathrm{d}x\right)^{\frac{p-\varepsilon}{2}} \\
&\times \left(\int_{\Omega_2} (|\nabla u_1| + |\nabla u_2|)^{p-\varepsilon} \mathrm{d}x\right)^{\frac{2-p}{2}} \left(\int_{\Omega_2} g(x)^{p-\varepsilon} \mathrm{d}x\right)^{\frac{\varepsilon}{2}} \\
\le & \left(\int_{\Omega_2} (|\nabla u_1| + |\nabla u_2|)^{p-2} |\nabla u_1 - \nabla u_2|^2 g(x)^{-\varepsilon} \mathrm{d}x\right)^{\frac{p-\varepsilon}{2}} \\
&\times \left(\int_{\Omega_2} (|\nabla u_1| + |\nabla u_2|)^{p-\varepsilon} \mathrm{d}x\right)^{\frac{2-p}{2}} \left(\int_{\Omega_2} |\nabla u_1 - \nabla u_2|^{p-\varepsilon} \mathrm{d}x\right)^{\frac{\varepsilon}{2}},
\end{aligned}$$

that is,

$$\begin{aligned}
&\int_{\Omega_2} |\nabla u_1 - \nabla u_2|^{p-\varepsilon} \mathrm{d}x \\
\le & \left(\int_{\Omega_2} (|\nabla u_1| + |\nabla u_2|)^{p-2} |\nabla u_1 - \nabla u_2|^2 g(x)^{-\varepsilon} \mathrm{d}x\right)^{\frac{p-\varepsilon}{2-\varepsilon}} \\
&\times \left(\int_{\Omega_2} (|\nabla u_1| + |\nabla u_2|)^{p-\varepsilon} \mathrm{d}x\right)^{\frac{2-p}{2-\varepsilon}}.
\end{aligned} \tag{3.11}$$

Combining (3.5), (3.6), (3.7) and (3.11), there have

$$\begin{aligned}
&\int_{\Omega_2} |\nabla u_1 - \nabla u_2|^{p-\varepsilon} \mathrm{d}x \\
\le & \left[C(\frac{\varepsilon}{1-\varepsilon})^{\frac{p-\varepsilon}{2-\varepsilon}} \left(\int_{\Omega_2} (|\nabla u_1| + |\nabla u_2|)^{p-\varepsilon} \mathrm{d}x\right)^{\frac{p-1}{2-\varepsilon}} \left(\int_{\Omega_2} |\nabla u_1 - \nabla u_2|^{p-\varepsilon} \mathrm{d}x\right)^{\frac{1-\varepsilon}{2-\varepsilon}}\right. \\
&\left. + C\left(\frac{\varepsilon\varepsilon'\varepsilon''}{1-\varepsilon}\right)^{\frac{p-\varepsilon}{2-\varepsilon}} \left(\int_{\Omega_2} (|\nabla u_1| + |\nabla u_2|)^{p-\varepsilon} \mathrm{d}x\right)^{\frac{p-\varepsilon}{2-\varepsilon}} + \left(\frac{\varepsilon\varepsilon' C(\varepsilon'')|\Omega_2|}{1-\varepsilon}\right)^{\frac{p-\varepsilon}{2-\varepsilon}}\right] \\
&\times \left(\int_{\Omega_2} (|\nabla u_1| + |\nabla u_2|)^{p-\varepsilon} \mathrm{d}x\right)^{\frac{p-\varepsilon}{2-\varepsilon}}.
\end{aligned} \tag{3.12}$$

Similar to Case 1, we let $\varepsilon \to 0$, and Theorem 2.1 is proved.

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非齐次椭圆方程很弱解的比较原理

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摘要: 在一些适当的假设条件下, 通过McShane扩张定理构造Lipschitz连续检验函数, 本文得到了拟线性椭圆方程$-\mathrm{div}A(x,\nabla u(x)) = f(x,u)$很弱解的比较原理, 推广了齐次方程的相关结果.

关键词: 很弱解; 比较原理; Lipschitz连续检验函数; McShane扩张定理