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A m, p-Laplacian Parabolic Equation with Nonlinear Absorption and Boundary Flux

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Abstract: In this paper, we deal with a m, p-Laplacian equation of parabolic type in $\mathbb{R}^+ \times \mathbb{R}^+$ ($p > 2$ and $m > 1$) with inner absorption term ($-\lambda u^\kappa$) and nonlinear boundary flux u^q. If $q < q^*$, every solution remains global. If $q > q^*$, both global solutions and blow-up solutions could exist depending on the choosing of initial data. In the balanced case $q = q^*$, the size of the coefficient of the absorption term plays a fundamental role in distinguishing global solutions from blow-up solutions. All solutions exist globally for $\kappa \leq 1$. If $1 < \kappa < m(p-1)+p$, large λ leads to blow-up of solutions for any initial data, i.e., Fujita-type blow-up, while small λ leads to global existence of solutions. Moreover, we give the quantitative estimates of the coefficient. If $\kappa > m(p-1)+p$, there exist both global and blow-up solutions.

Key words: m, p-Laplacian parabolic equation; Fujita-type blow-up; Global existence
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1. Introduction

In this paper, we consider a m, p-Laplacian equation of parabolic type

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(\left|\frac{\partial u^m}{\partial x}\right|^{p-2}\frac{\partial u^m}{\partial x}\right) = -\lambda u^\kappa, \quad (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+, \tag{1.1}$$

with nonlinear boundary flux

$$-\left(\left|\frac{\partial u^m}{\partial x}\right|^{p-2}\frac{\partial u^m}{\partial x}\right)(0,t) = u^q(0,t), \quad t \in \mathbb{R}^+, \tag{1.2}$$

and the initial datum

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^+, \tag{1.3}$$

where constants $p > 2$, $m > 1$, κ, q, $\lambda > 0$; u_0 is nonnegative and continuous with compact support in $\mathbb{R}^+$. Since the nonlinear diffusion of m, p-Laplacian type may be degenerated at $u = 0$ or $(u^m)_x = 0$, such problem does not admit classical solutions in general. The local existence and the comparison principle of weak solutions have been established in [7].

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The nonlinear diffusion equations (1.1) can be used to describe the nonstationary flow in a porous medium of fluids with a power dependence of the tangential stress on the velocity of displacement under polytropic conditions.

The equation (1.1) without absorption, subject to nonlinear boundary flux (1.2), has been studied by WANG, et al. in [6] (i.e., $\lambda = 0$). It was proved that the blow-up phenomena are caused by the nonlinear boundary flux and the Fujita exponent is denoted by $q_c := (m+1)(p-1)$. The results are obtained as follows.

(i) If $0 \le q \le q_0 := (m+1)(p-1)/p$, the solution is global;

(ii) If $q_0 < q < q_c$, every solution blows up for nontrivial, nonnegative initial data;

(iii) If $q > q_c$, both global solutions and blow-up solutions could exist.

Recently, JIN, et al.[3] considered the parabolic equation of p-Laplacian type,

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right) = -\lambda u^{\kappa}, \quad (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+,$$

subject to the nonlinear boundary flux $-\left(\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right)(0,t) = u^q(0,t)$, $t \in \mathbb{R}^+$, where constants $p > 2$, κ, q, $\lambda > 0$. The critical Fujita absorption exponent has been firstly introduced in the critical case, which can be seen from the results:

(i) If $q < q^* = \max\{2(p-1)/p, (\kappa+1)(p-1)/p\}$, then every nontrivial and nonnegative solution exists globally in time;

(ii) If $q > q^*$, then there exist both global solutions and blow-up solutions;

(iii) If $q = q^*$ with λ large, then all solutions exist globally;

(iv) Let $q = q^*$ with λ small.

If $\kappa \le 1$, all solutions exist globally.

If $1 < \kappa < 2p-1$, the solutions blow up under any nontrivial nonnegative initial data.

If $\kappa > 2p-1$, there could be both global solutions and blow-up solutions for small and large initial data, respectively.

The other works about the parabolic equations of p-Laplacian type can be found in [1-2,4-5,8-10] and the papers cited there.

To our knowledge, the system (1.1)-(1.3) has not been considered before. In this paper, we want to determine the critical Fujita absorption exponent for such m, p-Laplacian equation (1.1) with absorption and boundary flux, and study how the absorption affects the global and blow-up solutions, inspired by [3, 6]. Moreover, we try to obtain the quantitative description about the coefficient λ of the absorption term in distinguishing the existence of global solutions from blow-up solutions.

The main results will be given in the next section. We also give some remarks about the influence of coefficients, the absorption and the boundary flux on the existence of global solutions and blow-up solutions of system (1.1)-(1.3). The proof of the main results can be found in Sections 3-6, respectively.

2. Main Results and Remarks

For convenience, we denote a positive constant

$$q^* := \max\left\{\frac{(m+1)(p-1)}{p}, \frac{(\kappa+m)(p-1)}{p}\right\}.$$

It can be found that $q^* = (m+1)(p-1)/p$ for $\kappa < 1$ and $q^* = (\kappa+m)(p-1)/p$ for $\kappa \geq 1$, respectively. The main results are as follows.

Theorem 2.1 (i) If $q < q^*$, then every nontrivial nonnegative solution of system (1.1)-(1.3) exists globally in time.

(ii) If $q > q^*$, then both global solutions and blow-up solutions could exist.

(iii) Let $q = q^*$. If one of the following conditions holds, then all solutions exist globally.

• $\kappa \leq 1$;

• $1 < \kappa < m(p-1)$ and

$$\lambda \geq (mr)^{-1}(rm-1)M^{r[m(p-1)-\kappa]-p}, \tag{2.1}$$

where positive constants $r := p/[m(p-1)-\kappa]$ and M satisfies that

$$M^{r[m(p-1)-q]-(p-1)} \geq 1 \quad \text{and} \quad [M - x/(rm)]_+^r \geq u_0(x)$$

for any compactly supported $u_0(x)$;

• $\kappa > m(p-1)$ and $\lambda > (\kappa+m)(p-1)/p$;

• $\kappa = m(p-1)$ and $\lambda > m^p(p-1)$;

(iv) Let $q = q^*$.

• If $1 < \kappa < m(p-1)$, the solutions blow up under any nontrivial nonnegative initial data provided that λ satisfies that

$$\lambda \leq \min\left\{1,\ C^{-r(\kappa-1)-1}(1-\sigma)m^{p-1}r^{p-1}A^{m(p-1)-\kappa}(rm-1)(p-1)\right\}, \tag{2.2}$$

where constants $r, C, A, \alpha, \beta, \sigma$ satisfy that for small $\sigma \in (0,1)$ and large constant C,

$$\begin{aligned} r :=& \frac{p-1}{m(p-1)-1}, \quad A := m^{-\frac{p}{m(p-1)-\kappa}} r^{-\frac{p}{m(p-1)-\kappa}} C^{-r+\frac{p}{m(p-1)-\kappa}}, \\ \alpha :=& \frac{1}{\kappa-1}, \quad \beta := -\frac{\alpha(m(p-1)-q)}{p-1}, \\ \alpha C \leq& \sigma m^{p-1}r^{p-1}A^{m(p-1)-1}(rm-1)(p-1); \end{aligned}$$

• If $m(p-1) \leq \kappa < m(p-1)+p$ and $\lambda < (m+1)(p-1)/(mp)$, the solutions blow up under any nontrivial nonnegative initial data;

• If $\kappa > m(p-1)+p$, global solutions exist for small initial data while blow-up solutions exist for large initial data, respectively.

Remark 2.1 The results in Theoreom 2.1 are compatible with the ones of [3] if taking $m = 1$.

Remark 2.2 There is a Fujita-type blow-up result in the exponent region

$$\left\{ q = \frac{(\kappa+m)(p-1)}{p},\ 1 < \kappa < m(p-1)+p \right\},$$

which is equivalent to $\{(p-1)(m+1) < pq < (p-1)p(m+1)\}$.

Remark 2.3 The results in Theoreom 2.1 show that the existence of global and blow-up solutions was influenced not only by the exponents p, q, m but also by the coefficient of the absorption term in the equation (1.1). If $q < q^*$, every solution exists globally, while $q > q^*$, both global solutions and blow-up solutions exist depending on the choosing of initial data. The balanced case $q = q^*$ is more interesting. The size of the coefficient of the absorption term plays a fundamental role in distinguishing global solutions from non-global solutions. In fact, for $q = (\kappa+m)(p-1)/p$,

• $1 < \kappa < m(p-1)$ and λ is large satisfying (2.1): global existence.

• $1 < \kappa < m(p-1)$ and λ is small satisfying (2.2): blow-up.

• $m(p-1) \le \kappa < m(p-1)+p$ and $\lambda > \max\{(\kappa+m)(p-1)/p,\ m^p(p-1)\}$: global existence.

• $m(p-1) \le \kappa < m(p-1)+p$ and $\lambda < (m+1)(p-1)/(mp)$: blow-up.

3. Proof of Theorem 2.1 (i)

It is worthy of noticing the coefficient $\lambda > 0$ can be scaled except for the case $q = \frac{(\kappa+m)(p-1)}{p}$. So we may assume $\lambda = 1$ whenever $q \ne \frac{(\kappa+m)(p-1)}{p}$.

Proof of Theorem 2.1 (i) We consider the case $q < q^*$. It is known from [6] that system (1.1)-(1.3) without absorption only possesses global solutions if $q \le \frac{(m+1)(p-1)}{p}$, and so does system (1.1)-(1.3) itself by the comparison principle. We just need to consider the case $\kappa > 1$, namely, $\frac{(m+1)(p-1)}{p} < q < \frac{(\kappa+m)(p-1)}{p}$. Define

$$\overline{u}(x,t) := \mathrm{e}^{\alpha t} f(\eta), \quad \eta := \mathrm{e}^{\beta t} x, \tag{3.1}$$

with constants α and β to be determined. A simple calculation yields that

$$-\left(\left|\frac{\partial \overline{u}^m}{\partial x}\right|^{p-2}\frac{\partial \overline{u}^m}{\partial x}\right)(0,t) \ge \overline{u}^q(0,t), \quad t > 0, \tag{3.2}$$

is equivalent to

$$-m^{p-1}\mathrm{e}^{[m(p-1)-q]\alpha t+(p-1)\beta t}(|f^{m-1}f'|^{p-2}f^{m-1}f')(0) \ge f^q(0), \tag{3.3}$$

and

$$\frac{\partial \overline{u}}{\partial t} \ge \frac{\partial}{\partial x}\left(\left|\frac{\partial \overline{u}^m}{\partial x}\right|^{p-2}\frac{\partial \overline{u}^m}{\partial x}\right) - u^\kappa \tag{3.4}$$

is equivalent to

$$\alpha f(\eta) + \beta\eta f'(\eta) \ge m^{p-1}\mathrm{e}^{[m(p-1)-1]\alpha t+p\beta t}(|f^{m-1}f'|^{p-2}f^{m-1}f')'(\eta) - \mathrm{e}^{\alpha(\kappa-1)t}f^\kappa(\eta). \tag{3.5}$$

Take $f(\eta) := (M+\mathrm{e}^{-\kappa\eta})^r$. Then the inequalities (3.3) and (3.5) are equivalent to

$$m^{p-1}\mathrm{e}^{[m(p-1)-q]\alpha t+(p-1)\beta t}(\kappa r)^{p-1}(M+1)^{(p-1)[r(m-1)+r-1]-rq} \ge 1,$$

and

$$\begin{aligned}&\alpha\mathrm{e}^{-\alpha[m(p-1)-1]t-\beta pt}(M+\mathrm{e}^{-\kappa\eta}) + \mathrm{e}^{-\alpha[m(p-1)-\kappa]t-\beta pt}(M+\mathrm{e}^{-\kappa\eta})^{r(\kappa-1)+1}\\ \ge\, & r\beta\eta\kappa\mathrm{e}^{-\kappa\eta}\mathrm{e}^{-\alpha[m(p-1)-1]t-\beta pt} + m^{p-1}\kappa^p r^{p-1}(p-1)(rm-1)\mathrm{e}^{-\kappa p\eta}(M+\mathrm{e}^{-\kappa\eta})^{r[m(p-1)-1]-p+1}\\ &+ m^{p-1}\kappa^p r^{p-1}(p-1)\mathrm{e}^{-\kappa(p-1)\eta}(M+\mathrm{e}^{-\kappa\eta})^{r[m(p-1)]-p+2}.\end{aligned}$$

Noticing $\frac{\alpha[q-m(p-1)]}{p-1} < \frac{\alpha[\kappa-m(p-1)]}{p}$ by $q < \frac{(\kappa+m)(p-1)}{p}$, we take

$$\frac{\alpha[q-m(p-1)]}{p-1} < \beta < \frac{\alpha[\kappa-m(p-1)]}{p}.$$

Thus (3.3) is true provided that

$$(M+1)^{1-\frac{r[m(p-1)-q]}{p-1}} \ge r\kappa m,$$

and (3.5) is ensured by

$$\alpha M \ge r\beta \quad \text{and} \quad (M+\mathrm{e}^{-\kappa\eta})^{r[\kappa-m(p-1)]+p-1} \ge (m\kappa r)^p(p-1).$$

Take $r > \frac{p-1}{(p-1)(\kappa+1)-pq}$ to get

$$\frac{r[\kappa-m(p-1)]+p-1}{p} \ge 1 + \frac{r[q-m(p-1)]}{p-1}.$$

Then for any large constant M, there exists some constant κ such that the above three inequalities hold. This proves (3.2) and (3.4). In addition, $\bar{u}(x,0) = (M+\mathrm{e}^{-\kappa x})^r \ge u_0(x)$, whenever M is large enough. We conclude that $\bar{u}$ is a global super-solution to system (1.1)-(1.3) by using the comparison principle.

4. Proof of Theorem 2.1 (ii)

Next consider the case $q > q^*$.

Proof of Theorem 2.1 (ii) Define

$$\underline{u}(x,t) := (T-t)^{-\alpha}h(\eta), \quad \eta := (T-t)^{-\beta}x.$$

Then $\underline{u}$ is a subsolution whenever the function h satisfies the inequalities

$$\begin{aligned} &\alpha h(\eta) + \beta\eta h'(\eta) + (T-t)^{\alpha(1-\kappa)+1}h^\kappa(\eta) \\ \le& m^{p-1}(T-t)^{1-(m(p-1)-1)\alpha-p\beta}(|h^{m-1}h'|^{p-2}h^{m-1}h')'(\eta), \quad (4.1)\\ &-m^{p-1}(T-t)^{\alpha q-(m\alpha+\beta)(p-1)}(|h^{m-1}h'|^{p-2}h^{m-1}h')(0) \le h^q(0), \quad (4.2)\\ &T^{-\alpha}h(T^{-\beta}x) \le u_0(x). \quad (4.3)\end{aligned}$$

Let $h := A(M-\eta)_+^r$ with $f_+ := \max\{f, 0\}$. Then the inequalities (4.1) and (4.2) are equivalent to

$$\begin{aligned} &\alpha A(M-\eta) - rA\beta\eta + A^\kappa(T-t)^{\alpha-\alpha\kappa+1}(M-\eta)^{r(\kappa-1)+1} \\ \le& m^{p-1}A^{m(p-1)}r^{p-1}(p-1)(rm-1)(T-t)^{1-(m(p-1)-1)\alpha-p\beta}(M-\eta)^{r[m(p-1)-1]-p+1}, \quad (4.4)\end{aligned}$$

$$m^{p-1}A^{m(p-1)}r^{p-1}(T-t)^{\alpha q-(m\alpha+\beta)(p-1)}M^{(rm-1)(p-1)-rq} \le A^q. \quad (4.5)$$

Noticing that $q > \frac{(p-1)(\kappa+m)}{p}$, and letting

$$\begin{cases} \dfrac{1}{m} < r < \min\left\{\dfrac{p}{m(p-1)-1}, \dfrac{p}{m(p-1)-\kappa}\right\}, & k < m(p-1), \\ \dfrac{1}{m} < r < \dfrac{p}{m(p-1)-1}, & k \ge m(p-1), \end{cases}$$

and $\alpha > \frac{p-1}{pq-(m+1)(p-1)}$, we have

$$\frac{\alpha[q-m(p-1)]}{p-1} > \frac{1-\alpha[m(p-1)-1]}{p},$$

and choose

$$\frac{1-\alpha[m(p-1)-1]}{p} < \beta < \frac{\alpha[q-m(p-1)]}{p-1}.$$

Then (4.4)-(4.5) are ensured by

$$\alpha - r\beta \le \frac{2}{3}m^{p-1}A^{m(p-1)-1}r^{p-1}(rm-1)(p-1)T^{1-[m(p-1)-1]\alpha-\beta p}M^{r(m(p-1)-1)-p},$$

$$1 \le \frac{1}{3}m^{p-1}A^{m(p-1)-\kappa}r^{p-1}(p-1)(rm-1)T^{\alpha(\kappa-m(p-1))-\beta p}M^{r[m(p-1)-\kappa]-p},$$

$$m^{p-1}A^{m(p-1)-q}r^{p-1}T^{\alpha q-(m\alpha+\beta)(p-1)}M^{r[m(p-1)-q]-(p-1)} \le 1,$$

which do hold for appropriately small $T > 0$. Thus, u blows up at some time $T^* \le T$ by the comparison principle provided that $u_0(x)$ is so large such that (4.3) is satisfied.

Next, we consider the global existence of solutions. Let $\overline{u} := (M+\varepsilon x)^{-r}$ with constant $r > 0$ to be determined. We see that

$$\frac{\partial\overline{u}}{\partial t} \ge \frac{\partial}{\partial x}\left(\left|\frac{\partial\overline{u}^m}{\partial x}\right|^{p-2}\frac{\partial\overline{u}^m}{\partial x}\right) - \overline{u}^\kappa$$

is equivalent to

$$\varepsilon(p-1)(rm+1)(rm\varepsilon)^{p-1}(M+\varepsilon x)^{r[\kappa-m(p-1)]-p} \le 1,$$

and

$$-\left(\left|\frac{\partial \overline{u}^m}{\partial x}\right|^{p-2}\frac{\partial \overline{u}^m}{\partial x}\right)(0,t) \geq \overline{u}^q(0,t)$$

is ensured by

$$(rm\varepsilon)^{p-1} \geq M^{(rm+1)(p-1)-rq}.$$

Let $r < \frac{p}{\kappa - m(p-1)}$ with $\kappa > m(p-1)$, and take

$$\varepsilon := \frac{1}{rm} M^{rm+1-\frac{rq}{p-1}}.$$

Then the above two inequalities are satisfied provided that

$$\frac{(p-1)(rm+1)}{rm} M^{\frac{rp}{p-1}\left(\frac{(\kappa+m)(p-1)}{p}-q\right)} \leq 1,$$

which does hold for appropriately large constant M. We conclude that $\overline{u}$ is a time-independent supersolution provided that the initial data are so small such that $u_0(x) \leq \overline{u}$.

5. Proof of Theorem 2.1 (iii)

In this section, we treat the critical case of $q = q^* := \max\left\{\frac{(\kappa+m)(p-1)}{p}, \frac{(m+1)(p-1)}{p}\right\}$.

Proof of Theorem 2.1 (iii) If $\kappa \leq 1$, then $q^* = (m+1)(p-1)/p$, and the solution always exists globally for any $\lambda \geq 0$ by the arguments for the case (i). So, it suffices to consider the case $\kappa > 1$, that is, $q^* = \frac{(\kappa+m)(p-1)}{p} > \frac{(m+1)(p-1)}{p}$.

Firstly, consider the case $1 < \kappa < m(p-1)$. Take $\overline{u}$ as that in the proof for (i) with $\beta := \frac{\alpha(\kappa - m(p-1))}{p} < 0$. Clearly, (3.3) and (3.5) are ensured by

$$\begin{cases} m^{p-1}(|f^{m-1}f'|^{p-2}f^{m-1}f')'(\eta) \leq \lambda f^\kappa(\eta), \\ -m^{p-1}(|f^{m-1}f'|^{p-2}f^{m-1}f')(0) \geq f^q(0). \end{cases} \tag{5.1}$$

We claim that (3.5) admits global nonnegative solutions with compact support if and only if $\lambda \geq q$. In fact, we might take

$$f(0) = A, \quad -mf^{m-1}f'(0) \geq A^{\frac{q}{p-1}}.$$

By a simple calculation, we obtain that

$$|f^{m-1}f'|^p(\eta) \geq m^p A^{\kappa+m} - m^{1-p}\frac{\lambda}{q}A^{\kappa+m} + m^{1-p}\frac{\lambda}{q}f^{\kappa+m}.$$

If $\lambda < q$, we have

$$|f^{m-1}f'|^p(\eta) \geq m^p A^{\kappa+m} - m^{1-p}\frac{\lambda}{q}A^{\kappa+m},$$

which means there exists some constant $\eta_0 > 0$ such that $f(\eta_0) = 0$ with $f'(\eta_0) < 0$. If $\lambda \geq q$, let $f := (M - C\eta)_+^r$. Then the above inequalities hold if and only if

$$\begin{cases} m^{p-1}C^p r^{p-1}(rm-1)M^{r[m(p-1)-\kappa]-p} \leq \lambda, \\ (mCr)^{p-1}M^{r[m(p-1)-q]-(p-1)} \geq 1, \end{cases} \tag{5.2}$$

which are true with $r := \frac{p}{m(p-1)-\kappa}$ and $C := (mr)^{-1}$. In addition, due to $\overline{u}(x,0) = (M - \frac{x}{rm})_+^r$, we see, for any $u_0(x)$ compactly supported, there exists some positive constant $M > 0$ such that $\overline{u}(x,0) \geq u_0(x)$, and thus the solution exists globally according to the comparison principle.

Next, treat the case $\kappa \geq m(p-1)$ with $\lambda > q^*$. Define $\overline{u}$ as the one in (3.1) with $\beta := \frac{\alpha[\kappa - m(p-1)]}{p}$. Then $\overline{u}$ is a supersolution if and only if

$$-m^{p-1}(|f^{m-1}f'|^{p-2}f^{m-1}f')(0) \geq f^q(0),$$

$$\mathrm{e}^{-\alpha[m(p-1)-1]t-\beta pt}(\alpha f(\eta)+\beta\eta f'(\eta))+\lambda f^{\kappa}(\eta)\geq m^{p-1}(|f^{m-1}f'|^{p-2}f^{m-1}f')'(\eta),$$
$$\overline{u}(x,0)\geq u_0(x).$$

If $\kappa>m(p-1)$, take $f:=\sigma b^{-r}+(b+\frac{2\eta}{r})^{-r}$ with $r:=\frac{p}{\kappa-m(p-1)}$. Then the first two inequalities are ensured by

$$m^{p-1}2^{p-1}\geq(\sigma+1)^{q-(m-1)(p-1)}\quad\text{and}\quad\lambda\geq q^*(\sigma+1).$$

Noticing that $\lambda>q^*$. Then we can choose some positive constant σ appropriately small such that the above inequalities hold. In addition, we note that for any $u_0(x)$, there exists some constant b small enough such that $\overline{u}(x,0)\geq u_0(x)$. Thus, $\overline{u}$ is a global supersolution by the comparison principle.

If $\kappa=m(p-1)$, take $f:=A\mathrm{e}^{-\eta}$. Then $\overline{u}$ is a supersolution if $\lambda>m^p(p-1)$, with $\overline{u}(x,0)=A\mathrm{e}^{-x}\geq u_0(x)$. Clearly, we have $A\mathrm{e}^{-x}\geq u_0(x)$ whenever constant $A>0$ is large enough, and thus the solution u exists globally for any initial datum.

6. Proof of Theorem 2.1 (iv)

At the last section, we pay attention to the more interesting balanced case $q=q^*$.

Proof of Theorem 2.1 (iv) If $\kappa\leq1$, then $q^*=(m+1)(p-1)/p$, and the solution always exists globally for any $\lambda\geq0$ by the arguments for (i). Hence, it suffices to consider the case $\kappa>1$, that is $q^*=\frac{(\kappa+m)(p-1)}{p}>\frac{(m+1)(p-1)}{p}$. Take $\underline{u}$ as that for (ii) with

$$\beta:=\frac{\alpha(q-m(p-1))}{p-1},\quad\alpha:=\frac{1}{\kappa-1},\quad r:=\frac{p}{m(p-1)-1}.$$

Then (4.1)-(4.2) are equivalent to

$$\begin{cases}\alpha h(\eta)+\beta\eta h'(\eta)+\lambda h^{\kappa}(\eta)\leq m^{p-1}(|h^{m-1}h'|^{p-2}h^{m-1}h')'(\eta),\\ -m^{p-1}(|h^{m-1}h'|^{p-2}h^{m-1}h')(0)\leq h^q(0).\end{cases}\tag{6.1}$$

Assume $\kappa\geq m(p-1)$. By taking $h:=A(M-\eta)_+^r$, (6.1) becomes

$$\begin{aligned}&\alpha\leq\sigma m^{p-1}A^{m(p-1)-1}r^{p-1}(rm-1)(p-1),\\&\lambda\leq(1-\sigma)m^{p-1}A^{m(p-1)-\kappa}r^{p-1}(rm-1)(p-1)M^{-p-r[\kappa-m(p-1)]},\\&m^{p-1}r^{p-1}M^{-r[q-m(p-1)]-(p-1)}\leq A^{q-m(p-1)}.\end{aligned}$$

If $\kappa>m(p-1)$, the last inequality holds with

$$A:=m^{\frac{p-1}{q-m(p-1)}}r^{\frac{p-1}{q-m(p-1)}}M^{-r-\frac{p-1}{q-m(p-1)}},$$

and the second inequality is equivalent to $\lambda\leq(1-\sigma)\frac{(m+1)(p-1)}{mp}$, which holds for some positive constant $\sigma\in(0,1)$ if $\lambda<\frac{(m+1)(p-1)}{mp}$. In addition, for the fixed $\sigma>0$, the first inequality holds for appropriately small constant M. If $\kappa=m(p-1)$, (4.1)-(4.2) are ensured by

$$\begin{aligned}\alpha\leq&\sigma m^{p-1}r^{p-1}(rm-1)(p-1)A^{m(p-1)-1},\\ \lambda\leq&(1-\sigma)m^{p-1}r^{p-1}(rm-1)(p-1)M^{-p},\qquad rm\leq M.\end{aligned}$$

For $M=rm$, appropriately large, the above inequalities hold with $\lambda<\frac{(m+1)(p-1)}{mp}$. So, $\underline{u}$ is a blow-up subsolution if $u_0(x)\geq\underline{u}(x,0)$. However, we note that $\beta=0$ when $\kappa=m(p-1)$, and $\underline{u}(x,0)=T^{-\alpha}A(M-x)_+^r$. By the finite propagation property of disturbances, there must exist some constant t_0 such that $\sup u(\cdot,t_0)>2M$ for any nontrivial initial datum $u_0(x)$. Thus, we can consider this problem for $t\geq t_0$ with a suitable T, and suppose $\underline{u}(x,0)\leq u(x,t_0)$. We have obtained the solution u blows up for suitable initial data when $\kappa\geq m(p-1)$, and the blow-up occurs under any nontrivial nonnegative initial datum if $\kappa=m(p-1)$.

Next, show that if $m(p-1) < \kappa < m(p-1) + p$ with $\lambda < \frac{(m+1)(p-1)}{mp}$, the solution will blow up for any initial data. Let

$$\widehat{u} := (T+t)^{-\alpha} f(\eta), \quad \eta := (T+t)^{-\beta} x,$$

where $\widehat{\alpha} := \frac{1}{\kappa-1}, \widehat{\beta} := \frac{\kappa - m(p-1)}{p(\kappa-1)}$. Then $\widehat{u}$ is a sub-solution of the problem (1.1)-(1.3) if and only if

$$-\widehat{\beta}\eta f'(\eta) + \lambda f^{\kappa}(\eta) \le m^{p-1}(|f^{m-1}f'|^{p-2}f^{m-1}f')'(\eta) + \widehat{\alpha} f(\eta),$$
$$-m^{p-1}(|f^{m-1}f'|^{p-2}f^{m-1}f')(0) \le f^q(0), \qquad \widehat{u}(x,0) \le u_0(x).$$

Let $f(\eta) := \widehat{A}(b^{\frac{p}{p-1}} - \eta^{\frac{p}{p-1}})^{\frac{p-1}{m(p-1)-1}}$. Then the above three inequalities are ensured by

$$\lambda \widehat{A}^{\kappa-1}(b^{\frac{p}{p-1}} - \eta^{\frac{p}{p-1}})^{\frac{(p-1)(\kappa-1)}{m(p-1)-1}+1} + \left[m^{p-1}\widehat{A}^{m(p-1)-1}(\frac{p}{m(p-1)-1})^{p-1} - \widehat{\alpha}\right](b^{\frac{p}{p-1}} - \eta^{\frac{p}{p-1}})$$
$$\le (m^{p-1}\widehat{A}^{m(p-1)-1}(\frac{p}{m(p-1)-1})^p - \widehat{\beta}\frac{p}{m(p-1)-1})\eta^{\frac{p}{p-1}},$$

and

$$\widehat{T}^{-\widehat{\alpha}}\widehat{A}(b^{\frac{p}{p-1}} - (\widehat{T}^{-\widehat{\beta}}x)^{\frac{p}{p-1}})^{\frac{p-1}{m(p-1)-1}} \le u_0(x).$$

Take

$$\widehat{A} := \widehat{\beta}^{\frac{1}{m(p-1)-1}}\left[\frac{m(p-1)-1}{p}\right]^{\frac{p-1}{m(p-1)-1}} m^{\frac{p-1}{m(p-1)-1}}.$$

Then the two inequalities are equivalent to

$$\lambda \widehat{A}^{\kappa-1}(b^{\frac{p}{p-1}} - \eta^{\frac{p}{p-1}})^{\frac{(p-1)(\kappa-1)}{m(p-1)-1}} \le \widehat{\alpha} - \widehat{\beta}.$$

We note that $\kappa < m(p-1)+p$ and $\widehat{\alpha} > \widehat{\beta}$. Then for any given $\widehat{T} > 0$, there exists appropriately small constant b such that the above inequalities hold. For any initial datum, there exists sufficiently small constant b such that $\widehat{u}$ is a subsolution. Recalling the blow-up subsolution $\underline{u}(x,t)$, we note that for $\kappa > m(p-1)$, u blows up if $\underline{u}(x,0) \le u_0(x)$. For appropriately large T, we might take $t^* = T - \widehat{T}$, and noticing that $\alpha = \widehat{\alpha}$ and $\beta = \widehat{\beta}$. Then for appropriately small b, we must have $\underline{u}(x,0) \le \widehat{u}(x,t^*) \le u(x,t^*)$, which means that u blows up before $T+t^*$.

In what follows, we consider the case of $1 < \kappa < m(p-1)$. Let

$$\underline{u}(x,t) := (T-t)^{-\alpha} h(\eta), \quad \eta := (T-t)^{-\beta} x$$

with $\alpha := \frac{1}{\kappa-1}$ and $\beta := -\frac{\alpha(m(p-1)-q)}{p-1}$. Then $\underline{u}(x,t)$ is a subsolution if $\underline{u}(x,0) \le u_0(x)$ and

$$\begin{cases} \alpha h(\eta) + \beta\eta h'(\eta) + \lambda h^{\kappa}(\eta) \le m^{p-1}(|h^{m-1}h'|^{p-2}h^{m-1}h')'(\eta), \\ m^{p-1}(|h^{m-1}h'|^{p-2}h^{m-1}h')(0) \le h^q(0). \end{cases}$$

Take $h := A(C-\eta)_+^r$. Then the above inequalities are transformed into

$$\begin{cases} \alpha(C-\eta)_+ - r\beta\eta + \lambda A^{\kappa-1}(C-\eta)_+^{r(\kappa-1)+1} \\ \quad \le m^{p-1}r^{p-1}A^{m(p-1)-1}(rm-1)(p-1)(C-\eta)_+^{(rm-1)(p-1)-r}, \\ m^{p-1}r^{p-1}A^{m(p-1)-q}C^{r[m(p-1)-q]-(p-1)} \le 1. \end{cases}$$

Noticing $m(p-1) - q = \frac{p-1}{p}(m(p-1)-\kappa)$ and letting

$$r := \frac{p-1}{m(p-1)-1} \text{ and } A := m^{-\frac{p}{m(p-1)-\kappa}} r^{-\frac{p}{m(p-1)-\kappa}} C^{-r+\frac{p}{m(p-1)-\kappa}},$$

we see that the above inequalities are ensured by

$$\lambda C^{r(\kappa-1)+1} \le (1-\sigma)m^{p-1}r^{p-1}A^{m(p-1)-\kappa}(rm-1)(p-1), \tag{6.2}$$

$$\alpha C \le \sigma m^{p-1}r^{p-1}A^{m(p-1)-1}(rm-1)(p-1). \tag{6.3}$$

For $\lambda < 1$, there exists some positive constant $\sigma \in (0,1)$ such that $\lambda \leq 1-\sigma$, and thus (6.2) is true. Furthermore, a direct calculation yields that (6.3) holds for appropriately large constant C. In addition, for any $u_0(x) \neq 0$, we have $\underline{u}(x,0) \leq u_0(x)$ provided that T is large enough, which means that $\underline{u}$ is a blow-up sub-solution according to the comparison principle.

Finally, if $\kappa > m(p-1)+p$, that is, $q > (m+1)(p-1)$, it is known from [6] that the problem (1.1)-(1.3) with $\lambda = 0$ admits global solutions for small initial data, and so does (1.1)-(1.3) with $\lambda > 0$ according to the comparison principle.

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具有非线性吸收项和边界流的m,p-Laplacian抛物方程研究

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摘要: 本文研究了一类在$\mathbb{R}^+ \times \mathbb{R}^+$中的$m,p$-Laplacian抛物方程($p > 2$, $m > 1$), 其具有非线性内部吸收项($-\lambda u^\kappa$)和非线性边界流u^q. 当$q < q^*$时, 任意解都是整体存在的. 当$q > q^*$时, 根据初值的选取, 爆破解和整体解都可能存在. 在临界情形$q = q^*$, 吸收项系数的大小在决定解的整体存在和爆破现象方面发挥决定性作用. 当$\kappa \leq 1$时, 所有解整体存在. 当$1 < \kappa < m(p-1)+p$时, 对于任意初值, 大的λ可以导致解发生有限时刻爆破, 即Fujita爆破, 而小的λ可以导致解整体存在. 而且, 我们给出了系数大小的定量估计. 当$\kappa > m(p-1)+p$时, 爆破解和整体解都是可以存在的.

关键词: m,p-Laplacian抛物方程; Fujita爆破; 整体存在