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Meromorphic Solutions of a Type of Complex Differential-Difference Equations

REN Guozhen(任国珍), GAO Lingyun(高凌云)
(Department of Mathematics, Jinan University, Guangzhou 510632, China)

Abstract: Using Nevanlinna theory of the value distribution of meromorphic functions, we investigate the properties and expression of meromorphic solutions of a type of complex differential-difference equation, and we obtain some results, which are the extensions of complex difference equations to differential-difference equations. Example shows that our results are meaningful.
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1. Introduction

Let $f(z)$ be a meromorphic function in the whole complex plan $\mathbf{C}$. We assume that the reader is familiar with the standard notation and results of the Nevanlinna theory of meromorphic functions[1−2], such as the characteristic function $T(r,f)$, proximity function $m(r,f)$, counting function $N(r,f)$.

The notation $S(r,f)$ denotes any quantity that satisfies the condition:$S(r,f)=o(T(r,f))$ as $r\to\infty$ possibly outside an exceptional set E of r of finite linear measure $\lim\limits_{r\to\infty}\int_{[1,\infty)\cap E}\frac{\mathrm{d}r}{r}<\infty$. A meromorphic function $a(z)$ is called a small function of $f(z)$ if and only if $T(r,a(z))=S(r,f)$.Moreover, we use the notation $\deg_f P$ for the degree of $P(z,f)$ with respect to f and the order of f is defined by

$$\rho(f)=\limsup_{r\to\infty}\frac{\log T(r,f)}{\log r}.$$

Many papers investigate solutions of some types of complex differential equations, and obtain some results[3−8]. Recently, there has been renewed interests in meromorphic solutions of complex difference equations, in addition to the complex differential equation. Many authors have investigated complex difference equations, and obtained some results[9−15].

In 2005, Laine and Rieppo[14] considered a type of complex differential equations of the following form

$$\sum_{\{J\}}\alpha_J(z)\left(\prod_{j\in J}f(z+c_j)\right)=R(z,f(z))=\frac{P(z,f(z))}{Q(z,f(z))},$$

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Biography: REN Guozhen, female, Han, Shanxi, major in complex analysis and differential equation.

where P and Q are relatively prime polynomials in f over the field of rational functions, the coefficients α_J are rational functions and $q = \deg_f Q > 0$. They obtained the following results.

Theorem A[14] Assume $f(z)$ is a transcendental meromorphic solution of the above difference equation, and $f(z)$ has finitely many poles, it must be of the form

$$f(z) = r(z)\mathrm{e}^{g(z)} + s(z),$$

where $r(z)$ and $s(z)$ are rational functions and $g(z)$ is a transcendental entire function satisfying a difference equation of the form either

$$\sum_{j\in J} g(z + c_j) = (j_0 - q)g(z) + d$$

or

$$\sum_{j\in J} g(z + c_j) = \sum_{j\in I} g(z + c_j) + d,$$

here J and I are non-empty disjoint subsets of $\{1, 2, \cdots, n\}$, $c_j \in \mathbf{C} \setminus \{0\}$, $j_0 \in \{0, 1, \cdots, p\}$, $p = \deg_f P$, and $d \in \mathbf{C}$.

2. Some Lemmas

In order to prove our results, we need the following lemmas.

Lemma 2.1[14] Let f be a non-constant meromorphic function and let $P(z, f)$, $Q(z, f)$ be two polynomials in f with meromorphic coefficients small relative to f. If P and Q have no common factors of positive degree in f over the field of small functions relative to f, then

$$\overline{N}\left(r, \frac{1}{Q(z, f)}\right) \le \overline{N}\left(r, \frac{P(z, f)}{Q(z, f)}\right) + S(r, f).$$

Remark 2.1[14] If $f(z)$ is a transcendental meromorphic function and $P(z, f)$, $Q(z, f)$ have rational coefficients, then $P(z, f(z))$ and $Q(z, f(z))$ has only finitely many common zeros.

Lemma 2.2[16] Let $f(z)$ be a meromorphic function and let $\varPhi$ be given by

$$\varPhi = f^n + a_{n-1}f^{n-1} + \cdots + a_0,$$
$$T(r, a_j) = S(r, f), j = 0, \cdots, n - 1.$$

Then either

$$\varPhi \equiv \left(f + \frac{a_{n-1}}{n}\right)^n,$$

or

$$T(r, f) \le \overline{N}(r, \frac{1}{\varPhi}) + \overline{N}(r, f) + S(r, f).$$

Lemma 2.3[1] Suppose that $a_1, a_2, \cdots, a_n, n \ge 2$, are meromorphic functions and let $g_1, g_2, \cdots, g_n$ be entire functions satisfying following conditions

1) $\sum\limits_{j=1}^{n} a_j(z)\mathrm{e}^{g_j(z)} \equiv 0$;

2) $g_j(z) - g_k(z)$ are not constants for $1 \le j < k \le n$;

3) if $1 \le j \le n, 1 \le h < k \le n, T(r, a_j) = o\{T(r, \mathrm{e}^{g_h - g_k})\}(r \to \infty, r \notin E)$.

Then $a_j(z) \equiv 0 (j = 1, 2, \cdots, n)$.

3. Main Results and Proofs

In this paper, letting

$$P(z, f) = a_p(z)f^p + a_{p-1}(z)f^{p-1} + \cdots + a_0(z),$$

$$Q(z,f)=f^q+b_{q-1}(z)f^{q-1}+\cdots+b_0(z),$$

we will investigate some properties of the following complex differential-difference equations:

$$\sum_{\{J\}}\alpha_J(z)\left(\prod_{j\in J}f'(z+c_j)\right)=R(z,f(z))=\frac{P(z,f(z))}{Q(z,f(z))},\tag{3.1}$$

where P,Q are relatively prime polynomials in f over the field of rational functions, Q is a monic polynomial, the coefficients $\{a_i(z)\},\{b_i(z)\}$ are small functions relative to f, $c_j\in\mathbf{C}\setminus\{0\}(j=1,2,\cdots,n)$ are distinct non-zero complex numbers. We denote $p=\deg_f P$ and $q=\deg_f Q$.

We will prove

Theorem 1.1 Suppose that f is a transcendental meromorphic solution of (3.1) and for all $r\geq r_0$, f has infinitely many poles, the coefficients $\alpha_J(z)$ are rational and non-vanishing. Let $C=\max\{|c_1|,\cdots,|c_n|\}$, if P,Q satisfies the condition $p\geq q+2$, then

$$n(r,f')\geq K(p-q)^{\frac{r}{C}}$$

for some constant $K>0$.

Proof Suppose that f, the solution of (3.1), is transcendental meromorphic with infinitely many poles. Moreover, denote $m=p-q>1$.

Case 1) If $q=0$, we rewrite (3.1) as follows

$$\sum_{\{J\}}\alpha_J(z)\left(\prod_{j\in J}f'(z+c_j)\right)=P(z,f(z))=\sum_{j=0}^{p}a_j(z)f(z)^j.\tag{3.2}$$

Choose a pole z_0 of f having multiplicity $\tau\geq 1$ such that z_0 is not a zero of $a_p(z)$. Then the right-hand side of (3.2) has a pole of multiplicity $p\tau$ at z_0. Theorefore, there must exist at least one index $l_1\in\{1,\cdots,n\}$ such that $z_0+c_{l_1}$ is a pole of f of multiplicity $v_1\geq p\tau$.

Substitute $z_0+c_{l_1}$ for z in (3.2) to obtain

$$\sum_{\{J\}}\alpha_J(z_0+c_{l_1})\left(\prod_{j\in J}f'(z_0+c_{l_1}+c_j)\right)=\sum_{j=0}^{p}a_j(z_0+c_{l_1})f(z_0+c_{l_1})^j.\tag{3.3}$$

We will discuss two following possibilities:

(i) If $z_0+c_{l_1}$ is a zero of $a_p(z)$, this process will be terminated, we have to choose another pole z_0 of f and continue the above process.

(ii) If $z_0+c_{l_1}$ is not a zero of $a_p(z)$, then the right-hand side of (3.3) has a pole of multiplicity pv_1 at $z_0+c_{l_1}$. Hence, there exists at least one index $l_2\in\{1,\cdots,n\}$ such that $z_0+c_{l_1}+c_{l_2}$ is a pole of multiplicity $v_2\geq pv_1\geq p^2\tau$. Now we continue inductively to construct poles. Since f has infinitely many poles, we can find a pole z_0 of f such that

$$\zeta_k=z_0+c_{l_1}+\cdots+c_{l_k}$$

is a pole of f of multiplicity v_k for all $k\in\mathbb{N}$. Since $v_k\geq p^k\tau\to\infty$ as $k\to\infty$, and f is a transcendental meromorphic solution of (3.1), we have

$$|\zeta_k|\leq|z_0|+kC\to\infty$$

as $k\to\infty$.

So, for k sufficiently large, say $k\geq k_0$,

$$p^k\tau\leq(1+p+\cdots+p^k)\tau\leq n(|\zeta_k|,f)\leq n(|z_0|+kC,f)\leq n(t+kC,f)\leq n(t+kC,f'),$$

where t may be chosen arbitrarily from the interval $[|z_0|, |z_0| + C)$. Letting $k \to \infty$ for each t, we get for all $r \geq r_0 = (k_0 + 1)C + |z_0|$, there holds

$$n(r, f') \geq Kp^{\frac{r}{C}},$$

where $K = \tau p^{-\frac{|z_0|+C}{C}}$.

Case 2) If $q \geq 1$, now choose a pole z_0 of f of multiplicity $\tau \geq 1$ such that z_0 is neither a zero nor a pole of any of the coefficients of $R(z, f)$, then we can see that the right-hand side of (3.1) has a pole of multiplicity $m\tau$ at z_0. Hence, there exists index $l_1 \in \{1, \cdots, n\}$ such that $z_0 + c_{l_1}$ is a pole of f of multiplicity $v_1 \geq m\tau$.

We proceed to follow the steps (i) and (ii) in Case 1). We can inductively construct poles

$$\zeta_k = z_0 + c_{l_1} + \cdots + c_{l_k}$$

of f of multiplicity $v_k \geq m^k\tau \to \infty$ as $k \to \infty$, and $|\zeta_k| \to \infty$ as $k \to \infty$. According to the above analysis, similarly, for $k \geq k_0$,

$$m^k\tau \leq (1 + m + \cdots + m^k)\tau \leq n(|\zeta_k|, f) \leq n(|z_0| + kC, f) \leq n(t + kC, f) \leq n(t + kC, f'),$$

where $t \in [|z_0|, |z_0| + C)$ may be chosen arbitrarily. Letting $k \to \infty$ for each t, the following assertion as in Case 1). The proof is complete.

Theorem 1.2 Suppose that f is a transcendental meromorphic solution of (3.1) and f has finitely many poles, the coefficients $\alpha_J(z)$ are rational functions and $q > 0$. Then, $f(z)$ must be of the form

$$f(z) = r(z)\mathrm{e}^{g(z)} + s(z),$$

where $r(z)$ and $s(z)$ are rational functions and $g(z)$ is a transcendental entire function satisfying a difference equation of the form either

$$\sum_{j\in J} g(z + c_j) = (j_0 - q)g(z) + d$$

or

$$\sum_{j\in J} g(z + c_j) = \sum_{j\in I\cup U} g(z + c_j) + d,$$

here J, I and U are non-empty disjoint subsets of $\{1, 2, \cdots, n\}$, $j_0 \in \{0, 1, \cdots, p\}$ and $d \in \mathbf{C}$.

Proof Let $f(z)$ is a transcendental meromorphic solution of (3.1), then according to Remark 2.1, we get that $P(z, f(z))$ and $Q(z, f(z))$ has only finitely many common zeros.

By using the assumptions and Lemma 2.1, we obtain

$$\begin{aligned} N\left(r, \frac{1}{Q(z, f(z))}\right) &\leq N\left(r, \frac{P(z, f(z))}{Q(z, f(z))}\right) + O(\log r) \\ &\leq N\left(r, \sum_{J\in\{J\}} \alpha_J(z)\left(\prod_{j\in J} f'(z + c_j)\right)\right) + O(\log r) \\ &\leq N\left(r, \sum_{J\in\{J\}} \alpha_J(z)\left(\prod_{j\in J} f(z + c_j)\right)\right) \\ &\quad + \overline{N}\left(r, \sum_{J\in\{J\}} \alpha_J(z)\left(\prod_{j\in J} f(z + c_j)\right)\right) + O(\log r) \\ &= O(\log r). \end{aligned} \tag{3.4}$$

Applying Lemma 2.2, we yield that either

$$Q(z,f(z)) = (f(z)-s(z))^q,$$

where $s(z)$ is a rational function, or

$$T(r,f) \le \overline{N}\left(r, \frac{1}{Q(z,f)}\right) + \overline{N}(r,f) + S(r,f). \tag{3.5}$$

Combining the inequalities (3.4) and (3.5), we deduce

$$T(r,f) = S(r,f),$$

which is a contradiction with assumption. Therefore, we can assume that

$$Q(z,f(z)) = (f(z)-s(z))^q.$$

According to inequality (3.4), we obtain that $Q(z,f(z))$ has finitely many zeros. Since $f(z)$ is a transcendental meromorphic function with finitely many poles, $(f(z)-s(z))^q$ also has finitely many poles, there exist a rational function $h(z)$ and a non-constant entire function $\overline{g}(z)$ such that

$$f(z)-s(z) = \alpha(h(z))^{\frac{1}{q}} \mathrm{e}^{\frac{\overline{g}(z)}{q}},$$

where α is a qth root of unity. Now denoting that

$$g(z) = \frac{\overline{g}(z)}{q},$$

we observe that

$$r(z) = \alpha(h(z))^{\frac{1}{q}}$$

is a rational function. We can get

$$f(z) = r(z)\mathrm{e}^{g(z)} + s(z). \tag{3.6}$$

Substituting (3.6) into the equation (3.1), we obtain

$$\sum_{\{J\}} \alpha_J(z)\left(\prod_{j\in J}\left(r'(z+c_j)\mathrm{e}^{g(z+c_j)} + r(z+c_j)g'(z+c_j)\mathrm{e}^{g(z+c_j)} + s'(z+c_j)\right)\right)$$
$$= \frac{\sum\limits_{j=0}^{p} \overline{p}_j(z)\mathrm{e}^{jg(z)}}{(r(z)\mathrm{e}^{g(z)})^q} \tag{3.7}$$

i.e.,

$$\alpha_M(z)\left(\prod_{j\in M} r'(z+c_j)\mathrm{e}^{qg(z)+\sum\limits_{j\in M} g(z+c_j)} + \prod_{j\in M} r(z+c_j)g'(z+c_j)\mathrm{e}^{qg(z)+\sum\limits_{j\in M} g(z+c_j)}\right)$$
$$+\sum_{J\in\{U\}} R_J(z)\mathrm{e}^{qg(z)+\sum\limits_{j\in J} g(z+c_j)} + \sum_{J\in\{I\}} H_J(z)\mathrm{e}^{qg(z)+\sum\limits_{j\in J} g(z+c_j)} = \sum_{j=0}^{p} \overline{p}_j(z)\mathrm{e}^{jg(z)}, \tag{3.8}$$

where the cardinality of the set $M \in \{J\}$ is maximal among the sets in the collection $\{J\}$ such that $\alpha_J \ne 0$, $\{U\}$ and $\{I\}$ are two collections of non-empty subsets of $\{1,2,\cdots,n\}$ such that $M \notin \{U\}$, $M \notin \{I\}$. For every J, $R_J(z)$ is a meromorphic function respect to $g(z)$ and $g'(z)$, $H_J(z)$ is a meromorphic function respect to $s'(z)$, $\overline{p}_p(z) \not\equiv 0$.

Further,

$$A_1(z)\mathrm{e}^{qg(z)+\sum\limits_{j\in M} g(z+c_j)} + A_2(z)\mathrm{e}^{qg(z)+\sum\limits_{j\in M} g(z+c_j)} + A_3(z)\mathrm{e}^{qg(z)+\sum\limits_{j\in J} g(z+c_j)}$$
$$+A_4(z)\mathrm{e}^{qg(z)+\sum\limits_{j\in J} g(z+c_j)} + A_5(z)\mathrm{e}^{jg(z)} \equiv 0, \tag{3.9}$$

where

$$A_1(z)=\alpha_M(z)\prod_{j\in M}r'(z+c_j),\quad A_2(z)=\alpha_M(z)\prod_{j\in M}r(z+c_j)g'(z+c_j),$$

$$A_3(z)=\sum_{J\in\{U\}}R_J(z),\quad A_4(z)=\sum_{J\in\{I\}}H_J(z),\quad A_5(z)=-\sum_{j=0}^{p}\overline{p}_j(z).$$

Since $\rho(g(z))=\rho(g'(z))$ and $\rho(g(z))<\rho(\mathrm{e}^{g(z)})$, we get

$$T(r,A_j)=o\left(T\left(r,\mathrm{e}^{\sum\limits_{j\in M}g(z+c_j)-\sum\limits_{j\in J_0}g(z+c_j)}\right)\right);$$

$$T(r,A_j)=o\left(T\left(r,\mathrm{e}^{\sum\limits_{j\in M}g(z+c_j)-\sum\limits_{j\in J_1}g(z+c_j)}\right)\right);$$

$$\vdots$$

$$T(r,A_j)=o\left(T\left(r,\mathrm{e}^{\sum\limits_{j\in J_2}g(z+c_j)+(q-j_0)g(z)}\right)\right),$$

$$j=1,2,3,4,5,$$

where $J_0\in\{I\}, J_1\in\{U\}, J_2\in\{I\}\cup\{M\}$ and $j_0\in\{0,1,\cdots,p\}$. By Lemma 2.3, we see that there must exist at least two exponents in the equation (3.7) such that

$$\sum_{j\in M}g(z+c_j)-\sum_{j\in J_3}g(z+c_j)=d, \tag{3.10}$$

or

$$\sum_{j\in J_4}g(z+c_j)+(q-j_0)g(z)=d, \tag{3.11}$$

where $J_3\in\{I\}\cup\{U\}, J_4\in\{I\}\cup\{U\}\cup\{M\}$, $j_0\in\{0,1,\cdots,p\}$ and $d\in\mathbf{C}$.

We need to prove that $g(z)$ must be transcendental. In fact, suppose that $g(z)$ is a non-constant polynomial, then for every $j\in\{1,\cdots,n\}$, $g(z+c_j)$ can be written as follows

$$g(z+c_j)=g(z)+g_j(z),$$

where $\deg_z g_j<\deg_z g$. Substituting (3.6) into the equation (3.1), we obtain

$$\left(\sum_{\{J\}}\alpha_J(z)\prod_{j\in J}\left([r'(z+c_j)+r(z+c_j)(g'(z)+g_j'(z))]\mathrm{e}^{g(z)}\mathrm{e}^{g_j(z)}+s'(z+c_j)\right)\right)(r(z)\mathrm{e}^{g(z)})^q$$

$$=\sum_{j=0}^{p}a_j(z)(r(z)\mathrm{e}^{g(z)}+s(z))^j, \tag{3.12}$$

where the rational functions a_j are the coefficients of the polynomial $P(z,f)$.

Since the polynomials P and Q are relatively prime, we can see that P and Q have no common factor of positive degree in f and $\sum\limits_{j=0}^{p}a_j(z)s(z)^j\not\equiv 0$. Therefore, we obtain a non-trivial algebraic equation for $\mathrm{e}^{g(z)}$ with coefficients small relative to $\mathrm{e}^{g(z)}$, which is a contradiction. The proof is complete.

Example 1.2 $f(z)=\frac{1}{z}\mathrm{e}^{\sin z}+\frac{1}{z-\pi}$ is a transcendental meromorphic solutions of the following complex differential-difference equation

$$f'(z-\pi)-\frac{z+\pi}{z-\pi}f'(z+\pi)$$

$$=\frac{2\pi(3\pi^2z^3-z^5-2\pi^4z)(f-\frac{1}{z-\pi})-2\pi z^2(z-2\pi)^2}{z^3(z-\pi)^2(z+\pi)(z-2\pi)^2(f-\frac{1}{z-\pi})}.$$

In this case, $q=1$, $c_1=-\pi$, $c_2=\pi$. $g(z)=\sin z$ is a transcendental entire function satisfying a difference equation of the form

$$g(z+\pi)-g(z-\pi)=\sin(z+\pi)-\sin(z-\pi)=0.$$

Remark 1.2 Example 1.2 shows that the result in Theorem 1.2 holds.

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一类复微分-差分方程的亚纯解

任国珍, 高凌云

(暨南大学数学系, 广东 广州 510632)

摘要: 本文研究一类复微分-差分方程的亚纯解的性质和表达式问题, 利用亚纯函数的Nevanlinna值分布理论来证明, 并得到一些结论, 所得结论是从复差分方程到复微分差分方程的推广. 例子表明我们的结果是有意义的.

关键词: 复微分-差分方程; 亚纯解; 值分布理论