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鲁棒凸优化问题拟近似解的刻划

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摘要：本文研究鲁棒凸优化问题拟近似解的最优性条件和对偶理论. 首先利用鲁棒优化方法, 在由约束函数的共轭函数的上图给出的闭凸锥约束规格条件下, 建立了拟近似解的最优性充要条件. 其次给出了鲁棒凸优化问题拟近似解在Wolf型和Mond-weir型对偶模型下的强(弱)对偶定理. 最后给出具体实例验证了本文获得的结果.

关键词：鲁棒凸优化; 次微分; 拟近似解; 最优性条件; 对偶理论
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1. 引言

令X,Y,Z为实Banach空间, $\mathbb{V}\subset Z$为不确定的紧凸集, $C\subset X$为闭凸集, $D\subset Y$为非空闭凸锥, 其对偶锥记为D^*. 设$f:X\to\mathbb{R}$为连续的凸函数, 向量值函数$g:X\to Y$称为D-凸函数, 也即: 如果对任意的$x,y\in X$, $t\in[0,1]$, 满足

$$g(tx+(1-t)y)-tg(x)-(1-t)g(y)\in -D.$$

若$-g$为D-凸函数, 则称g为D-凹函数, 考虑如下凸优化问题:

$$\text{(P)}\qquad \begin{cases}\inf & f(x)\\ \text{s.t.} & x\in C, -g(x)\in D.\end{cases}$$

问题(P)的约束条件中含有不确定数据, 也即$g:X\times Z\to Y$为连续的D-凸凹函数(任意的$v\in\mathbb{V}, g(\cdot,v)$为$D$-凸函数, 对任意的$x\in X, g(x,\cdot)$为$D$-凹函数), 此时问题(P)变为如下的不确定性优化问题[1]:

$$\text{(UP)}\qquad \begin{cases}\inf & f(x)\\ \text{s.t.} & x\in C, -g(x,v)\in D.\end{cases}$$

不确定性优化问题是优化问题的一个重要分支, 目前处理不确定性优化问题的方法很多, 其中鲁棒优化是一种较为常用的办法. 鲁棒优化问题[1]致力于保证最坏的解不受不确定性数据对该优化问题的干扰. 针对不确定性凸优化问题(UP), 其对应的鲁棒凸优化问题描述如下:

$$\text{(RUP)}\qquad \begin{cases}\inf & f(x)\\ \text{s.t.} & x\in C, -g(x,v)\in D, \forall\ v\in\mathbb{V},\end{cases}$$

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其中$v \in \mathbb{V}$为不确定参数, (RUP)的可行集和最优解集分别记为

$$\mathscr{F} := \{x \in C : -g(x, v) \in D, \forall\ v \in \mathbb{V}\}$$

和

$$S := \{x \in \mathscr{F} : f(x) \le f(y), \forall y \in \mathscr{F}\}.$$

近些年来, 问题(RUP)引起了国内外学者的广泛关注, 文[1-3]对鲁棒优化问题精确解的最优性条件和对偶理论进行了研究. 但众所周知, 许多实际优化问题的精确解并不存在, 而且在算法设计中大多数得到的也是近似解, 因此研究优化问题的近似解很有必要, 故对问题(RUP)近似解的研究十分有意义. 目前, 一些学者已经致力于问题(RUP)近似解的研究, JIAO和Lee[4]针对半无限鲁棒凸优化问题的近似解建立了相应的最优性条件和对偶定理, Son[5]等针对无限维约束的非凸规划问题建立了近似解的最优性条件和对偶, Lee和Lee[6]刻划了含有不确定数据的凸优化问题的近似解, Dutta[7]等给出了优化问题近似KKT点的终止原则, Chuong和Kim[8]给出了非光滑多目标优化问题帕累托近似解的Fritz-John最优性条件, Lee和JIAO[9]针对不确定数据的鲁棒凸优化问题, 建立了拟近似解的最优性条件和对偶理论(拟近似解的概念见第2节). 本文是对问题(RUP)的进一步研究, 我们将致力于通过最优性条件和对偶刻划鲁棒凸优化问题的拟近似解.

文章内容安排如下: 第2节, 给出本文用到的记号、基本概念和引理等; 第3节,在文[13]给出的一种闭凸锥约束规格条件下, 建立鲁棒凸优化问题关于拟近似解的最优性条件; 最后, 在Wolf型和Mond-Weir型对偶模型下建立拟近似解的对偶理论, 并举例说明所获得的结果.

2.预备知识

实Banach空间X的对偶空间记为X^*. 设$A \subset X$, A的闭包和凸包分别记为$\mathrm{cl}(A), \mathrm{conv}(A)$. 对任意的$a_1, a_2 \in A$, $t \in [0,1]$, 若$ta_1 + (1-t)a_2 \in A$, 则称集合A是凸集. 扩充实值函数$f : X \to \mathbb{R} \cup \{+\infty\}$的有效域和上图分别定义为

$$\mathrm{dom} f := \{x \in X : f(x) < +\infty\}$$

和

$$\mathrm{eip} f := \{(x, r) \in X \times \mathbb{R} : f(x) \le r\}.$$

扩充实值函数$f : X \to \mathbb{R} \cup \{+\infty\}$, 如果对于$t \in [0,1]$, $x, y \in X$有$f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$成立, 称函数$f$是凸函数. 若$-f$是凸函数, 则称$f$是凹函数. f是凸函数等价于$\mathrm{epi} f$是凸集. 扩充实值函数$f : X \to \mathbb{R} \cup \{+\infty\}$在$x \in X$处的次微分定义为

$$\partial f(x) := \begin{cases} \{x^* \in X : \langle x^*, y - x\rangle \le f(y) - f(x), \forall y \in X\}, & x \in \mathrm{dom} f, \\ \varnothing, & \text{否则}. \end{cases}$$

对于$\varepsilon \ge 0$, 扩充实值函数$f : X \to \mathbb{R} \cup \{+\infty\}$在$x \in X$处的$\epsilon$-次微分定义为

$$\partial_\epsilon f(x) := \begin{cases} \{x^* \in X : \langle x^*, y - x\rangle \le f(y) - f(x) + \epsilon, \forall y \in X\}, & x \in \mathrm{dom} f, \\ \varnothing, & \text{否则}. \end{cases}$$

若对于所有的$x \in X$, 有$\mathrm{liminf}_{y \to x} f(y) \ge f(x)$成立, 则称函数$f$是下半连续函数. 对于凸函数$f : X \to \mathbb{R} \cup \{+\infty\}$, 其共轭函数$f^* : X \to \mathbb{R} \cup \{+\infty\}$定义为

$$f^*(x^*) := \sup\{\langle x^*, x\rangle - f(x) : x \in X\}, x^* \in X^*.$$

下面给出问题(RUP)拟近似解的定义和相关的一些引理.

定义2.1[4]　对于给定$\varepsilon \ge 0$, $\overline{x} \in \mathscr{F}$, 称$\overline{x}$是问题(RUP)的拟$\epsilon$-解, 若满足$f(\overline{x}) \le f(x) + \sqrt{\epsilon}\|x - \overline{x}\|, \forall x \in \mathscr{F}$.

注2.1[10]　若$\overline{x}$是问题(RUP)的拟ϵ-解, 且扩充实值函数$f : X \to \mathbb{R} \cup \{+\infty\}$在有效域内可微, 则$\|\nabla f(\overline{x})\| \le \sqrt{\epsilon}$.

约束品行(Constraint Qualification)是刻划数学规划问题最优性的重要条件, 本文采用如下约束品行.

定义2.2[13] 对于问题(RUP), 若$\forall x\in\mathscr{F}$, $\bigcup\limits_{v\in\mathbb{V},\lambda\in D^*}\mathrm{epi}(\lambda g(x,v))^*$是闭凸集, 则称$\bigcup\limits_{v\in\mathbb{V},\lambda\in D^*}\mathrm{epi}(\lambda g(x,v))^*$闭凸锥约束品行在$x\in\mathscr{F}$ 处成立, 记为(CQ)成立.

引理2.1[11] 设$f:X\to\mathbb{R}\cup\{+\infty\}$是下半连续凸函数, $\overline{x}\in\mathrm{dom}f$, 则

$$\mathrm{epi}f^*=\bigcup_{\epsilon\geq 0}\{(a,\langle a,\overline{x}\rangle+\epsilon-f(\overline{x}):a\in\partial_\epsilon f(\overline{x})\}.$$

引理2.2[12] 令$f,g:X\to\mathbb{R}\cup\{+\infty\}$是下半连续凸函数, 且$\mathrm{dom}f\cap\mathrm{dom}g\neq\varnothing$, 则有

$$\mathrm{epi}(f+g)^*=\mathrm{cl}(\mathrm{epi}f^*+\mathrm{epi}g^*).$$

若函数f,g中至少有一个是连续函数，则

$$\mathrm{epi}(f+g)^*=\mathrm{epi}f^*+\mathrm{epi}g^*.$$

引理2.3[13] 设$g:X\times Z\to Y$是连续的D-凸凹函数, 则

$$\bigcup_{v\in\mathbb{V},\lambda\in D^*}\mathrm{epi}(\lambda g(x,v))^*$$

是凸集.

引理2.4[13] $g:X\times Z\to Y$是连续的D-凸凹函数, 如果存在$x\in X,v\in\mathbb{V}$使得$g(x,v)<0$, 则

$$\bigcup_{v\in\mathbb{V},\lambda\in D^*}\mathrm{epi}(\lambda g(x,v))^*$$

是闭集.

3. 最优性条件

本节建立拟近似解的最优性条件, 首先给出如下引理.

引理3.1[14] 设$f:X\to\mathbb{R}$是凸函数, $g:X\times Z\to Y$为连续的D-凸凹函数, 若$\mathbb{V}\subset Z$且$\mathscr{F}\neq\varnothing$. 则下述结论等价:

1) $-g(x,v)\in D,x\in C,v\in\mathbb{V}\Rightarrow f(x)\geq 0$;

2) $(0,0)\in\mathrm{epi}f^*+\mathrm{cl}(\mathrm{conv}\bigcup\limits_{v\in\mathbb{V},\lambda\in D^*}\mathrm{epi}(\lambda g(x,v))^*)$.

定理3.1 设$f:X\to\mathbb{R}$是连续凸函数, $g:X\times Z\to Y$是连续函数, 且对于任意的$v\in V,g(\cdot,v)$是D-凸函数, 假设(CQ)成立, 则下述结论等价:

1) $\overline{x}$是问题(RUP)的拟ϵ-解;

2) 存在$\overline{v}\in\mathbb{V},\overline{\lambda}\in D^*$, 使得$(0,-\sqrt{\epsilon}\|\overline{x}\|-f(\overline{x}))\in\mathrm{epi}f^*+\mathrm{epi}(\overline{\lambda}g(x,\overline{v}))^*+\sqrt{\epsilon}\mathbb{B}\times\mathbb{R}_+$, $\mathbb{B}$表示单位闭球.

证 由1)$\Rightarrow$ 2)

设$\overline{x}$是问题(RUP)的拟ϵ-解, 则对于任意的$x\in\mathscr{F}$, $f(x)+\sqrt{\epsilon}\|x-\overline{x}\|\geq f(\overline{x})$. 所以有$\mathscr{F}\subseteq\{x\in C:f(x)+\sqrt{\epsilon}\|x-\overline{x}\|-f(\overline{x})\geq 0\}$. 令$\phi(x)=f(x)+\sqrt{\epsilon}\|x-\overline{x}\|-f(\overline{x})$. 由引理3.1有

$$(0,0)\in\mathrm{epi}\phi^*+\mathrm{cl}(\mathrm{conv}\bigcup_{v\in\mathbb{V},\lambda\in D^*}\mathrm{epi}(\lambda g(x,v))^*).$$

由假设条件,

$$(0,0)\in\mathrm{epi}\phi^*+\bigcup_{v\in\mathbb{V},\lambda\in D^*}\mathrm{epi}(\lambda g(x,v))^*.$$

所以存在$\overline{\lambda}\in D^*,\overline{v}\in\mathbb{V}$使得

$$(0,0)\in\mathrm{epi}\phi^*+\mathrm{epi}(\overline{\lambda}g(x,\overline{v}))^*.\tag{3.1}$$

下面证明

$$\mathrm{epi}\phi^* = \mathrm{epi}f^* + \sqrt{\epsilon}\mathbb{B} \times [\sqrt{\epsilon}\|\overline{x}\| + f(\overline{x}), +\infty).$$

由引理2.2知,

$$\begin{aligned}\mathrm{epi}\phi^* &= \mathrm{epi}(f(\cdot) + \sqrt{\epsilon} \times \|\cdot - \overline{x}\| - f(\overline{x}))^* \\ &= \mathrm{epi}f^* + \mathrm{epi}(\sqrt{\epsilon} \times \|\cdot - \overline{x}\| - f(\overline{x}))^*.\end{aligned} \tag{3.2}$$

因为

$$(\sqrt{\epsilon} \times \|\cdot - \overline{x}\| - f(\overline{x}))^* = \begin{cases} \sqrt{\epsilon}\|\overline{x}\| + f(\overline{x}), \text{如果}\|a\| \leq \sqrt{\epsilon}, \\ +\infty, \qquad\qquad \text{否则}. \end{cases}$$

由(3.2)知

$$\mathrm{epi}\phi^* = \mathrm{epi}f^* + \sqrt{\epsilon}\mathbb{B} \times [\sqrt{\epsilon}\|\overline{x}\| + f(\overline{x}), +\infty).$$

因此, 由(3.1)可得存在$\overline{v} \in \mathbb{V}, \overline{\lambda} \in D^*$, 使得

$$(0, -\sqrt{\epsilon}\|\overline{x}\| - f(\overline{x})) \in \mathrm{epi}f^* + \mathrm{epi}(\overline{\lambda}g(x,\overline{v}))^* + \sqrt{\epsilon}\mathbb{B} \times \mathbb{R}_+.$$

由2) $\Rightarrow$ 1)

由2)知存在$\overline{v} \in \mathbb{V}, \overline{\lambda} \in D^*$, 使得

$$(0, -\sqrt{\epsilon}\|\overline{x}\| - f(\overline{x})) \in \mathrm{epi}f^* + \mathrm{epi}(\overline{\lambda}g(x,\overline{v}))^* + \sqrt{\epsilon}\mathbb{B} \times \mathbb{R}_+.$$

则存在$u \in X, \alpha \geq 0, \omega \in X, \beta \geq 0, b \in \mathbb{B}, r \in \mathbb{R}_+$使得

$$(0, -\sqrt{\epsilon}\|\overline{x}\| - f(\overline{x})) = (u, f^*(u) + \alpha) + \overline{\lambda}(\omega, g^*(\omega,\overline{v}) + \beta) + (\sqrt{\epsilon}b, r).$$

所以有

$$0 = u + \overline{\lambda}\omega + \sqrt{\epsilon}b$$

和

$$-\sqrt{\epsilon}\|\overline{x}\| - f(\overline{x}) = f^*(u) + \alpha + \overline{\lambda}(g^*(\omega,\overline{v}) + \beta) + r.$$

故对任意的$x \in X$,

$$\begin{aligned}-\langle\overline{\lambda}\omega, x\rangle - \langle\sqrt{\epsilon}b, x\rangle - f(x) &= \langle u, x\rangle - f(x) \leq f^*(u) \\ &= -\sqrt{\epsilon}\|\overline{x}\| - f(\overline{x}) - \alpha - \overline{\lambda}(g^*(\omega,\overline{v}) + \beta) - r.\end{aligned}$$

因此对任意的$x \in X$, 有

$$\begin{aligned}f(\overline{x}) \leq &\langle\overline{\lambda}\omega, x\rangle + \langle\sqrt{\epsilon}b, x\rangle - \sqrt{\epsilon}\|\overline{x}\| + f(x) - \alpha - \overline{\lambda}g^*(\omega,\overline{v}) - \overline{\lambda}\beta - r \\ \leq &\langle\overline{\lambda}\omega, x\rangle + \langle\sqrt{\epsilon}b, x\rangle - \sqrt{\epsilon}\|\overline{x}\| + f(x) - \overline{\lambda}g^*(\omega,\overline{v}) \\ \leq &\langle\overline{\lambda}\omega, x\rangle + \sqrt{\epsilon}\|x - \overline{x}\| + f(x) - \overline{\lambda}g^*(\omega,\overline{v}) \\ \leq &f(x) + \overline{\lambda}g(\omega,\overline{v}) + \sqrt{\epsilon}\|x - \overline{x}\|.\end{aligned}$$

又$\overline{\lambda}g(\omega,\overline{v}) \leq 0$, 故$f(\overline{x}) \leq f(x) + \sqrt{\epsilon}\|x - \overline{x}\|$. 所以$\overline{x}$是问题(RUP)的拟$\epsilon$-解. 证毕.

由引理3.1和定理3.1可得问题(RUP)的拟ϵ-解的最优性条件.

定理3.2　设$f: X \to \mathbb{R}$是连续凸函数, $g: X \times Z \to Y$是连续函数，且对于任意的$v \in V, g(\cdot, v)$是D-凸函数, 假设(CQ)成立, 则下述结论等价:

1) $\overline{x}$是问题(RUP)的拟ϵ-解;

2) 存在$\overline{v} \in \mathbb{V}, \overline{\lambda} \in D^*$, 使得

$$0 \in \partial f(\overline{x}) + \partial((\overline{\lambda}g)(\cdot,\overline{v}))(\overline{x}) + \sqrt{\epsilon}\mathbb{B}, \tag{3.3}$$

$$(\overline{\lambda}g)(\overline{x},\overline{v}) = 0. \tag{3.4}$$

证　由1) $\Leftrightarrow$ 2)

设$\overline{x}$是问题(RUP)的拟ϵ-解, 由定理3.1知存在$\overline{v}\in\mathbb{V},\overline{\lambda}\in D^*$, 使得

$$(0,-\sqrt{\epsilon}\|\overline{x}\|-f(\overline{x}))\in \mathrm{epi}f^*+\mathrm{epi}(\overline{\lambda}g(x,\overline{v}))^*+\sqrt{\epsilon}\mathbb{B}\times\mathbb{R}_+.$$

由引理2.1, 上式等价于存在$\overline{v}\in\mathbb{V},\overline{\lambda}\in D^*,\epsilon_0\geq 0,\ \epsilon\geq 0$, 使得

$$\begin{aligned}(0,-\sqrt{\epsilon}\|\overline{x}\|-f(\overline{x}))\in &\bigcup_{\epsilon_0\geq 0}\{(\xi_0,\langle\xi_0,\overline{x}\rangle+\epsilon_0-f(\overline{x})):\xi_0\in\partial_{\epsilon_0}f(\overline{x})\}\\ &+\{(\xi,\langle\xi,\overline{x}\rangle+\epsilon-\overline{\lambda}g(\overline{x},\overline{v})):\xi\in\partial_{\epsilon}(\overline{\lambda}g)(\cdot,\overline{v})(\overline{x})\}+\sqrt{\epsilon}\mathbb{B}\times\mathbb{R}_+.\end{aligned}$$

即存在$\overline{v}\in\mathbb{V},\overline{\lambda}\in D^*,\overline{\xi_0}\in\partial_{\overline{\epsilon_0}}f(\overline{x}),\overline{\xi}\in\partial_{\overline{\epsilon}}(\overline{\lambda}g)(\cdot,\overline{v})(\overline{x}),b\in\mathbb{B},r\in\mathbb{R}_+,\overline{\epsilon}\geq 0$使得

$$(0,-\sqrt{\epsilon}\|\overline{x}\|-f(\overline{x}))=(\overline{\xi_0},\langle\overline{\xi_0},\overline{x}\rangle+\overline{\epsilon_0}-f(\overline{x}))+(\overline{\xi},\langle\overline{\xi},\overline{x}\rangle+\overline{\epsilon}-(\overline{\lambda}g)(\overline{x},\overline{v}))+(\sqrt{\epsilon}b,r).$$

上式等价于存在$\overline{v}\in\mathbb{V},\overline{\lambda}\in D^*,\overline{\xi_0}\in\partial_{\overline{\epsilon_0}}f(\overline{x}),\overline{\xi}\in\partial_{\overline{\epsilon}}(\overline{\lambda}g)(\cdot,\overline{v})(\overline{x}),b\in\mathbb{B},r\in\mathbb{R}_+,\overline{\epsilon}\geq 0$使得

$$0=\overline{\xi_0}+\overline{\xi}+\sqrt{\epsilon}b$$

和

$$-\sqrt{\epsilon}\|\overline{x}\|-f(\overline{x})=\langle\overline{\xi_0},\overline{x}\rangle+\overline{\epsilon_0}-f(\overline{x})+\langle\overline{\xi},\overline{x}\rangle+\overline{\epsilon}-(\overline{\lambda}g)(\overline{x},\overline{v})+r$$

成立. 即

$$\overline{\lambda}g(\overline{x},\overline{v})=\overline{\epsilon_0}+\overline{\epsilon}+\langle\overline{\xi_0}+\overline{\xi},\overline{x}\rangle+\sqrt{\epsilon}\|\overline{x}\|+r.$$

故有

$$0\leq\overline{\epsilon_0}+\overline{\epsilon}\leq\overline{\epsilon_0}+\overline{\epsilon}-\sqrt{\epsilon}\langle b,\overline{x}\rangle+\sqrt{\epsilon}\|\overline{x}\|+r=(\overline{\lambda}g)(\overline{x},\overline{v})\leq 0$$

成立. 故存在$\overline{v}\in\mathbb{V},\overline{\lambda}\in D^*$, 使得

$$0\in\partial f(\overline{x})+\partial((\overline{\lambda}g)(\cdot,\overline{v}))(\overline{x})+\sqrt{\epsilon}\mathbb{B},\quad(\overline{\lambda}g)(\overline{x},\overline{v})=0.$$

证毕.

4. 对偶定理

本节将先引入问题(RUP)的Wolfe型鲁棒对偶问题(WD)和Mond-Weir型鲁棒对偶问题(MWD), 之后分别建立对应的弱对偶定理和强对偶定理. Wolfe型鲁棒对偶问题(WD)如下

$$(\mathrm{WD})\qquad\begin{cases}\max & f(y)+(\lambda g)(y,v)\\ \mathrm{s.t.} & 0\in\partial f(y)+\partial((\lambda g)(\cdot,v))(y)+\sqrt{\epsilon}\mathbb{B},\\ & \lambda\in D^*,y\in X,v\in\mathbb{V},\epsilon\geq 0.\end{cases}$$

定理4.1(近似弱对偶定理) 给定问题(RUP)的任意可行解x以及对偶问题(WD)的任意可行解(y,v,λ),

$$f(x)\geq f(y)+(\lambda g)(y,v)-\sqrt{\epsilon}\|x-y\|$$

总成立.

证 设x和(y,v,λ)分别是问题(RUP)和对偶问题(WD)的可行解. 则存在$\overline{\xi_0}\in\partial f(y),\overline{\xi}\in\partial((\lambda g)(\cdot,v))(y),b\in\mathbb{B}$使得$\overline{\xi_0}+\overline{\xi}+\sqrt{\epsilon}b=0$. 所以有

$$\begin{aligned}f(x)-(f(y)+(\lambda g)(y,v))&\geq\langle\overline{\xi_0},x-y\rangle-(\lambda g)(y,v)\\ &=-\langle\xi+\sqrt{\epsilon}b,x-y\rangle-\lambda g(y,v)\\ &\geq-((\lambda g)(x,v)-(\lambda g)(y,v))-\langle\sqrt{\epsilon}b,x-y\rangle-(\lambda g)(y,v)\\ &\geq-(\lambda g)(x,v)-\sqrt{\epsilon}\|b\|\cdot\|x-y\|\\ &\geq-\sqrt{\epsilon}\|x-y\|.\end{aligned}$$

因此$f(x)\geq f(y)+(\lambda g)(y,v)-\sqrt{\epsilon}\|x-y\|$. 证毕.

定理4.2(近似强对偶定理)　设$f:X\to\mathbb{R}$是连续凸函数, $g:X\times Z\to Y$是连续函数, 且对于任意的$v\in\mathbb{V}, g(\cdot,v)$是$D$-凸函数, 假设(CQ)成立, 如果$\overline{x}$是问题(RUP)的拟$\epsilon$-解, 则存在$\overline{v}\in\mathbb{V},\overline{\lambda}\in D^*$使得$(\overline{x},\overline{v},\overline{\lambda})$是对偶问题(WD)的拟$\epsilon$-解.

证　设$\overline{x}\in\mathscr{F}$是问题(RUP)的拟$\epsilon$-解. 由定理3.2知, 存在$\overline{v}\in\mathbb{V},\overline{\lambda}\in D^*$使得

$$0\in\partial f(\overline{x})+\partial((\overline{\lambda}g)(\cdot,\overline{v}))(\overline{x})+\sqrt{\epsilon}\mathbb{B}, (\overline{\lambda}g)(\overline{x},\overline{v})=0.$$

这说明$(\overline{x},\overline{v},\overline{\lambda})$是对偶问题(WD)的可行解, 根据定理4.1, 对对偶问题(WD)的任一可行解(y,v,λ)有

$$f(\overline{x})+(\overline{\lambda}g)(\overline{x},\overline{v})-\{f(y)+(\lambda g)(y,v)\}\geq-\sqrt{\epsilon}\|\overline{x}-y\|+(\overline{\lambda}g)(\overline{x},\overline{v})\geq-\sqrt{\epsilon}\|\overline{x}-y\|.$$

故$(\overline{x},\overline{v},\overline{\lambda})$是对偶问题(WD)的拟$\epsilon$-解.

下面给出Mond-Weir型鲁棒对偶问题(MWD)

$$\text{(MWD)}\qquad\begin{cases}\max & f(y)\\ \text{s.t.} & 0\in\partial f(y)+\partial((\lambda g)(\cdot,v))(y)+\sqrt{\epsilon}\mathbb{B},\\ & (\lambda g)(y,v)\geq0,\lambda\in D^*,y\in X,v\in\mathbb{V},\epsilon\geq0.\end{cases}$$

定理4.3(近似弱对偶定理)　给定问题(RUP)的任意可行解x以及对偶问题(MWD)的任意可行解(y,v,λ),

$$f(x)\geq f(y)-\sqrt{\epsilon}\|x-y\|$$

总成立.

证　设x和(y,v,λ)分别是问题(RUP)和对偶问题(MWD)的可行解. 则存在$\overline{\xi_0}\in\partial f(y),\overline{\xi}\in\partial((\lambda g)(\cdot,v))(y), b\in\mathbb{B}$使得$\overline{\xi_0}+\overline{\xi}+\sqrt{\epsilon}b=0$. 所以有

$$\begin{aligned}f(x)-f(y)&\geq\langle\overline{\xi_0},x-y\rangle=-\langle\xi+\sqrt{\epsilon}b,x-y\rangle\\&\geq-((\lambda g)(x,v)-(\lambda g)(y,v))-\langle\sqrt{\epsilon}b,x-y\rangle\\&\geq-(\lambda g)(x,v)-\sqrt{\epsilon}\|b\|\cdot\|x-y\|+(\lambda g)(y,v)\\&\geq-\sqrt{\epsilon}\|x-y\|.\end{aligned}$$

因此$f(x)\geq f(y)-\sqrt{\epsilon}\|x-y\|$. 证毕.

定理4.4(近似强对偶定理)　设$f:X\to\mathbb{R}$是连续凸函数, $g:X\times Z\to Y$是连续函数, 且对于任意的$v\in\mathbb{V}, g(\cdot,v)$是$D$-凸函数, 假设(CQ)成立, 如果$\overline{x}$是问题(RUP)的拟$\epsilon$-解, 则存在$\overline{v}\in\mathbb{V},\overline{\lambda}\in D^*$使得$(\overline{x},\overline{v},\overline{\lambda})$是对偶问题(MWD)的拟$\epsilon$-解.

证　设$\overline{x}\in\mathscr{F}$是问题(RUP)的拟$\epsilon$-解. 由定理3.2知, 存在$\overline{v}\in\mathbb{V},\overline{\lambda}\in D^*$使得

$$0\in\partial f(\overline{x})+\partial((\overline{\lambda}g)(\cdot,\overline{v}))(\overline{x})+\sqrt{\epsilon}\mathbb{B},\quad(\overline{\lambda}g)(\overline{x},\overline{v})=0.$$

这说明$(\overline{x},\overline{v},\overline{\lambda})$是对偶问题(MWD)的可行解, 根据定理4.3, 对偶问题(MWD)的任一可行解(y,v,λ)有

$$f(\overline{x})-f(y)\geq-\sqrt{\epsilon}\|\overline{x}-y\|.$$

故$(\overline{x},\overline{v},\overline{\lambda})$是对偶问题(MWD)的拟$\epsilon$-解.证毕.

下面给出一个例子说明Wolfe型鲁棒对偶问题(WD)和Mond-weir型鲁棒对偶问题(MWD)的近似对偶理论.

例4.1　考虑信息不确定下的鲁棒凸优化问题

$$\text{(RUP)}\qquad\begin{cases}\min & x_1^2+x_2^2\\ \text{s.t.} & x_1^2-3v_1x_1\leq0, v_1\in[-1,1].\end{cases}$$

解　设$f(x_1,x_2)=x_1^2+x_2^2, g_1((x_1,x_2),v_1)=x_1^2-3v_1x_1$. 问题(RUP)的可行集

$$\mathscr{F}:=\{(x_1,x_2)\in\mathbb{R}^2:x_1^2-3v_1x_1\leq0,v_1\in[-1,1]\}=\{(x_1,x_2)\in\mathbb{R}^2:x_1=0,x_2\in\mathbb{R}\}.$$

问题(RUP)的拟ϵ-解集

$$S:=\{(x_1,x_2)\in\mathbb{R}^2:x_1=0,-\frac{\sqrt{\epsilon}}{2}\le x_2\le\frac{\sqrt{\epsilon}}{2}\}.$$

当$\lambda_1\ge 0, v_1\in V_1$,

$$(\lambda_1 g_1(\cdot,v_1))^*(a)=\begin{cases}0, & \text{如果}\lambda_1=0,\\ \dfrac{(a+3v_1\lambda_1)^2}{4\lambda_1}, & \text{如果}\lambda_1>0.\end{cases}$$

$$\bigcup_{\lambda_1\ge 0,v_1\in V_1}\mathrm{epi}(\lambda_1 g_1(\cdot,v_1))^*=\bigcup_{\lambda_1>0,v_1\in[-1,1]}\{(a,r):r\ge\frac{(a+3v_1\lambda_1)^2}{4\lambda_1}\}\cup\{0\}\times\mathbb{R}_+=\mathbb{R}\times\mathbb{R}_+.$$

所以锥

$$\bigcup_{\lambda_1\ge 0,v_1\in V_1}\mathrm{epi}(\lambda_1 g_1(\cdot,v_1))^*=\mathbb{R}\times\mathbb{R}_+$$

是闭凸集.

问题(RUP)的Wolf型对偶问题(WD)如下

$$(\mathrm{WD})\qquad\begin{cases}\max & f(y_1,y_2)+\lambda_1 g_1((y_1,y_2),v_1)\\ \text{s.t.} & 0\in\partial f(y_1,y_2)+\partial(\lambda_1 g_1(\cdot,v_1))(y_1,y_2)+\sqrt{\epsilon}\mathbb{B},\\ & \lambda_1\ge 0, v_1\in[-1,1],\epsilon\ge 0.\end{cases}$$

对偶问题(WD)的可行集

$$\begin{aligned}\mathscr{F}_{WD}:=&\{((y_1,y_2),v_1,\lambda_1):0\in\partial f(y_1,y_2)+\partial(\lambda_1 g_1((y_1,y_2),v_1))\\ &+\sqrt{\epsilon}\mathbb{B},\lambda_1\ge 0,v_1\in[-1,1],\epsilon\ge 0\}.\\ =&\{((y_1,y_2),v_1,\lambda_1):0=2y_1+2\lambda_1 y_1-3v_1\lambda_1+\sqrt{\epsilon}b_1,\\ &y_2=-\frac{\sqrt{\epsilon}}{2}b_2,b_1^2+b_2^2\le 1,\lambda_1\ge 0,v_1\in[-1,1],\epsilon\ge 0\}.\end{aligned}$$

对于任意的$(x_1,x_2)\in\mathscr{F}$, $((y_1,y_2),v_1,\lambda_1)\in\mathscr{F}_{\mathrm{WD}}$, 有

$$\begin{aligned}&f(x_1,x_2)-[f(y_1,y_2)+\lambda_1 g_1((y_1,y_2),v_1)-\sqrt{\epsilon}\|(x_1,x_2)-(y_1,y_2)\|]\\ =&x_2^2-y_1^2-y_2^2-\lambda_1 y_1^2+3v_1\lambda_1 y_1+\sqrt{\epsilon}\sqrt{y_1^2+(x_2-y_2)^2}\\ =&x_2^2-y_2^2+(3v_1\lambda_1-\lambda_1 y_1-y_1)y_1++\sqrt{\epsilon}\sqrt{y_1^2+(x_2-y_2)^2}\\ =&x_2^2-\frac{\epsilon}{4}b_2^2+\lambda_1 y_1^2+y_1^2+\sqrt{\epsilon}b_1y_1+\sqrt{\epsilon}\sqrt{y_1^2+(x_2+\frac{\sqrt{\epsilon}}{2}b_2)^2}\\ \ge&(x_2+\frac{\sqrt{\epsilon}}{2}b_2)^2+\lambda_1 y_1^2+\sqrt{\epsilon}(b_1y_1-b_2(x_2+\frac{\sqrt{\epsilon}}{2}b_2))+\sqrt{\epsilon}\sqrt{y_1^2+(x_2+\frac{\sqrt{\epsilon}}{2}b_2)^2}\\ \ge&\sqrt{\epsilon}(b_1y_1-b_2(x_2+\frac{\sqrt{\epsilon}}{2}b_2))+\sqrt{\epsilon}\sqrt{y_1^2+(x_2+\frac{\sqrt{\epsilon}}{2}b_2)^2}\\ \ge&-\sqrt{\epsilon}\sqrt{b_1^2+b_2^2}\sqrt{y_1^2+(x_2+\frac{\sqrt{\epsilon}}{2}b_2)^2}+\sqrt{\epsilon}\sqrt{y_1^2+(x_2+\frac{\sqrt{\epsilon}}{2}b_2)^2}\ge 0.\end{aligned}$$

故定理4.1成立.

再设问题(RUP)的拟ϵ-解$(\overline{x_1},\overline{x_2})\in S$, 则$\overline{x_1}=0,-\frac{\sqrt{\epsilon}}{2}\le\overline{x_2}\le\frac{\sqrt{\epsilon}}{2}$. 令$\overline{\lambda_1}=\frac{\sqrt{\epsilon}}{2},\overline{v_1}=b_1$. 则$((\overline{x_1},\overline{x_2}),\overline{v_1},\overline{\lambda_1})\in\mathscr{F}_{\mathrm{WD}}$, 且对于任意的$((y_1,y_2),v_1,\lambda_1)\in\mathscr{F}_{\mathrm{WD}}$,

$$f(\overline{x_1},\overline{x_2})+\overline{\lambda_1}g_1((\overline{x_1},\overline{x_2}),\overline{v_1})-[f(y_1,y_2)+\lambda_1 g_1((y_1,y_2),v_1)]$$

$$\begin{aligned}&\geq -\sqrt{\epsilon}\|(\overline{x_1},\overline{x_2})-(y_1,y_2)\|+\overline{\lambda_1}g_1((\overline{x_1},\overline{x_2}),\overline{v_1})\\&=-\sqrt{\epsilon}\|(\overline{x_1},\overline{x_2})-(y_1,y_2)\|+\overline{\lambda_1}(\overline{x_1}^2-2\overline{v_1}\,\overline{x_1})\\&=-\sqrt{\epsilon}\|(\overline{x_1},\overline{x_2})-(y_1,y_2)\|.\end{aligned}$$

这说明$((\overline{x_1},\overline{x_2}),\overline{v_1},\overline{\lambda_1})$是对偶问题(WD)的拟$\epsilon$-解, 故定理4.2成立.

问题(RUP)的Mond-weir型对偶问题(MWD)如下

$$(\mathrm{MWD})\qquad\begin{cases}\max & f(y_1,y_2)\\ \text{s.t.} & 0\in\partial f(y_1,y_2)+\partial(\lambda_1g_1(\cdot,v_1))(y_1,y_2)+\sqrt{\epsilon}\mathbb{B},\\ & \lambda_1g_1((y_1,y_2),v_1)\geq 0,\lambda_1\geq 0,v_1\in[-1,1],\epsilon\geq 0.\end{cases}$$

对偶问题(MWD)的可行集

$$\begin{aligned}\mathscr{F}_{\mathrm{MWD}}:=&\{((y_1,y_2),v_1,\lambda_1):0\in\partial f(y_1,y_2)+\partial(\lambda_1g_1((y_1,y_2),v_1))\\&+\sqrt{\epsilon}\mathbb{B},\lambda_1g_1((y_1,y_2),v_1)\geq 0,\lambda_1\geq 0,v_1\in[-1,1],\epsilon\geq 0\}.\\=&\{((y_1,y_2),v_1,\lambda_1):0=2y_1+2\lambda_1y_1-3v_1\lambda_1+\sqrt{\epsilon}b_1,\\ y_2=-\frac{\sqrt{\epsilon}}{2}b_2,&b_1^2+b_2^2\leq 1,\lambda_1g_1((y_1,y_2),v_1)\geq 0,\lambda_1\geq 0,v_1\in[-1,1],\epsilon\geq 0\}.\end{aligned}$$

对于任意的$(x_1,x_2)\in\mathscr{F}$,$((y_1,y_2),v_1,\lambda_1)\in\mathscr{F}_{\mathrm{MWD}}$, 有

$$\begin{aligned}&f(x_1,x_2)-[f(y_1,y_2)+\sqrt{\epsilon}\|(x_1,x_2)-(y_1,y_2)\|]\\&\geq f(x_1,x_2)-[f(y_1,y_2)+\lambda_1g_1((y_1,y_2),v_1)+\sqrt{\epsilon}\|(x_1,x_2)-(y_1,y_2)\|]\\&=x_2^2-y_2^2-y_1^2-\lambda_1y_1^2+3v_1\lambda_1y_1+\sqrt{\epsilon}\sqrt{y_1^2+(x_2-y_2)^2}\\&=x_2^2-\frac{\epsilon}{4}b_2^2+\lambda_1y_1^2+y_1^2+\sqrt{\epsilon}b_1y_1+\sqrt{\epsilon}\sqrt{y_1^2+(x_2+\frac{\sqrt{\epsilon}}{2}b_2)^2}\\&\geq(x_2+\frac{\sqrt{\epsilon}}{2}b_2)^2+\lambda_1y_1^2+\sqrt{\epsilon}(b_1y_1-b_2(x_2+\frac{\sqrt{\epsilon}}{2}b_2))+\sqrt{\epsilon}\sqrt{y_1^2+(x_2+\frac{\sqrt{\epsilon}}{2}b_2)^2}\\&\geq\sqrt{\epsilon}(b_1y_1-b_2(x_2+\frac{\sqrt{\epsilon}}{2}b_2))+\sqrt{\epsilon}\sqrt{y_1^2+(x_2+\frac{\sqrt{\epsilon}}{2}b_2)^2}\\&\geq-\sqrt{\epsilon}\sqrt{b_1^2+b_2^2}\sqrt{y_1^2+(x_2+\frac{\sqrt{\epsilon}}{2}b_2)^2}+\sqrt{\epsilon}\sqrt{y_1^2+(x_2+\frac{\sqrt{\epsilon}}{2}b_2)^2}\geq 0.\end{aligned}$$

故定理4.3成立.

再设问题(RUP)的拟ϵ-解$(\overline{x_1},\overline{x_2})\in S$, 则$\overline{x_1}=0,-\frac{\sqrt{\epsilon}}{2}\leq\overline{x_2}\leq\frac{\sqrt{\epsilon}}{2}$. 令$\overline{\lambda_1}=\frac{\sqrt{\epsilon}}{2},\overline{v_1}=b_1$. 则$((\overline{x_1},\overline{x_2}),\overline{v_1},\overline{\lambda_1})\in\mathscr{F}_{\mathrm{MWD}}$, 且对于任意的$((y_1,y_2),v_1,\lambda_1)\in\mathscr{F}_{\mathrm{MWD}}$,

$$f(\overline{x_1},\overline{x_2})-f(y_1,y_2)\geq-\sqrt{\epsilon}\|(\overline{x_1},\overline{x_2})-(y_1,y_2)\|.$$

这说明$((\overline{x_1},\overline{x_2}),\overline{v_1},\overline{\lambda_1})$是对偶问题(MWD)的拟$\epsilon$-解, 故定理4.4成立.

5. 结论

本文对鲁棒优化问题的拟近似解进行了研究, 借助约束函数的共轭函数的上图给出的闭凸锥约束规格条件, 构建了鲁棒优化问题的最优性条件, 之后给出了鲁棒凸优化问题拟近似解在Wolf型和Mond-weir型对偶模型下的对偶定理. 能否引入其它的约束规格条件建立鲁棒优化问题的最优性条件, 以及能否研究鲁棒优化问题其它解的最优性条件是值得考虑的问题.

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Characterizations of Quasi Approximate Solution for Robust Convex Optimization Problems

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Abstract: In this paper, we study quasi approximate solution for a robust convex optimization problem in the face of data uncertainty. By using the robust optimization approach, we first establish optimality conditions for quasi approximate solution under a closed convex cone constraint qualification which is given by conjugate function for the constraint function. In addition, we also characterize Wolf type and Mond-weir type duality theorems for quasi approximate solution on the robust convex optimization problem. Moreover, some examples are given to illustrate the obtained results.

Key words: Robust convex optimization; Subdifferential; Quasi approximate solution; Optimality condition; Duality theorem