

Global Existence, Asymptotic Behavior and Uniform Attractor for a Non-Autonomous Thermoelastic Timoshenko System of Type I with a Memory Term

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Abstract: In this article, we consider a non-autonomous Timoshenko system of thermoelasticity of type I with memory type. We first establish the global existence of solutions using the semi-group theory. And then we obtain asymptotic behavior of the solution. Last, we prove the existence of a uniform attractor by using the method of uniform contractive functions.

Key words: Timoshenko system; Global existence; Asymptotic behavior; Uniform attractor; Semigroup method

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1. Introduction

Timoshenko^[17] developed a simple model to describe the transverse vibration of a beam, whose system of coupled hyperbolic equations are given by

$$\begin{cases} \rho u_{tt} = (K(u_x - \varphi))_x, & \text{in } (0, L) \times \mathbb{R}_+, \\ I_\rho \varphi_{tt} = (EI\varphi_x)_x + K(u_t - \varphi), & \text{in } (0, L) \times \mathbb{R}_+, \end{cases} \quad (1.1)$$

together with boundary conditions of the form

$$EI\varphi_x|_{x=0} = 0, \quad (u_x - \varphi)|_{x=0} = 0,$$

where t is the time variable and x denotes the space variable along the beam of length L , u is the transverse displacement of the beam and φ denotes the rotation angle of the filament of the beam. The coefficients ρ, I_ρ, E, I and K are respectively the density (the mass per unit length), the polar moment of inertia of a cross section, Young's modulus of elasticity, the moment of inertia of a cross, and the shear modulus. System (1.1) with the above boundary conditions, is conservative and the natural energy of the beam is given by

$$E(t) = \frac{1}{2} \int_0^L (\rho|u_t|^2 + I_\rho|\varphi_t|^2 + EI|\varphi_x|^2 + K|u_x - \varphi|^2) dx.$$

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System (1.1) with various initial conditions and boundary conditions has been studied by many mathematicians and asymptotic behavior of solutions has been established. Kim and Renardy^[7] considered system (1.1) with two boundary controls of the form

$$\begin{cases} K\varphi(t, L) - Ku_x(t, L) = \alpha u_t(t, L), & \text{in } (0, \infty), \\ EI\varphi_x(t, L) = -\beta\varphi_t(t, L), & \text{in } (0, \infty), \end{cases} \quad (1.2)$$

and used the multiplier techniques to establish an exponential decay result for the natural energy of system (1.1). They also provided numerical estimates to the eigenvalues of the operator associated with system (1.1).

Concerning stabilization via classical heat effect, Muñoz Rivera and Racke^[10] investigated the system

$$\begin{cases} \rho_1\varphi_{tt} - \sigma(\varphi_x, \psi)_x = 0, & \text{in } (0, L) \times \mathbb{R}_+, \\ \rho_2\psi_{tt} - b\psi_{xx} + K(\varphi_x + \psi) + \gamma\theta_x = 0, & \text{in } (0, L) \times \mathbb{R}_+, \\ \rho_3\theta_t - K\theta_{xx} + \gamma\psi_{xt} = 0, & \text{in } (0, L) \times \mathbb{R}_+, \end{cases} \quad (1.3)$$

where φ, ψ and θ are functions of (x, t) which model the transverse displacement of the beam, the rotation angle of the filament and difference temperature, respectively. Under appropriate conditions on $\sigma, \rho_1, b, K, \gamma$, they proved several exponential decay results for the linearized system and non-exponential stability results for the case of different wave speeds.

Messaoudi, Pokojovy and Said-Houari^[8] studied Timoshenko systems of thermoelasticity with second sound

$$\begin{cases} \rho_1\varphi_{tt} - \sigma(\varphi_x, \psi)_x + \mu\varphi_t = 0, & \text{in } (0, L) \times \mathbb{R}_+, \\ \rho_2\psi_{tt} - b\psi_{xx} + K(\varphi_x + \psi) + \beta\theta_x = 0, & \text{in } (0, L) \times \mathbb{R}_+, \\ \rho_3\theta_t + \gamma q_x + \delta\psi_{xt} = 0, & \text{in } (0, L) \times \mathbb{R}_+, \\ \tau_0 q_t + q + \kappa\theta_x = 0, & \text{in } (0, L) \times \mathbb{R}_+, \end{cases} \quad (1.4)$$

where $\varphi = \varphi(x, t)$ is the displacement vector, $\psi = \psi(x, t)$ is the rotation angle of the filament, $\theta = \theta(x, t)$ is the temperature, $q = q(x, t)$ is the heat flux vector and $\rho_1, \rho_2, \rho_3, b, K, \gamma, \delta, \kappa, \mu, \tau_0$ are positive constants. The nonlinear function σ is assumed to be sufficiently smooth and satisfies

$$\sigma_{\varphi_x}(0, 0) = \sigma_{\psi}(0, 0) = K,$$

and

$$\sigma_{\varphi_x\varphi_x}(0, 0) = \sigma_{\varphi_x\psi}(0, 0) = \sigma_{\psi\psi} = 0.$$

Ammar-Khodja et al.^[2] considered a linear Timoshenko-type system with memory of the form

$$\begin{cases} \rho_1\varphi_{tt} - K(\varphi_x + \psi)_x = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + \int_0^t g(t-s)\psi_{xx}(s)ds + K(\varphi_x + \psi) = 0, \\ \varphi(x, 0) = \varphi_0(x), \quad \varphi_t(x, 0) = \varphi_1(x), \\ \psi(x, 0) = \psi_0(x), \quad \psi_t(x, 0) = \psi_1(x), \\ \varphi(0, t) = \varphi(1, t) = \psi(0, t) = \psi(1, t) = 0, \end{cases} \quad (1.5)$$

in $(0, L) \times \mathbb{R}_+$, and used the multiplier techniques to prove that system is uniformly stable if and only if the wave speeds are equal ($K/\rho_1 = b/\rho_2$) and g decays uniformly. More precisely, they proved an exponential decay if g decays in an exponential rate and polynomially if g

decays in a polynomial rate. They also required some extra technical conditions on both g' and g'' to obtain their results. Apalara^[1] considered a linear thermoelastic Timoshenko system with memory effects where the thermoelastic coupling is acting on shear force and obtained a general stability result irrespective of the wave speeds of the system. Also, Muñoz Rivera and Racke^[9] treated a nonlinear Timoshenko-type system of the form

$$\begin{cases} \rho_1 \varphi_{tt} - \sigma_1(\varphi_x, \psi)_x = 0, \\ \rho_2 \psi_{tt} - \chi \psi_{xx} + \sigma_2(\varphi_x, \psi) + d\psi_t = 0 \end{cases} \tag{1.6}$$

in a one-dimensional bounded domain. The dissipation is produced here through a frictional damping which is only present in the equation for the rotation angle.

Recently, Fernández Sare and Rivera^[4] considered a Timoshenko-type system with a past history of the form

$$\begin{cases} \rho_1 \varphi_{tt} - K(\varphi_x + \psi)_x = 0, \\ \rho_2 \psi_{tt} - b\psi_{xx} + \int_0^t g(s)\psi_{xx}(t-s, \cdot)ds + K(\varphi_x + \psi) = 0, \end{cases} \tag{1.7}$$

where ρ_1, ρ_2, K, b are positive constants and g is a positive twice differentiable function satisfying, for some constants $k_0, k_1, k_2 > 0$,

$$\begin{aligned} g(t) > 0, \quad -k_0 g(t) \leq g'(t) \leq -k_1 g(t), \quad |g''| \leq k_2 g(t), \quad \forall t \geq 0, \\ \check{b} = b - \int_0^\infty g(s)ds > 0, \end{aligned} \tag{1.8}$$

and showed that the dissipation given by the history term is strong enough to stabilize the system exponentially if and only if the wave speeds are equal. They also proved that the solution decays polynomially for the case of different wave speeds.

Guesmia^[5] consider a Timoshenko system in one-dimensional bounded domain with infinite memory and distributed time delay both acting on the equation of the rotation angle

$$\begin{cases} \rho_1 \varphi_{tt} - k_1(\varphi_x + \psi)_x = 0, \\ \rho_2 \psi_{tt} - k_2 \psi_{xx} + k_1(\varphi_x + \psi) + \int_0^\infty g(s)\psi_{xx}(x, t-s)ds + \int_0^\infty f(s)\psi_t(x, t-s)ds = 0, \\ \varphi(0, t) = \psi(0, t) = \varphi(L, t) = \psi(L, t) = 0, \\ \varphi(x, 0) = \varphi_0(x), \quad \varphi_t(x, 0) = \varphi_1(x), \\ \psi(x, -t) = \psi_0(x, t), \quad \psi_t(x, -t) = \psi_1(x, t), \end{cases} \tag{1.9}$$

where $(x, t) \in [0, L] \times \mathbb{R}_+$, $(\varphi_0, \psi_0, \varphi_1, \psi_1)$ are given initial data belonging to a suitable space, L, ρ_1, ρ_2, k_1 and k_2 are positive constants. The stability results show that the only dissipation resulting from the infinite memory guarantees the asymptotic stability of the system regardless to the speeds of wave propagation and in spite of the presence of a distributed time delay.

Recently, QIN and WEI^[16] established the global existence and asymptotic behavior of solutions by using the semigroup method and multiplicative techniques, then proved the existence of a uniform attractor for a non-autonomous thermoelastic system by using the method of uniform contractive functions. QIN and MA^[15] presented recent results on some global well-posedness and asymptotic behavior of the solutions to non-classical thermo (visco) elastic models. They established the global existence result for the higher-dimensional linear thermoviscoelastic equations of type III by using a semigroup approach. Using the multiplier techniques and Lyapunov methods, they proved that the energy goes to zero exponentially

by introducing a velocity feedback on a part of the boundary of a thermoelastic body, which is clamped along the rest of its boundary to increase the loss of energy. For more related results, we refer to [12-14,18].

In the present paper, we consider a non-autonomous Timoshenko-type system of thermoelasticity of type I with memory term:

$$\begin{cases} \rho_1 \varphi_{tt} - K(\varphi_x + \psi)_x + \gamma \theta_x = f(x, t), \\ \rho_2 \psi_{tt} - b \psi_{xx} + K(\varphi_x + \psi) - \gamma \theta + \int_0^t g(t-s) \psi_{xx} ds = z(x, t), \\ \rho_3 \theta_t - \beta \theta_{xx} + \gamma(\varphi_x + \psi)_t = h(x, t), \end{cases} \quad (1.10)$$

where $\varphi = \varphi(x, t)$, $\psi = \psi(x, t)$, $\theta = \theta(x, t)$, $(x, t) \in (0, 1) \times (0, +\infty)$ and $\rho_1, \rho_2, \rho_3, K, b, \beta, \gamma$ are positive constants. In this system, $\int_0^t g(t-s) \psi_{xx}(x, s) ds$ represents a memory term, and f, z, h are forcing terms. We consider (1.10) coupled with initial data

$$\begin{cases} \varphi(x, 0) = \varphi_0(x), \varphi_t(x, 0) = \varphi_1(x), & x \in (0, 1), \\ \psi(x, 0) = \psi_0(x), \psi_t(x, 0) = \psi_1(x), \theta(x, 0) = \theta_0(x), & x \in (0, 1), \end{cases} \quad (1.11)$$

and Neumann-Dirichlet-Dirichlet boundary conditions

$$\varphi_x(0, t) = \varphi_x(1, t) = \psi(0, t) = \psi(1, t) = \theta(0, t) = \theta(1, t) = 0, \quad t \geq 0, \quad (1.12)$$

and establish a general stability result irrespective of the wave speeds of the system. It is imperative to mention that the boundary conditions ($\varphi_x(0, t) = \varphi_x(1, t) = \psi(0, t) = \psi(1, t) = 0$) are equivalent to the sliding end boundary condition on Timoshenko system.

2. Preliminaries

In this section, we present some materials needed for our main results. For simplicity of notations, hereafter we denote by $\|\cdot\|_q$ the norm of Lebesgue space $L^q(\Omega)$, and by $\|\cdot\|_2$ the norm of Lebesgue space $L^2(\Omega)$.

In order to deal with the memory term, we assume that the function g satisfies the following assumptions:

(H1) $g: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a C^1 decreasing function satisfying:

$$g(0) > 0, \quad b - \int_0^\infty g(s) ds = l > 0.$$

(H2) There exists a nonincreasing differentiable function $\xi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$, and a constant M , satisfying:

$$|\xi(t)| < M, \quad \text{and} \quad g'(t) \leq -\xi(t)g(t), \quad \forall t \geq 0.$$

Meanwhile, from the first equation in (1.10) and the boundary conditions (1.12), it follows that

$$\frac{d^2}{dt^2} \int_0^1 \varphi(x, t) dx = 0. \quad (2.1)$$

Obviously, using the initial data of φ , we obtain

$$\int_0^1 \varphi(x, t) dx = t \int_0^1 \varphi_1(x) dx + \int_0^1 \varphi_0(x) dx.$$

Consequently, as in [6], if we let

$$\bar{\varphi}(x, t) = \varphi(x, t) - t \int_0^1 \varphi_1(x) dx - \int_0^1 \varphi_0(x) dx,$$

then we obtain

$$\int_0^1 \bar{\varphi}(x, t) dx = 0, \quad \forall t \geq 0.$$

Therefore, the use of Poincaré’s inequality for $\bar{\varphi}$ is justified. In addition, a simple substitution shows that $(\bar{\varphi}, \psi, \theta)$ satisfies system (1.10) with initial data for $\bar{\varphi}$ given by

$$\bar{\varphi}_0(x) = \varphi_0(x) - \int_0^1 \varphi_0(x) dx \quad \text{and} \quad \bar{\varphi}_1(x) = \varphi_1(x) - \int_0^1 \varphi_1(x) dx.$$

Henceforth, we work with $\bar{\varphi}$ instead of φ , but write φ for simplicity of notation.

In what follows, we consider (φ, ψ, θ) to be a solution of system (1.10) with the regularity needed to justify the calculations in this paper.

3. Global Well-Posedness of the Problem

In this section, we shall investigate the global well-posedness of problem (1.10) with the variable norm technique, we start with the vector function $U = (\varphi, u, \psi, v, \theta)^T$, where $u = \varphi_t, v = \psi_t$. We introduce as in [6],

$$L_*^2(0, 1) \equiv \{w \in L^2(0, 1) \mid \int_0^1 w(s) ds = 0\},$$

$$H_*^1(0, 1) \equiv H^1(0, 1) \cap L_*^2(0, 1).$$

Then system (1.10)-(1.12) is converted to the following abstract ODE

$$\begin{cases} \frac{dU}{dt} + \mathcal{A}U = F, & t > 0, \\ U(0) = (\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0)^T, \end{cases} \tag{3.1}$$

where operator $\mathcal{A}U$ and F are defined by

$$\mathcal{A}U = \begin{pmatrix} \varphi_t \\ \frac{K}{\rho_1}(\varphi_x + \psi)_x - \frac{\gamma}{\rho_1} \\ \psi_t \\ \frac{b}{\rho_2}\psi_{xx} - \frac{K}{\rho_2}(\varphi_x + \psi) + \frac{\gamma\theta}{\rho_2} - \frac{1}{\rho_2} \int_0^t g(t-s)\psi_{xx}(x, s) ds \\ \frac{\beta}{\rho_3}\theta_{xx} - \frac{\gamma}{\rho_3}(\varphi_x + \psi)_t \end{pmatrix},$$

$$F = \begin{pmatrix} 0 \\ \frac{f}{\rho_1} \\ 0 \\ \frac{z}{\rho_2} \\ \frac{h}{\rho_3} \end{pmatrix}.$$

Then we define the energy space

$$H \equiv H_*^1(0, 1) \times L_*^2(0, 1) \times H_0^1(0, 1) \times L^2(0, 1) \times L^2(0, 1).$$

We set

$$D(\mathcal{A}) = \left\{ U \in H \mid \varphi \in H^2(0, 1), \varphi_t \in H_*^1(0, 1), \psi \in H_0^1(0, 1) \cap H^2(0, 1), \right. \\ \left. \psi_t \in H_0^1(0, 1), \theta \in H_0^1(0, 1) \cap H^2(0, 1) \right\}. \tag{3.2}$$

Thus $\mathcal{A} : D(\mathcal{A}) \rightarrow H$ is a linear operator.

For $U = (\varphi, u, \psi, v, \theta)^T$ and $\tilde{U} = (\tilde{\varphi}, \tilde{u}, \tilde{\psi}, \tilde{v}, \tilde{\theta})^T$, we equip H with the inner product

$$(U, \tilde{U})_H = \int_0^1 [\rho_1 u \tilde{u} + \rho_2 v \tilde{v} + \rho_3 \theta \tilde{\theta} + (b - \int_0^t g(s) ds) \psi_x \tilde{\psi}_x]$$

$$+K(\varphi_x + \psi)(\tilde{\varphi}_x + \tilde{\psi})dx + g \circ \psi_x. \quad (3.3)$$

Now we state the following existence and regularity result.

Theorem 3.1 Assume that $f(x, t), z(x, t), h(x, t) \in C^1([0, +\infty), L^2(0, 1))$, and (H1)(H2) are satisfied. Then, for any $U_0 \in D(\mathcal{A})$, there exists a unique solution $U(t)$ of problem (3.1) satisfying

$$U(t) \in C([0, +\infty), D(\mathcal{A})) \cap C^1([0, +\infty), H),$$

equivalently

$$\begin{aligned} \varphi &\in C([0, +\infty), H^2(0, 1)) \cap C^1([0, +\infty), H_*^1(0, 1)), \\ u &\in C([0, +\infty), H_*^1(0, 1)) \cap C^1([0, +\infty), L_*^2(0, 1)), \\ \psi &\in C([0, +\infty), H_0^1(0, 1) \cap H^2(0, 1)) \cap C^1([0, +\infty), H_0^1(0, 1)), \\ v &\in C([0, +\infty), H_0^1(0, 1)) \cap C^1([0, +\infty), L^2(0, 1)), \\ \theta &\in C([0, +\infty), H_0^1(0, 1) \cap H^2(0, 1)) \cap C^1([0, +\infty), L^2(0, 1)). \end{aligned}$$

In order to complete the proof of Theorem 3.1, we will need the following lemmas. For an abstract initial problem

$$\begin{cases} \frac{dy}{dt} + \mathcal{A}y = F(t), \\ y(0) = y_0, \end{cases} \quad (3.4)$$

where \mathcal{A} is a maximal accretive operator defined in a dense subset $D(\mathcal{A})$ of a Banach space H , we have

Lemma 3.1^[19] Let \mathcal{A} be a linear operator defined in a Hilbert space $H, \mathcal{A}: D(\mathcal{A}) \subset H \rightarrow H$. Then the necessary and sufficient conditions for $D(\mathcal{A})$ being maximal accretive are:

- (i) $\operatorname{Re}(\mathcal{A}x, x) \leq 0, \quad \forall x \in D(\mathcal{A}),$
- (ii) $R(\lambda I - \mathcal{A}) = H.$

Lemma 3.2^[18] Assume that \mathcal{A} is m-accretive in a Banach space H , and

$$F(t) \in C^1([0, +\infty), H), \quad y_0 \in D(\mathcal{A}).$$

Then Problem (3.4) has a unique classical solution $y(t)$ such that

$$y(t) \in C^1([0, +\infty], H) \cap C([0, \infty), D(\mathcal{A})),$$

which can be expressed as

$$y(t) = S(t)y_0 + \int_0^t S(t-\tau)F(\tau)d\tau. \quad (3.5)$$

Proof of Theorem 3.1 By Lemma 3.1, we can know that \mathcal{A} is an maximal monotone operator (see also [11]). By the assumptions, we have $(\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0)^T \in D(\mathcal{A})$, and then by Lemma 3.2, we complete the proof.

4. Uniform Stability

In this section, we shall state and prove our decay results. To this end, we need now to establish several lemmas.

Lemma 4.1 Let (φ, ψ, θ) be the solution of problem (1.10)-(1.12). Then the energy functional defined by

$$E(t) = \frac{1}{2} \int_0^1 [\rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + \rho_3 \theta^2 + (b - \int_0^t g(s)ds)\psi_x^2 + K(\varphi_x + \psi)^2]dx + \frac{1}{2}g \circ \psi_x, \quad (4.1)$$

satisfies

$$E'(t) = -\beta \int_0^1 \theta_x^2 dx + \frac{1}{2} g' \circ \psi_x - \frac{1}{2} g(t) \int_0^1 \psi_x^2 dx + \int_0^1 f \varphi_t dx + \int_0^1 z \psi_t dx + \int_0^1 h \theta dx, \tag{4.2}$$

where

$$(g \circ \psi_x)(t) = \int_0^1 \int_0^t g(t-s)(\psi_x(t) - \psi_x(s))^2 ds dx.$$

Proof Multiplying the first three equations in (1.10) by φ_t, ψ_t and θ , respectively, and integrating by parts over $(0,1)$, and using the boundary conditions, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_0^1 [\rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + \rho_3 \theta^2 + (b - \int_0^t g(s) ds) \psi_x^2 + K(\varphi_x + \psi)^2] dx \\ & - \int_0^1 \psi_{xt} \int_0^t g(t-s) \psi_x(s) ds dx \\ & = -\beta \int_0^1 \theta_x^2 dx + \int_0^1 f \varphi_t dx + \int_0^1 z \psi_t dx + \int_0^1 h \theta dx. \end{aligned} \tag{4.3}$$

The last term in the left-hand side of (4.3) is estimated as follows

$$\begin{aligned} & - \int_0^1 \psi_{xt} \int_0^t g(t-s) \psi_x(s) ds dx \\ & = \int_0^1 \psi_{xt} \int_0^t g(t-s)(\psi_x(t) - \psi_x(s)) ds dx - \int_0^t g(s) ds \int_0^1 \psi_x \psi_{tx} dx \\ & = \frac{1}{2} \frac{d}{dt} g \circ \psi_x - \frac{1}{2} \frac{d}{dt} \int_0^t g(s) ds \int_0^1 \psi_x^2 dx - \frac{1}{2} g' \circ \psi_x + \frac{1}{2} g(t) \int_0^1 \psi_x^2 dx. \end{aligned} \tag{4.4}$$

Substituting of (4.4) into (4.3), and bearing in mind (4.1), we have (4.2).

Lemma 4.2 Let (φ, ψ, θ) be the solution of problem (1.10)-(1.12). Then, the functional

$$F_1(t) := -\rho_3 \int_0^1 \theta \left(\int_0^x \varphi_t(s, t) ds \right) dx,$$

satisfies, for any positive constant ε_1 , the estimate

$$\begin{aligned} F_1'(t) & \leq -\frac{\gamma}{4} \int_0^1 \varphi_t^2 dx + \varepsilon_1 \int_0^1 (\varphi_x + \psi)^2 dx + c \int_0^1 \psi_t^2 dx \\ & \quad + c \left(1 + \frac{1}{\varepsilon} \right) \int_0^1 \theta_x^2 dx + \frac{1}{\gamma} \int_0^1 (f^2 + h^2) dx. \end{aligned} \tag{4.5}$$

Proof A direct computations, and using (1.10), we set

$$\begin{aligned} F_1'(t) & = -\rho_3 \int_0^1 \theta_t \left(\int_0^x \varphi_t ds \right) dx - \rho_3 \int_0^1 \theta \left(\int_0^x \varphi_{tt}(s, t) ds \right) dx \\ & = \int_0^1 (-\beta \theta_{xx} + \gamma (\varphi_x + \psi)_t - h) \left(\int_0^x \varphi_t ds \right) dx \\ & \quad - \rho_3 \int_0^1 \theta \left[\int_0^x \frac{1}{\rho_1} (K(\varphi_x + \psi)_x - \gamma \theta + f) ds \right] dx \\ & = \int_0^1 \beta \theta_x \varphi_t dx - \gamma \int_0^1 \varphi_t^2 + \gamma \int_0^1 \psi_t \left(\int_0^x \varphi_t ds \right) dx - \int_0^1 h \left(\int_0^x \varphi_t ds \right) dx \\ & \quad - \frac{\rho_3 K}{\rho_1} \int_0^1 \theta (\varphi_x + \psi) dx + \frac{\rho_3 \gamma}{\rho_1} \int_0^1 \theta^2 dx - \frac{\rho_3}{\rho_1} \int_0^1 \theta \left(\int_0^x f ds \right) dx. \end{aligned}$$

Using Young's inequality, for any $\varepsilon_1 > 0$, we obtain

$$\begin{aligned} F_1'(t) &\leq \frac{\beta^2}{\gamma} \int_0^1 \theta_x^2 dx + \frac{\gamma}{4} \int_0^1 \varphi_t^2 dx - \gamma \int_0^1 \varphi_t^2 dx + \gamma \int_0^1 \psi_t^2 dx \\ &\quad + \frac{\gamma}{4} \int_0^1 \left(\int_0^x \varphi_t(s, t) ds \right)^2 dx + \frac{1}{\gamma} \int_0^1 h^2 dx + \frac{\gamma}{4} \int_0^1 \left(\int_0^x \varphi_t(s, t) ds \right)^2 dx \\ &\quad + c \left(1 + \frac{1}{\varepsilon_1} \right) \int_0^1 \theta^2 dx + \varepsilon_1 \int_0^1 (\varphi_x + \psi)^2 dx + \frac{1}{\gamma} \int_0^1 \left(\int_0^x f(s, t) ds \right)^2 dx. \end{aligned} \quad (4.6)$$

By the Cauchy-Schwarz inequality, it is clear that

$$\left(\int_0^x \varphi_t(s, t) ds \right)^2 \leq \left(\int_0^1 |\varphi_t| dx \right)^2 \leq \int_0^1 \varphi_t^2 dx. \quad (4.7)$$

Substituting (4.7) into (4.6), and using Poincaré's inequality, we obtain (4.5).

Lemma 4.3 Let (φ, ψ, θ) be the solution of problem (1.10)-(1.12). Then, the functional

$$F_2(t) := -\rho_1 \int_0^1 (\varphi_x + \psi) \left(\int_0^x \varphi_t(s, t) ds \right) dx,$$

satisfies the estimate

$$F_2'(t) \leq -\frac{K}{4} \int_0^1 (\varphi_x + \psi)^2 dx + c \int_0^1 \psi_t^2 dx + c \int_0^1 \varphi_t^2 dx + c \int_0^1 \theta_x^2 dx + \frac{1}{K} \int_0^1 f^2 dx. \quad (4.8)$$

Proof Taking a derivative of F_2 in t , using (1.10) and integrating by parts, we obtain

$$\begin{aligned} F_2'(t) &= -\rho_1 \int_0^1 (\varphi_x + \psi)_t \left(\int_0^x \varphi_t(s, t) ds \right) dx - \rho_1 \int_0^1 (\varphi_x + \psi) \left(\int_0^x \varphi_{tt} ds \right) dx \\ &= \rho_1 \int_0^1 \varphi_t^2 dx - \rho_1 \int_0^1 \psi_t \left(\int_0^x \varphi_t(s, t) ds \right) dx \\ &\quad - \int_0^1 (\varphi_x + \psi) \left(\int_0^x (f + K(\varphi_x + \psi)_x - \gamma \theta_x) ds \right) dx \\ &= \rho_1 \int_0^1 \varphi_t^2 - \rho_1 \int_0^1 \psi_t \left(\int_0^x \varphi_t ds \right) dx - K \int_0^1 (\varphi_x + \psi)^2 dx \\ &\quad + \gamma \int_0^1 \theta (\varphi_x + \psi)^2 dx - \int_0^1 (\varphi_x + \psi) \left(\int_0^x f ds \right) dx. \end{aligned}$$

Using Young's inequality, we obtain

$$\begin{aligned} F_2'(t) &\leq \rho_1 \int_0^1 \varphi_t^2 dx + \frac{\rho_1}{2} \int_0^1 \psi_t^2 dx + \frac{\rho_1}{2} \int_0^1 \left(\int_0^x \varphi_t(s, t) ds \right)^2 dx \\ &\quad - K \int_0^1 (\varphi_x + \psi)^2 dx + \frac{K}{2} \int_0^1 (\varphi_x + \psi)^2 dx + \frac{\gamma^2}{2K} \int_0^1 \theta^2 dx \\ &\quad + \frac{K}{4} \int_0^1 (\varphi_x + \psi)^2 dx + \frac{1}{K} \int_0^1 \left(\int_0^x f(s, t) ds \right)^2 dx. \end{aligned} \quad (4.9)$$

In the view of (4.7) and using Poincaré's inequality, we obtain (4.8).

Lemma 4.4 Let (φ, ψ, θ) be the solution of problem (1.10)-(1.12). Then, for any $t_0 > 0$, the functional

$$F_3(t) := -\rho_2 \int_0^1 \psi_t \left(\int_0^t g(t-s)(\psi(t) - \psi(s)) ds \right) dx$$

satisfies, for any positive constants $\varepsilon_1, \varepsilon_2$, the estimate

$$F_3'(t) \leq -\frac{\rho_2 g_0}{2} \int_0^1 \psi_t^2 dx + \varepsilon_1 \int_0^1 \psi_x^2 dx + \frac{\varepsilon_2}{2} \int_0^1 (\varphi_x + \psi)^2 dx + c \int_0^1 \theta_x^2 dx$$

$$+ c(1 + \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2})g \circ \psi_x - cg' \circ \psi_x + \frac{\varepsilon_2}{2} \int_0^1 z^2 dx, \tag{4.10}$$

where

$$g_0 = \int_0^{t_0} g(s)ds \text{ for some } t_0 > 0.$$

Proof Differentiating F_3 in t , taking into account (1.10), and using integrating by parts together with the boundary conditions, we obtain

$$\begin{aligned} F_3'(t) &= -\rho_2 \int_0^t \psi_{tt} (\int_0^t g(t-s)(\psi(t) - \psi(s))ds)dx - \rho_2 \int_0^1 \psi_t (\int_0^t g'(t-s)(\psi(t) - \psi(s))ds)dx \\ &\quad - \rho_2 \int_0^1 \psi_t (\int_0^t g(t-s)\psi_t(t)ds)dx \\ &= -\rho_2 \int_0^t g(s)ds \int_0^1 \psi_t^2 dx - \rho_2 \int_0^1 \psi_t (\int_0^t g'(t-s)(\psi(t) - \psi(s))ds)dx \\ &\quad + b \int_0^1 \psi_x (\int_0^t g(t-s)(\psi_x(t) - \psi_x(s))ds)dx - \gamma \int_0^1 \theta (\int_0^t g(t-s)(\psi(t) - \psi(s))ds)dx \\ &\quad + K \int_0^1 (\varphi_x + \psi) (\int_0^t g(t-s)(\psi(t) - \psi(s))ds)dx - \int_0^1 z (\int_0^t g(t-s)(\psi(t) - \psi(s))ds)dx \\ &\quad - \int_0^1 \int_0^t g(t-s)\psi_x(s)ds \int_0^t g(t-s)(\psi_x(t) - \psi_x(s))dsdx. \end{aligned} \tag{4.11}$$

Now, we estimate each term in the right-hand side of (4.11), by using Young's, the Cauchy-Schwarz, and Poincaré's inequalities. So, for any positive constants δ_1, ε_1 , and ε_2 , we obtain

$$\begin{aligned} I_1 &= -\rho_2 \int_0^1 \psi_t (\int_0^t g'(t-s)(\psi(t) - \psi(s))ds)dx \\ &\leq \rho_2 \delta_1 \int_0^1 \psi_t^2 dx + \frac{\rho_2}{4\delta_1} \int_0^1 (\int_0^t g'(t-s)(\psi(t) - \psi(s))ds)^2 dx \\ &\leq \rho_2 \delta_1 \int_0^1 \psi_t^2 dx + \frac{\rho_2}{4\delta_1} \int_0^1 (-g'(s)) \int_0^t -g'(t-s)(\psi(t) - \psi(s))^2 dsdx \\ &\leq \rho_2 \delta_1 \int_0^1 \psi_t^2 dx - \frac{c}{\delta_1} g' \circ \psi_x, \end{aligned} \tag{4.12}$$

$$\begin{aligned} I_2 &= b \int_0^1 \psi_x (\int_0^t g(t-s)(\psi_x(t) - \psi_x(s))ds)dx \\ &\leq \frac{\varepsilon_1}{2} \int_0^1 \psi_x^2 dx + \frac{b^2}{2\varepsilon_1} \int_0^1 (\int_0^t g(t-s)(\psi_x(t) - \psi_x(s))dx)^2 dx \\ &\leq \frac{\varepsilon_1}{2} \int_0^1 \psi_x^2 dx + \frac{b^2}{2\varepsilon_1} \int_0^t g(s)ds \int_0^1 \int_0^t g(t-s)(\psi_x(t) - \psi_x(s))^2 dsdx \\ &\leq \frac{\varepsilon_1}{2} \int_0^1 \psi_x^2 dx + \frac{c}{\varepsilon_1} g \circ \psi_x. \end{aligned} \tag{4.13}$$

Similarly to I_2 , we have

$$\begin{aligned} I_3 &= K \int_0^1 (\varphi_x + \psi) \int_0^t g(t-s)(\psi(t) - \psi(s))dsdx \\ &\leq \frac{\varepsilon_2}{2} \int_0^1 (\varphi_x + \psi)^2 dx + \frac{c}{2\varepsilon_2} g \circ \psi_x, \end{aligned} \tag{4.14}$$

$$\begin{aligned}
I_4 &= -\gamma \int_0^1 \theta \int_0^t g(t-s)(\psi(t) - \psi(s)) ds dx \\
&\leq \frac{\gamma}{2} \int_0^1 \theta^2 dx + \frac{\gamma}{2} g \circ \psi \leq c \int_0^1 \theta_x^2 dx + cg \circ \psi_x,
\end{aligned} \tag{4.15}$$

$$\begin{aligned}
I_5 &= -\int_0^1 \int_0^t g(t-s)\psi_x(s) ds \int_0^t g(t-s)(\psi_x(t) - \psi_x(s)) ds dx \\
&= -\int_0^t g(s) ds \int_0^1 \psi_x \int_0^t g(t-s)(\psi_x(t) - \psi_x(s)) ds dx \\
&\quad + \int_0^t \left(\int_0^t g(t-s)(\psi_x(t) - \psi_x(s)) ds \right)^2 dx \\
&\leq \frac{\varepsilon_1}{2} \int_0^1 \psi_x^2 dx + \int_0^1 \left(\int_0^t g(t-s)(\psi_x(t) - \psi_x(s)) ds \right)^2 dx \\
&\quad + \frac{1}{2\varepsilon_1} \left(\int_0^t g(s) ds \right)^2 \int_0^1 \left(\int_0^t g(t-s)(\psi_x(t) - \psi_x(s)) ds \right)^2 dx \\
&\leq \frac{\varepsilon_1}{2} \int_0^1 \psi_x^2 dx + \frac{c}{\varepsilon_1} g \circ \psi_x,
\end{aligned} \tag{4.16}$$

$$I_6 = -\int_0^1 z \int_0^t g(t-s)(\psi(t) - \psi(s)) ds dx \leq \frac{\varepsilon_2}{2} \int_0^1 z^2 dx + \frac{c}{2\varepsilon_2} g \circ \psi_x. \tag{4.17}$$

Since the function g is positive, continuous, and $g(0) > 0$, we have, for any $t \geq t_0 > 0$,

$$\int_0^t g(s) ds \geq \int_0^{t_0} g(s) ds = g_0. \tag{4.18}$$

Substituting (4.12)-(4.17) into (4.11), we obtain, for all $t \geq t_0$,

$$\begin{aligned}
F'_3(t) &\leq -\rho_2 [g_0 - \delta_1] \int_0^1 \psi_t^2 dx + \varepsilon_1 \int_0^1 \psi_x^2 dx + c \int_0^1 \theta_x^2 dx \\
&\quad + \frac{\varepsilon_2}{2} \int_0^1 (\varphi_x + \psi)^2 dx + c \left(1 + \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} \right) g \circ \psi_x - \frac{c}{\delta_1} g' \circ \psi_x + \frac{\varepsilon_2}{2} \int_0^1 z^2 dx.
\end{aligned}$$

Using (4.18) and letting $\delta_1 = \frac{g_0}{2}$, we obtain (4.10).

Lemma 4.5 Let (φ, ψ, θ) be the solution of problem (1.10)-(1.12). Then, the functional

$$F_4(t) := \rho_2 \int_0^1 \psi_t \psi dx$$

satisfies the estimate

$$\begin{aligned}
F'_4(t) &\leq -\frac{l}{2} \int_0^1 \psi_x^2 dx + \rho_2 \int_0^1 \psi_t^2 dx + c \int_0^1 (\varphi_x + \psi)^2 dx \\
&\quad + c \int_0^1 \theta_x^2 dx + cg \circ \psi_x + \frac{1}{4\delta_2} \int_0^1 z^2 dx.
\end{aligned} \tag{4.19}$$

Proof Direct differentiation of F_4 in t , using (1.10) and integration by parts, we obtain

$$\begin{aligned}
F'_4(t) &= -b \int_0^1 \psi_x^2 dx + \rho_2 \int_0^1 \psi_t^2 dx - K \int_0^1 (\varphi_x + \psi) \psi dx \\
&\quad + \gamma \int_0^1 \theta \psi dx + \int_0^1 \psi_x \int_0^t g(t-s)\psi_x(s) ds dx + \int_0^1 z \psi dx.
\end{aligned} \tag{4.20}$$

The last four terms on the right-hand side of (4.20) are estimated as follows:

$$-K \int_0^1 (\varphi_x + \psi) \psi dx \leq \frac{\delta_2}{2} \int_0^1 \psi_x^2 dx + \frac{K^2}{2\delta_2} \int_0^1 (\varphi_x + \psi)^2 dx, \tag{4.21}$$

$$\gamma \int_0^1 \psi \theta dx \leq \frac{\delta_2}{2} \int_0^1 \psi_x^2 dx + \frac{\gamma^2}{2\delta_2} \int_0^1 \theta_x^2 dx, \tag{4.22}$$

$$\begin{aligned} \int_0^1 \psi_x \int_0^t g(t-s)\psi_x(s) ds dx &= \int_0^t g(s) ds \int_0^1 \psi_x^2 dx - \int_0^1 \psi_x \int_0^t g(t-s)(\psi_x(t) - \psi_x(s)) ds dx \\ &\leq (\delta_2 + \int_0^t g(s) ds) \int_0^1 \psi_x^2 dx + \frac{1}{4\delta_2} (\int_0^t g(s) ds) g \circ \psi_x, \end{aligned} \tag{4.23}$$

$$\int_0^1 z \psi dx \leq \delta_2 \int_0^1 \psi_x^2 dx + \frac{1}{4\delta_2} \int_0^1 z^2 dx, \tag{4.24}$$

where we have used the Cauchy-Schwarz, Young's, and Poincaré's inequalities. Substituting (4.21)-(4.24) into (4.20), we obtain

$$\begin{aligned} F_4'(t) &\leq - (b - \int_0^t g(s) ds - 3\delta_2) \int_0^1 \psi_x^2 dx + \rho_2 \int_0^1 \psi_t^2 dx + \frac{c}{\delta_2} \int_0^1 \theta_x^2 dx \\ &\quad + \frac{c}{\delta_2} \int_0^1 (\varphi_x + \psi)^2 dx + \frac{c}{2\delta_2} g \circ \psi_x + \frac{1}{4\delta_2} \int_0^1 z^2 dx. \end{aligned} \tag{4.25}$$

Using the fact that $b - \int_0^t g(s) ds \geq l$, and letting $\delta_2 = \frac{l}{4}$, we obtain (4.19).

Lemma 4.6^[16] Suppose that $y(t) \in C^1(\mathbb{R}_+)$, $y(t) \geq 0, \forall t > 0$, and $y(t)$ satisfies the estimate

$$y'(t) \leq -C_0 y(t) + \lambda(t), \quad \forall t > 0, \tag{4.26}$$

where $0 \leq \lambda(t) \in L^1(\mathbb{R}_+)$ and C_0 is a positive constant. Then we have

$$\lim_{t \rightarrow +\infty} y(t) = 0. \tag{4.27}$$

Furthermore,

1) If $\lambda(t) \leq C_1 e^{-\alpha_0 t}, \forall t > 0$, with $C_1 > 0, \alpha_0 > 0$ being constants, then

$$y(t) \leq C_2 e^{-\alpha t}, \quad \forall t > 0, \tag{4.28}$$

with $C_2 > 0, \alpha > 0$ being constants;

2) If $\lambda(t) \leq C_3(1+t)^{-p}, \forall t > 0$, with $p > 1, C_3 > 0$ being constants, then

$$y(t) \leq C_4(1+t)^{-p+1}, \quad \forall t > 0, \tag{4.29}$$

with a constant $C_4 > 0$.

Now we state and prove our main results.

Theorem 4.1 Let $U_0 \in D(\mathcal{A})$, assume that hypotheses (H1) and (H2) hold, and (φ, ψ, θ) is the solution of the problem (1.10)-(1.12) and $f(x, t), z(x, t), h(x, t) \in C^1([0, +\infty), L^2(0, 1))$. Then, we have

$$\lim_{t \rightarrow +\infty} E(t) = 0. \tag{4.30}$$

If further

$$\|f\|_2^2 + \|z\|_2^2 + \|h\|_2^2 \leq C_0 e^{-\alpha_0 t}, \quad \forall t > 0, \tag{4.31}$$

with $C_0 > 0$ and $\alpha_0 > 0$ being constants, then there exist positive constants M, α such that the energy $E(t)$ satisfies

$$E(t) \leq M e^{-\alpha t}, \quad \forall t > 0. \tag{4.32}$$

If

$$\|f\|_2^2 + \|z\|_2^2 + \|h\|_2^2 \leq \frac{C'}{(1+t)^p}, \quad \forall t > 0, \tag{4.33}$$

with constants $C' > 0$, $p > 1$, then there exists a constant $C^* > 0$ such that

$$E(t) \leq C^*(1+t)^{-p+1}, \quad \forall t > 0. \tag{4.34}$$

To achieve our results, we define a Lyapunov functional $L(t)$, and show that it is equivalent to the energy function $E(t)$.

Lemma 4.7 For N sufficiently large, the functional defined by

$$L(t) := NE(t) + N_1F_1(t) + N_2F_2(t) + N_3F_3(t) + F_4(t), \tag{4.35}$$

where N_1, N_2 and N_3 are positive real numbers to be chosen appropriately later, satisfies

$$c_3E(t) \leq L(t) \leq c_4E(t), \quad \forall t > 0, \tag{4.36}$$

for two positive constants c_3 and c_4 .

Proof If we let $L(t) := NE(t) + N_1F_1(t) + N_2F_2(t) + N_3F_3(t) + F_4(t)$, then

$$\begin{aligned} |L(t)| \leq & \rho_3 N_1 \int_0^1 |\theta \int_0^x \varphi_t(s, t) ds| dx + \rho_2 N_3 \int_0^1 |\psi_t \int_0^x g(t-s)(\psi(t) - \psi(s)) ds| dx \\ & + \rho_1 N_2 \int_0^1 |(\varphi_x + \psi) \int_0^x \varphi_t(s, t) ds| dx + \rho_2 \int_0^1 |\psi_t \psi| dx. \end{aligned}$$

Exploiting Young's, Poincaré's, and the Cauchy-Schwarz inequalities, we obtain

$$|L(t)| \leq c \int_0^1 (\varphi_t^2 + \psi_t^2 + \psi_x^2 + (\varphi_x + \psi)^2 + \theta^2) dx + g \circ \psi_x \leq cE(t).$$

Consequently, we obtain

$$|L(t) - \widehat{N}E(t)| \leq cE(t),$$

that is,

$$(\widehat{N} - c)E(t) \leq L(t) \leq (\widehat{N} + c)E(t).$$

Choosing \widehat{N} large enough, (4.36) follows.

Next, we prove the main result of this section.

Proof of Theorem 4.1 Differentiating (4.35), recalling (4.5), (4.8), (4.10) and (4.19), and setting $\varepsilon_2 = \frac{KN_2}{8N_3}$, we obtain

$$\begin{aligned} L'(t) \leq & -[\beta N - cN_1(1 + \varepsilon_1) - cN_2 - cN_3 - c] \int_0^1 \theta_x^2 dx - \left[\frac{l}{2} - \varepsilon_1 N_3\right] \int_0^1 \psi_x^2 dx \\ & - \left[\frac{\gamma}{4} N_1 - cN_2\right] \int_0^1 \varphi_t^2 dx - \left[\frac{K}{8} N_2 - c - \varepsilon_1 N_1\right] \int_0^1 (\varphi_x + \psi)^2 dx \\ & - \left[\frac{\rho_2 g_0}{2} N_3 - cN_1 - cN_2 - \rho_2\right] \int_0^1 \psi_t^2 dx + c\left[1 + N_3\left(1 + \frac{1}{\varepsilon_1} + \frac{N_3}{N_2}\right)\right] g \circ \psi_x \\ & + \left[\frac{N}{2} - cN_3\right] g' \circ \psi_x. \end{aligned}$$

We choose N_2 large enough such that

$$\alpha_1 = \frac{K}{8} N_2 - c > 0,$$

and then choose N_1 large enough such that

$$\frac{\gamma}{4} N_1 - cN_2 > 0.$$

Similarly, we choose N_3 large enough such that

$$\frac{\rho_2 g_0}{2} N_3 - cN_1 - cN_2 - \rho_2 > 0,$$

and then, we choose ε_1 small enough such that

$$\varepsilon_1 < \min\left(\frac{\alpha_1}{N_1}, \frac{l}{2N_3}\right).$$

Finally, we choose N large enough such that (4.36) remains valid and

$$\beta N - cN_1\left(1 + \frac{1}{\varepsilon_1}\right) - cN_2 - cN_3 - c > 0, \quad \frac{N}{2} - cN_3 > 0.$$

Thus, using Poincaré’s inequality and (4.1), we obtain

$$L'(t) \leq -k_1E(t) + k_3(g \circ \psi_x)(t) + C(\|f\|_{L^2}^2 + \|z\|_{L^2}^2 + \|h\|_{L^2}^2), \quad \forall t \geq t_0, \tag{4.37}$$

for some positive constants k_1, k_3 and C .

Letting $F^2(t) = \|f\|_{L^2}^2 + \|z\|_{L^2}^2 + \|h\|_{L^2}^2$, and multiplying (4.37) by $\xi(t)$ and using (H2) and (4.2), we obtain

$$\xi(t)L'(t) \leq -k_1\xi(t)E(t) - 2k_3E'(t) + C\xi(t)F^2(t), \quad \forall t \geq t_0,$$

which can be rewritten as

$$(\xi(t)L(t) + 2k_3E(t))' - \xi'(t)L(t) \leq -k_1\xi(t)E(t) + C\xi(t)F^2(t), \quad \forall t \geq t_0.$$

Using the fact that $\xi'(t) \leq 0, \forall t \geq t_0$, we have

$$(\xi(t)L(t) + 2k_3E(t))' \leq -k_1\xi(t)E(t) + C\xi(t)F^2(t), \quad \forall t \geq t_0.$$

Exploiting (4.36), it can easily be shown that

$$R(t) := \xi(t)L(t) + 2k_3E(t) \sim E(t). \tag{4.38}$$

Consequently, for some positive constant λ_2 , we obtain

$$R'(t) \leq -\lambda_2\xi(t)E(t) + C\xi(t)(\|f\|_{L^2}^2 + \|z\|_{L^2}^2 + \|h\|_{L^2}^2). \tag{4.39}$$

Using (4.38) and (H2), we have

$$L'(t) \leq -\alpha L(t) + C'(\|f\|_{L^2}^2 + \|z\|_{L^2}^2 + \|h\|_{L^2}^2), \tag{4.40}$$

where α, C' are positive constants and C' is independent of initial data. Applying Lemma 4.6 and (H2) to (4.40), we can conclude (4.30)-(4.34).

5. Uniform Attractors

In this section, we shall establish the existence of uniform attractors for the system (1.10).

Setting $R_\tau = [\tau, +\infty), \tau \geq 0$, we consider the following system

$$\begin{cases} \rho_1\varphi_{tt} - K(\varphi_x + \psi)_x + \gamma\theta_x = f(x, t), \\ \rho_2\psi_{tt} - b\psi_{xx} + K(\varphi_x + \psi) - \gamma\theta + \int_0^t g(t-s)\psi_{xx}ds = z(x, t), \\ \rho_3\theta_t - \beta\theta_{xx} + \gamma(\varphi_x + \psi)_t = h(x, t), \end{cases} \tag{5.1}$$

together with the initial conditions and boundary conditions

$$\begin{cases} \varphi_x(0, t) = \varphi_x(1, t) = \psi(0, t) = \psi(1, t) = \theta(0, t) = \theta(1, t) = 0, \\ \varphi(x, \tau) = \varphi_\tau(x), \varphi_t(x, \tau) = \varphi_{1\tau}(x), \\ \psi(x, \tau) = \psi_\tau(x), \psi_t(x, \tau) = \psi_{1\tau}(x), \theta(x, \tau) = \theta_\tau(x). \end{cases} \tag{5.2}$$

Let

$$F = \begin{pmatrix} 0 \\ f \\ \rho_1 \\ 0 \\ z \\ \rho_2 \\ h \\ \rho_3 \end{pmatrix} \in Y = L^2(R_\tau, (L^2(0, L))^5), \tag{5.3}$$

$$\mathcal{H}_1 \equiv H_*^1(0, 1) \times L_*^2(0, 1) \times H_0^1(0, 1) \times L^2(0, 1) \times L^2(0, 1). \tag{5.4}$$

The energy of problem (5.1) is given by

$$E(t) = \frac{1}{2} \int_0^1 [\varphi_t^2 + \psi_t^2 + \theta^2 + (1 - \int_0^t g(s)ds)\psi_x^2 + (\varphi_x + \psi)^2]dx + \frac{1}{2}g \circ \psi_x. \tag{5.5}$$

For any $(\phi_\tau, \phi_{1\tau}, \psi_\tau, \psi_{1\tau}, \theta_\tau) \in \mathcal{H}_1$ and $F \in Y$, we define for $t \geq \tau, \tau \geq 0$,

$$U_F(t, \tau) : (\varphi_\tau, \varphi_{1\tau}, \psi_\tau, \psi_{1\tau}, \theta_\tau) \in \mathcal{H}_1 \mapsto (\varphi(t), \varphi_t(t), \psi(t), \psi_t(t), \theta(t)) \\ = U_F(t, \tau)(\varphi_\tau, \varphi_{1\tau}, \psi_\tau, \psi_{1\tau}, \theta_\tau),$$

where $(\varphi(t), \varphi_t(t), \psi(t), \psi_t(t), \theta(t))$ is a solution of problem (5.1).

Our results concern the uniform attractor in \mathcal{H}_1 , we define the hull of $F_0 \in Y$ as

$$\Sigma = H(F_0) = [F_0(t + h)|h \in \mathbb{R}_+]_Y,$$

where $[\cdot]_Y$ denotes the closure in Banach space Y .

We note that

$$F_0 \in Y \subseteq \hat{Y} = L_{loc}^2(\mathbb{R}_+, (L^2(0, L))^5),$$

where F_0 is a translation compact function in \hat{Y} in the weak topology, which means that $H(F_0)$ is compact in \hat{Y} . We consider the Banach space $L_{loc}^p(\mathbb{R}_+, Y_1)$ of functions $\sigma(s), s \in \mathbb{R}_+$ with values in Banach space Y_1 that are locally p -power integrable in the Bochner means. In particular, for any interval $[t_1, t_2] \subseteq \mathbb{R}_+$,

$$\int_{t_1}^{t_2} \|\sigma(s)\|_{Y_1}^p ds < +\infty.$$

Let $\sigma(s) \in L_{loc}^p(\mathbb{R}_+, Y_1)$, consider the quantity

$$\epsilon_\sigma(h) = \sup_{t \in \mathbb{R}_+} \int_t^{t+h} \|\sigma(s)\|_{Y_1}^p ds.$$

Lemma 5.1^[16] Let Σ be defined as before and $F_0 \in Y$, then

- 1) F_0 is a translation compact in \hat{Y} and any $F \in \Sigma = H(F_0)$ is also a translation compact function in \hat{Y} , moreover, $H(F) \subseteq H(F_0)$;
- 2) The set $H(F_0)$ is bounded in $L^2(\mathbb{R}_+, (L^2(0, 1))^5)$ such that

$$\epsilon_F(h) \leq \epsilon_{F_0}(h) < +\infty, \quad \forall F \in \Sigma.$$

Similarly to Theorem 3.1, we have following existence and uniqueness result.

Theorem 5.1 Let $\Sigma = [F_0(t + h)|h \in \mathbb{R}_+]_Y$, where $F_0 \in Y$ is an arbitrary but fixed symbol function. Then for any $F \in \Sigma$ and for any $(\varphi_\tau, \varphi_{\tau 1}, \psi_\tau, \psi_{\tau 1}, \theta_\tau) \in \mathcal{H}_1, \tau \geq 0$, the problem (5.1)-(5.2) has a unique global solution $(\varphi(t), \varphi_t(t), \psi(t), \psi_t(t), \theta(t)) \in \mathcal{H}_1$, which generates a unique semi-processes $\{U_F(t, \tau)\}(t \geq \tau, \tau \geq 0)$ on \mathcal{H}_1 of a two-parameter family of operators, such that, for any $t \geq \tau, \tau \geq 0$,

$$U_F(t, \tau)(\varphi_\tau, \varphi_{\tau 1}, \psi_\tau, \psi_{\tau 1}, \theta_\tau) = (\varphi(t), \varphi_t(t), \psi(t), \psi_t(t), \theta(t)) \in \mathcal{H}_1,$$

i.e.,

$$\varphi(t) \in C(R_\tau, H_*^1(0, 1)), \varphi_t(t) \in C(R_\tau, L_*^2(0, 1)), \psi(t) \in C(R_\tau, H_0^1(0, 1)), \\ \psi_t(t) \in C(R_\tau, L^2(0, 1)), \theta(t) \in C(R_\tau, L^2(0, 1)).$$

To obtain our results, we shall introduce some basic lemmas.

Let \mathcal{X} be a Banach space and $\hat{\Sigma}$ be a parameter set. The operators $\{U_F(t, \tau)\}(t \geq \tau, \tau \geq 0, F \in \hat{\Sigma})$ are said to be a family of processes in \mathcal{X} with symbol space $\hat{\Sigma}$ if for any $F \in \hat{\Sigma}$

$$U_F(t, s)U_F(s, \tau) = U_F(t, \tau), \quad \forall t \geq s \geq \tau \geq 0, \tag{5.6}$$

$$U_F(\tau, \tau) = \text{Id} \quad (\text{identity operator}), \quad \forall \tau \geq 0. \tag{5.7}$$

Lemma 5.2^[3] Let $\{U_F(t, \tau)\}(F \in \hat{\Sigma}, t \geq \tau, \tau \geq 0)$ be a family of processes satisfying the translation identities (5.6) and (5.7) on Banach space \mathcal{X} , and has a bounded uniformly (with respect to $F \in \hat{\Sigma}$) absorbing set $A_0 \subseteq \mathcal{X}$. Moreover, assuming that for any $\varepsilon > 0$, there exists a time $T = T(A_0, \varepsilon) > 0$ and a contractive function ϕ_T on $A_0 \times A_0$ such that

$$\|U_{F_1}(T, 0)x - U_{F_2}(T, 0)y\| \leq \varepsilon + \phi_T(x, y; F_1, F_2), \quad \forall x, y \in A_0, \quad \forall F_1, F_2 \in \hat{\Sigma}.$$

Then, $\{U_F(t, \tau)\}(F \in \hat{\Sigma}, t \geq \tau, \tau \geq 0)$ is uniformly (w.r.t. $F \in \hat{\Sigma}$) asymptotically compact in \mathcal{X} .

Lemma 5.3^[16] For every $\tau \in \mathbb{R}$, assume that ϕ_0 is non-negative locally summable function on $R_\tau \equiv [\tau, +\infty)$, and for every $\iota > 0$, we have

$$\sup_{t \geq \tau} \int_\tau^t \phi_0(s)e^{-\iota(t-s)} ds \leq \frac{1}{1 - e^{-\iota}} \sup_{t \geq \tau} \int_t^{t+1} \tau^{t+1} \phi_0(s) ds,$$

for all $t \geq \tau$.

Now, we shall establish that the family of semi-processes $\{U_F(t, \tau)\}$ has a bounded uniformly absorbing set given in the following theorem.

Theorem 5.2 Under the assumption (5.3), the family of processes $\{U_F(t, \tau)\}(F \in \Sigma, t \geq \tau, \tau \geq 0)$, corresponding to (5.1)-(5.2) has a bounded uniform absorbing set $A_0 \subseteq \mathcal{H}_1$.

Proof Similarly to the proof of Theorem 4.1, we obtain

$$\frac{dE(t)}{dt} \leq -\gamma E(t) + C_1(\|f\|_2^2 + \|z\|_2^2 + \|h\|_2^2),$$

where γ, C_1 are two positive constants with $C_1 > 0$ being independent of initial data. In the following, C denotes general positive constant, which may be different in different estimates.

Obviously, we have

$$E(t) \leq E(\tau)e^{-\gamma(t-\tau)} + C \int_\tau^t (\|f\|_2^2 + \|z\|_2^2 + \|h\|_2^2)e^{-\gamma(t-s)} ds. \tag{5.8}$$

Applying Lemmas 5.1 and 5.3 to (5.8), we conclude

$$\begin{aligned} E(t) &\leq E(\tau)e^{-\gamma(t-\tau)} + C \int_\tau^t \|F\|_Y^2 e^{-\gamma(t-s)} ds, \\ &\leq E(\tau)e^{-\gamma(t-\tau)} + C \frac{1}{1 - e^{-\gamma}} \sup_{t \geq \tau} \int_t^{t+1} \|F\|_Y^2 ds, \\ &\leq E(\tau)e^{-\gamma(t-\tau)} + C \frac{1}{1 - e^{-\gamma}} \epsilon_{F_0}(1). \end{aligned} \tag{5.9}$$

Now for any bounded set $A_0 \subseteq \mathcal{H}_1$, for any $(\varphi_\tau, \varphi_{\tau+1}, \psi_\tau, \psi_{\tau+1}, \theta_\tau) \in A_0, \tau > 0$, there exists a constant $C_{A_0} > 0$ such that $E(\tau) \leq C_{A_0}$. Taking

$$\begin{aligned} R_0^2 &= 2(C_1 \frac{\epsilon_{F_0}(1)}{1 - e^{-\gamma}} + 1), \\ t_0 &= t_0(\tau, F_0) = \tau - \gamma^{-1} \log\left(\frac{\epsilon_{F_0}(1) + 1}{C_{A_0}(1 - e^{-\gamma})}\right). \end{aligned}$$

Then, $A_0(0, R_0) = \{(\varphi(t), \varphi_t(t), \psi(t), \psi_t(t), \theta(t)) \in \mathcal{H}_1 : \|(\varphi(t), \varphi_t(t), \psi(t), \psi_t(t), \theta(t))\|_{\mathcal{H}_1^2} \leq R_0^2\} \subseteq \mathcal{H}_1$ is a uniformly absorbing set for any $F \in \Sigma$, that is, for any bounded subset $A \subseteq \mathcal{H}_1$, there exists a time $t_0 = t_0(\tau, F_0) \geq \tau$ such that, for all $t \geq t_0$,

$$\bigcup_{F \in \Sigma} U_F(t, \tau)A \subseteq A_0.$$

The proof is complete.

The main purpose of following part is to establish the uniformly asymptotic compactness (with respect to $F \in \Sigma$). Without loss of generality, we deal with the strong solutions in the following part, then the case for weak solutions follows easily for a similar argument.

For any $(\varphi_\tau^i, \varphi_{1\tau}^i, \psi_\tau^i, \psi_{1\tau}^i, \theta_\tau^i) \in A_0$, let $(\varphi_i(t), \varphi_{it}(t), \psi_i(t), \psi_{it}(t), \theta_i(t))$ be the corresponding solution to $F_i \in \Sigma$ with respect to initial data $(\varphi_\tau^i, \varphi_{1\tau}^i, \psi_\tau^i, \psi_{1\tau}^i, \theta_\tau^i), i = 1, 2$. Let

$$\begin{aligned} Q(t) &= (\pi(t), \omega(t), \alpha(t))^T = U_1(t) - U_2(t) \\ &= (\pi_1(t) - \pi_2(t), \omega_1(t) - \omega_2(t), \alpha_1(t) - \alpha_2(t))^T. \end{aligned} \tag{5.10}$$

Then $Q(t)$ satisfies

$$\begin{cases} \rho_1 \pi_{tt} - K(\pi_x + \omega)_x + \gamma \alpha_x = f_1(x, t) - f_2(x, t), \\ \rho_2 \omega_{tt} - b\omega_{xx} + K(\pi_x + \omega) - \gamma \alpha + \int_0^t g(t-s)\omega_{xx} ds = z_1(x, t) - z_2(x, t), \\ \rho_3 \alpha_t - \beta \alpha_{xx} + \gamma(\pi_x + \omega)_t = h_1(x, t) - h_2(x, t), \\ \pi_x(0, t) = \pi_x(1, t) = \omega(0, t) = \omega(1, t) = \alpha(0, t) = \alpha(1, t) = 0, \\ \pi(x, \tau) = \varphi_\tau^1(x) - \varphi_\tau^2(x); \pi_t(x, \tau) = \varphi_{1\tau}^1(x) - \varphi_{1\tau}^2(x), \\ \omega(x, \tau) = \psi_\tau^1(x) - \psi_\tau^2(x); \omega_t(x, \tau) = \psi_{1\tau}^1(x) - \psi_{1\tau}^2(x), \\ \alpha(x, \tau) = \theta_\tau^1(x) - \theta_\tau^2(x), \end{cases} \tag{5.11}$$

where $x \in (0, 1), t \in (\tau, +\infty)$.

Let

$$E_Q(t) = \frac{1}{2} \int_0^1 [\pi_t^2 + \omega_t^2 + \alpha^2 + (1 - \int_0^t g(s) ds)\omega_x^2 + (\pi_x + \omega)^2] dx + \frac{1}{2} g \circ \omega_x. \tag{5.12}$$

Without loss of generality, we assume that $\rho_1 = \rho_2 = \rho_3 = K = b = \gamma = \beta = 1$. Then, we have

$$\begin{aligned} \frac{dE_Q(t)}{dt} &= \frac{1}{2} \frac{d}{dt} \int_0^1 [\pi_t^2 + \omega_t^2 + \alpha^2 + (1 - \int_0^t g(s) ds)\omega_x^2 + (\pi_x + \omega)^2] dx + \frac{1}{2} \frac{d(g \circ \omega_x)}{dt} \\ &= - \int_0^1 \alpha_x^2 dx + \frac{1}{2} g' \circ \omega_x - \frac{1}{2} g(t) \int_0^1 \omega_x^2 dx \\ &\quad + \int_0^1 \pi_t(f_1 - f_2) dx + \int_0^1 \omega_t(z_1 - z_2) dx + \int_0^1 \alpha(h_1 - h_2) dx. \end{aligned} \tag{5.13}$$

Thus integrating (5.13) over $[\sigma, T]$, we arrive at

$$\begin{aligned} &E_Q(t) + \int_\sigma^T \int_0^1 \alpha_x^2 dx dt - \frac{1}{2} \int_\sigma^T \int_\sigma^T g' \circ \omega_x dt d\sigma + \int_\sigma^T \frac{1}{2} g(t) \int_0^1 \omega_x^2 dx dt \\ &= \int_\sigma^T \int_0^1 \pi_t(f_1 - f_2) dx dt + \int_\sigma^T \int_0^1 \omega_t(z_1 - z_2) dx dt \\ &\quad + \int_\sigma^T \int_0^1 \alpha(h_1 - h_2) dx dt + E_Q(\sigma), \end{aligned} \tag{5.14}$$

where $0 \leq \sigma \leq T$.

Integrating (5.14) over $[0, T]$ with respect to σ , we have

$$\begin{aligned} &TE_Q(t) + \int_0^T \int_\sigma^T \int_0^1 \alpha_x^2 dx dt d\sigma - \frac{1}{2} \int_0^T \int_\sigma^T \int_\sigma^T g' \circ \omega_x dt d\sigma + \int_0^T \int_\sigma^T \frac{1}{2} g(t) \int_0^1 \omega_x^2 dx dt d\sigma \\ &= \int_0^T \int_\sigma^T \int_0^1 \pi_t(f_1 - f_2) dx dt d\sigma + \int_0^T \int_\sigma^T \int_0^1 \omega_t(z_1 - z_2) dx dt d\sigma \end{aligned}$$

$$+ \int_0^T \int_\sigma^T \int_0^1 \alpha(h_1 - h_2) dx dt d\sigma + \int_0^T E_Q(\sigma) d\sigma. \tag{5.15}$$

Using Young’s inequality and Poincaré’s inequality, we obtain

$$\begin{aligned} & TE_Q(t) + \int_0^T \int_\sigma^T \int_0^1 \alpha_x^2 dx dt d\sigma \\ & \leq \int_0^T \int_\sigma^T \int_0^1 \pi_t(f_1 - f_2) dx dt d\sigma + \int_0^T \int_\sigma^T \int_0^1 \omega_t(z_1 - z_2) dx dt d\sigma \\ & \quad + \int_0^T \int_\sigma^T \int_0^1 (h_1 - h_2) \alpha dx dt d\sigma + \int_0^T E_Q(\sigma) d\sigma \\ & \leq \frac{1}{2} \int_0^T \int_\sigma^T \int_0^1 (f_1 - f_2)^2 dx dt d\sigma + \frac{1}{4} \int_0^T \int_\sigma^T \int_0^1 (h_1 - h_2)^2 dx dt d\sigma \\ & \quad + \int_0^T \int_\sigma^T \int_0^1 \alpha_x^2 dx dt d\sigma + \frac{1}{2} \int_0^T \int_\sigma^T \int_0^1 (z_1 - z_2)^2 dx dt d\sigma \\ & \quad + \frac{1}{2} \left(\int_0^T \int_\sigma^T \int_0^1 \omega_t^2 dx dt d\sigma + \int_0^T \int_\sigma^T \int_0^1 \pi_t^2 dx dt d\sigma \right) + \int_0^T E_Q(\sigma) d\sigma. \end{aligned} \tag{5.16}$$

Using integration by parts, we obtain

$$\begin{aligned} TE_Q(t) & \leq \frac{1}{2} \int_0^T \int_\sigma^T \int_0^1 (f_1 - f_2)^2 dx dt d\sigma + \frac{1}{4} \int_0^T \int_\sigma^T \int_0^1 (h_1 - h_2)^2 dx dt d\sigma \\ & \quad + \frac{1}{2} \int_0^T \int_\sigma^T \int_0^1 (z_1 - z_2)^2 dx dt d\sigma + \int_0^T \int_\sigma^T E_Q(t) dt d\sigma + \int_0^T E_Q(\sigma) d\sigma. \end{aligned} \tag{5.17}$$

According to (5.11)-(5.12), we obtain

$$\begin{aligned} \int_0^T |E_Q(\sigma)| d\sigma & = \int_0^T \frac{1}{2} \int_0^1 [\pi_t^2 + \omega_t^2 + \alpha^2 + (1 - \int_0^t g(s) ds) \omega_x^2 + (\pi_x + \omega)^2] dx d\sigma + \frac{1}{2} \int_0^T g \circ \omega_x d\sigma \\ & \leq \int_0^T (E_{U_1}(\sigma) + E_{U_2}(\sigma)) d\sigma \\ & \leq 2 \int_0^T [E(\tau) e^{-\gamma(t-\tau)} + C \int_\tau^\sigma (\|f\|_2^2 + \|z\|_2^2 + \|h\|_2^2) e^{-\gamma(t-s)} ds] d\sigma \\ & \leq C_1, \end{aligned} \tag{5.18}$$

where $C_1 = C(T, \tau, \gamma)$ is a positive constant.

Similarly to (5.18), we can obtain

$$\begin{aligned} \int_0^T \int_\sigma^T E_Q(t) dt d\sigma & \leq 2C \int_0^T \int_\sigma^T \int_\tau^t (\|f\|_2^2 + \|z\|_2^2 + \|h\|_2^2) e^{-\gamma(t-s)} ds dt d\sigma \\ & \quad + 2 \int_0^T \int_\sigma^T E(\tau) e^{-\gamma(t-\tau)} dt d\sigma \\ & \leq C_2, \end{aligned} \tag{5.19}$$

Thus we conclude

$$\begin{aligned} TE_Q(t) & \leq \frac{1}{2} \int_0^T \int_\sigma^T \int_0^1 (f_1 - f_2)^2 dx dt d\sigma + \frac{1}{2} \int_0^T \int_\sigma^T \int_0^1 (z_1 - z_2)^2 dx dt d\sigma \\ & \quad + \frac{1}{4} \int_0^T \int_\sigma^T \int_0^1 (h_1 - h_2)^2 dx dt d\sigma + C_M. \end{aligned} \tag{5.20}$$

Let

$$R((\varphi_0^1, \varphi_{10}^1, \psi_0^1, \psi_{10}^1, \theta_0^1), (\varphi_0^2, \varphi_{10}^2, \psi_0^2, \psi_{10}^2, \theta_0^2), F_1, F_2)$$

$$\begin{aligned}
&= \frac{1}{2} \int_0^T \int_\sigma^T \int_0^1 (f_1 - f_2)^2 dx dt d\sigma + \frac{1}{2} \int_0^T \int_\sigma^T \int_0^1 (z_1 - z_2)^2 dx dt d\sigma \\
&\quad + \frac{1}{4} \int_0^T \int_\sigma^T \int_0^1 (h_1 - h_2)^2 dx dt d\sigma.
\end{aligned} \tag{5.21}$$

Thus we get

$$E_Q(T) \leq \frac{C_M}{T} + \frac{1}{T} R((\varphi_0^1, \varphi_{10}^1, \psi_0^1, \psi_{10}^1, \theta_0^1), (\varphi_0^2, \varphi_{10}^2, \psi_0^2, \psi_{10}^2, \theta_0^2); F_1, F_2). \tag{5.22}$$

In the sequel, we are now ready to prove the uniformly asymptotic compactness in \mathcal{H}_1 .

Theorem 5.3 Assume that F satisfies (5.3), then the family of semi-process $\{U_F(t, \tau)\} (F \in \Sigma, t \geq \tau, \tau \geq 0)$, corresponding to (5.1), is uniformly (w.r.t. $F \in \Sigma$) asymptotically compact in \mathcal{H}_1 .

Proof Because the family of semi-process $\{U_F(t, \tau)\} (F \in \Sigma, t \geq \tau, \tau \geq 0)$ has a bounded uniformly absorbing set, by the definition of C_M , we know that for any fixed $\varepsilon > 0$, we can choose $T > 0$ so large that

$$\frac{C_M}{T} \leq \varepsilon.$$

Due to Lemma 5.1, it is sufficient to prove that $R(\cdot, \cdot, \cdot, \cdot, \cdot) \in \text{Contr}(A_0, \Sigma)$ for each fixed T .

From the proof of Theorem 5.1, we can deduce that for any fixed T , we have

$$\bigcup_{F \in \Sigma} \bigcup_{t \in [\tau, T]} U_F(t, \tau) A_0, \tag{5.23}$$

is bounded in \mathcal{H}_1 and the bound depend on T .

Let $(\varphi_i, \varphi_{it}, \psi_i, \psi_{it}, \theta_i)$ be the solutions corresponding to initial data $(\varphi_\tau^i, \varphi_{1\tau}^i, \psi_\tau^i, \psi_{1\tau}^i, \theta_\tau^i) \in A_0$ with respect to symbol $F_n \in \Sigma, n = 1, 2, \dots$. Then from (5.3), we can infer

$$\lim_{i \rightarrow \infty} \lim_{j \rightarrow \infty} \int_0^T \int_\sigma^T \int_0^1 (f_i - f_j)^2 dx dt d\sigma = 0, \tag{5.24}$$

$$\lim_{i \rightarrow \infty} \lim_{j \rightarrow \infty} \int_0^T \int_\sigma^T \int_0^1 (z_i - z_j)^2 dx dt d\sigma = 0, \tag{5.25}$$

$$\lim_{i \rightarrow \infty} \lim_{j \rightarrow \infty} \int_0^T \int_\sigma^T \int_0^1 (h_i - h_j)^2 dx dt d\sigma = 0. \tag{5.26}$$

According to from (5.22) and (5.23), we conclude that $R(\cdot, \cdot, \cdot, \cdot, \cdot) \in \text{Contr}(A_0, \Sigma)$ immediately.

Theorem 5.4 Assume that f, z, h satisfies (5.3) and Σ is defined above, then the family of processes $U_F(t, \tau) (F \in \Sigma, t \geq \tau, \tau \geq 0)$ corresponding to (5.1) has a compact uniform (w.r.t. $F \in \Sigma$) attractor \mathcal{A}_Σ .

Proof According to Theorems 5.2 and 5.3, we have the existence of a uniform attractor at once.

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带有记忆项的非自治第一类型热弹性Timoshenko系统解的整体存在性、渐近性及其一致吸引子

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摘要: 本文研究带有记忆项的非自治第一类型热弹性Timoshenko系统解的整体存在性、渐近性及其一致吸引子. 首先利用半群理论, 获得了解的整体存在性. 其次研究解的渐近行为. 最后利用一致压缩函数法证明一致吸引子的存在性.

关键词: Timoshenko系统; 解的整体存在性; 渐近性; 一致吸引子; 半群理论