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Non-$l_n^{(1)}$ Point and Uniformly Non-$l_n^{(1)}$ Point in Orlicz-Bochner Sequence Spaces

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Abstract: Some characterizations of non-$l_n^{(1)}$ point and uniformly non-$l_n^{(1)}$ point in Banach space were given. Moreover, in this paper some criteria were obtained for non-$l_n^{(1)}$ point and uniformly non-$l_n^{(1)}$ point in Orlicz-Bochner sequence spaces .

Key words: Non-$l_n^{(1)}$ point; Uniformly non-$l_n^{(1)}$ point; Orlicz-Bochner sequence space; Orlicz norm; Luxemburg norm
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1. Introduction

Uniformly non-$l_n^{(1)}$ $(n \geq 2, n \in \mathbb{N})$ property, a geometric property of Banach space introduced by Giesy[9, 10], plays important role in probability theory, fixed point theory and any other fields[1, 3, 7, 12]. Therefore in the near several decade years the characteristics of uniformly non-$l_n^{(1)}$ property were widely investigated in some Banach spaces, including Orlicz, Orlicz-Bochner and Musielak-Orlicz spaces[2,11,13−14,18]. Uniformly non-$l_2^{(1)}$ property is called uniform nonsquareness, defined by James[17], which implies the fixed point property[8].

In the past several years, the pointwise properties of locally uniform nonsquareness and nonsquareness were defined and studied in Orlicz, Orlicz-Bochner spaces by WANG and SHI etc.=5,19−23> The pointwise properties of locally uniformly non-$l_n^{(1)}$ $(n \geq 2, n \in \mathbb{N})$ property and non-$l_n^{(1)}$ property were defined and studied in Orlicz space by CHEN[5]. Inspired by the methods of [19-21], in this paper we give the criteria for non-$l_n^{(1)}$ point and uniformly non-$l_n^{(1)}$ point of Orlicz-Bochner sequence spaces.

Let $(X, \|\cdot\|)$ be a real Banach space, X^* its dual space. Denote by $B(X)$ and $S(X)$ the unit ball and the unit sphere of X respectively. A point $x \in S(X)$ is called non-$l_n^{(1)}$ $(n \geq 2, n \in \mathbb{N})$ point[5] if for any $v_2, v_3, \cdots, v_n \in S(X)$, the inequality $\|x \pm v_2 \pm \cdots \pm v_n\| < n$ holds for a certain choice of signs. If $n = 2$, then x is called non-square point[22]. A point $x \in S(X)$ is called uniformly non-$l_n^{(1)}$ $(n \geq 2, n \in \mathbb{N})$ point[5], if there exists $\delta \in (0, 1)$ such that for any

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$v_2, v_3, \cdots, v_n \in S(X)$, the inequality $\|x \pm v_2 \pm \cdots \pm v_n\| \le n - \delta$ holds true for some choice of signs. Similarly as above we have uniformly non-square point[22].

Let $\mathbb{R}$ be the set of real numbers. A function $M : \mathbb{R} \to \mathbb{R}^+ \bigcup \{0\}$ is called an Orlicz function if M is convex, even, $M(0) = 0, M(u) > 0$ for $u \neq 0$ and $\lim_{u\to 0} \frac{M(u)}{u} = 0$, $\lim_{u\to\infty} \frac{M(u)}{u} = \infty$. The complementary function N of M, in the sense of Young is defined by

$$N(v) = \sup_{u\in\mathbb{R}} \{uv - M(u)\}.$$

Let M be an Orlicz function. M is said to satisfy the δ_2-condition (shortly denoted by $M \in \delta_2$), if there exist $K > 2$ and $r_0 > 0$ such that $M(2u) \le KM(u)$ whenever $|u| \le r_0$.

Let X_i be a Banach space, $x_i \in X_i, i = 1, 2, \cdots$. For $x = (x_1, x_2, \cdots)$, we call $\rho_M(x) = \sum_{i=1}^{\infty} M(\|x_i\|)$ its modular. The linear space

$$l_M(X_i) = \{x = (x_1, x_2, \cdots) : \rho_M(\lambda x) < \infty \text{ for some } \lambda > 0\}$$

endowed the Orlicz norm

$$\|x\|_M = \inf_{h>0} \frac{1}{h} (1 + \rho_M(hx)) = \frac{1}{k} (1 + \rho_M(kx)) \tag{1.1}$$

or the Luxemburg norm

$$\|x\|_{(M)} = \inf \left\{\lambda > 0 : \rho_M(\frac{x}{\lambda}) \le 1\right\}$$

forms a Banach space, denoted by $l_M(X_i)$ or $l_{(M)}(X_i)$ respectively. In (1.1) the infimum is attained when $k \in [k^*, k^{**}]$, where $k^* = \inf\{k > 0 : \rho_N(p(kx)) \ge 1\}$, $k^{**} = \sup\{k > 0 : \rho_N(p(kx)) \le 1\}$. We called such space Orlicz-Bochner sequence space. For more references about Orlicz-Bochner space and non-$l_n^{(1)}$ property we refer to [4-5, 11, 14-16, 18-19].

2. Some Lemmas

Lemma 2.1 Let X be a Banach space, then $x \in S(X)$ is a non-$l_n^{(1)}$ point if and only if for any $v_2, \cdots, v_n \in X\backslash\{0\}$, the inequality

$$\|x \pm v_2 \pm \cdots \pm v_n\| < \|x\| + \|v_2\| + \cdots + \|v_n\|$$

holds for some choice of signs.

Proof The sufficient is obvious, we only need to prove the necessity. For any given $v_2, \cdots, v_n \in X\backslash\{0\}$, without loss of generality, we may assume

$$\min_{\pm}\{\|x \pm v_2 \pm \cdots \pm v_n\|\} = \|x + v_2 + \cdots + v_n\|.$$

The point $x \in S(X)$ being a non-$l_n^{(1)}$ point shows $n > \left\|x + \frac{v_2}{\|v_2\|} + \cdots + \frac{v_n}{\|v_n\|}\right\|$. Next we will divide the proof into two cases.

Case 1 $\|x\| = \min\{\|x\|, \|v_2\|, \cdots, \|v_n\|\}$. We have

$$\begin{aligned} n &> \left\|x + \frac{v_2}{\|v_2\|} + \cdots + \frac{v_n}{\|v_n\|}\right\| = \left\|x + v_2 + \cdots + v_n - \sum_{i=2}^{n} \left(v_i - \frac{v_i}{\|v_i\|}\right)\right\| \\ &\ge \|x + v_2 + \cdots + v_n\| - \left\|\sum_{i=2}^{n} \left(v_i - \frac{v_i}{\|v_i\|}\right)\right\| \ge \|x + v_2 + \cdots + v_n\| - \sum_{i=2}^{n} \left(1 - \frac{1}{\|v_i\|}\right) \|v_i\| \\ &= \|x + v_2 + \cdots + v_n\| - \sum_{i=2}^{n} \|v_i\| + n - 1, \end{aligned}$$

and so $\|x + v_2 + \cdots + v_n\| < \|x\| + \|v_2\| + \cdots + \|v_n\|$.

Case 2 $\|x\| > \min\{\|x\|, \|v_2\|, \cdots, \|v_n\|\}$. Without loss of generality, we may assume $\|v_n\| = \min\{\|x\|, \|v_2\|, \cdots, \|v_n\|\}$, then

$$\begin{aligned} n &> \left\| \frac{x}{\|x\|} + \frac{v_2}{\|v_2\|} + \cdots + \frac{v_n}{\|v_n\|} \right\| \\ &= \left\| \frac{1}{\|v_n\|}(x + v_2 + \cdots + v_n) + x\left(\frac{1}{\|x\|} - \frac{1}{\|v_n\|}\right) + \cdots + v_{n-1}\left(\frac{1}{\|v_{n-1}\|} - \frac{1}{\|v_n\|}\right) \right\| \\ &\geq \frac{1}{\|v_n\|}\|x + v_2 + \cdots + v_n\| - \left(\frac{1}{\|v_n\|} - \frac{1}{\|x\|}\right)\|x\| - \cdots - \left(\frac{1}{\|v_n\|} - \frac{1}{\|v_{n-1}\|}\right)\|v_{n-1}\| \\ &= \frac{1}{\|v_n\|}\|x + v_2 + \cdots + v_n\| - \frac{1}{\|v_n\|}(\|x\| + \|v_2\| + \cdots + \|v_{n-1}\|) + n - 1. \end{aligned}$$

It follows that $\|x + v_2 + \cdots + v_n\| < \|x\| + \|v_2\| + \cdots + \|v_n\|$.

Using the similar method as Lemma 2.1 we can get the following lemma, by the definition of uniformly non-$l_n^{(1)}$ point.

Lemma 2.2 Let X be a Banach space, $x \in X\backslash\{0\}$, then $\frac{x}{\|x\|}$ is a uniformly non-$l_n^{(1)}$ point if and only if there exists $\delta \in (0,1)$ such that for any $k > 0$ and any $v_2, \cdots, v_n \in X\backslash\{0\}$, the inequality

$$\|kx \pm v_2 \pm \cdots \pm v_n\| \leq (\|kx\| + \|v_2\| + \cdots + \|v_n\|)\left(1 - \frac{n\delta\min\{\|kx\|, \cdots, \|v_n\|\}}{\|kx\| + \cdots + \|v_n\|}\right)$$

holds for some choice of signs.

Lemma 2.3[5, 18] The following statements are equivalent:

1) $N \in \delta_2$;

2) For any $l > 1$ and $v_0 > 0$, there exists a $\delta > 1$ such that $N(\delta v) \leq l\delta N(v)$ for all $v \in (0, v_0)$;

3) For all $\alpha \in (0,1)$ and $u_0 > 0$, there exists a $\beta \in (0,1)$ such that $M(\alpha u) \leq \alpha\beta M(u)$ for all $u \in (0, u_0)$.

From [18] we know that in Lemma 2.3 above, for $u_0 > 0$,

$$f(\alpha) := \sup_{0<u<u_0} \frac{M(\alpha u)}{\alpha M(u)}$$

is non-decreasing with respect to $\alpha \in (0,1)$.

Lemma 2.4[5] Suppose $N \in \delta_2$ and $L \in \mathbb{R}^+$. Then the set $\{k \in K(x) : x \in l_M, \|x\|_M \leq L\}$ is bounded.

Lemma 2.5[5] The following statements are equivalent:

1) l_M is locally non-$l_n^{(1)}$;

2) $S(l_M)$ has a uniformly non-$l_n^{(1)}$ point;

3) $N \in \delta_2$.

Lemma 2.6[5] Let $x \in S(l_{(M)})$, the following statements are equivalent:

1) x is a uniformly non-$l_n^{(1)}$ point;

2) x is a non-$l_n^{(1)}$ point;

3) $\theta(x) := \inf\left\{\lambda > 0 : \rho_M(\frac{x}{\lambda}) < \infty\right\} < 1$.

Lemma 2.7 If $x = (x(1), \cdots, x(n), \cdots) \in S(l_M(X_i))$ is a uniformly non-$l_n^{(1)}$ point or a non-$l_n^{(1)}$ point, then $\|x(\cdot)\|$ is a uniformly non-$l_n^{(1)}$ point or a non-$l_n^{(1)}$ point in l_M, respectively.

Proof Suppose that x is a uniformly non-$l_n^{(1)}$ point. Arbitrarily choosing $y_i \in S(X_i)$, we set

$$u(i) = \begin{cases} \dfrac{x(i)}{\|x(i)\|}, & i \in \text{supp}x \\ y_i, & i \notin \text{supp}x \end{cases},$$

where $\text{supp}x = \{i \in \mathbb{N} : x(i) \neq 0\}$. Define $T : l_M \to l_M(X_i)$ by

$$T(h)(i) = h(i) \cdot u(i).$$

Obviously $T(\|x(\cdot)\|) = x$, and $\|T(h)\|_M = \|h\|_M$ for all $h \in l_M$. Therefore for any $x_2, \cdots, x_n \in S(l_M)$, we have $\|T(x_2)\|_M = \cdots = \|T(x_n)\|_M = 1$ and, by the definition of uniformly non-$l_n^{(1)}$ point,

$$\begin{aligned} &\big\|\|x(\cdot)\| \pm x_2 \pm \cdots \pm x_n\big\|_M = \big\|T\big(\|x(\cdot)\| \pm x_2 \pm \cdots \pm x_n\big)\big\|_M \\ =&\|x \pm T(x_2) \pm \cdots \pm T(x_n)\|_M \leq n - \delta \end{aligned}$$

for some choice of signs. Hence $\|x(\cdot)\|$ is a uniformly non-$l_n^{(1)}$ point in l_M.

Similarly $\|x(\cdot)\|$ is a uniformly non-$l_n^{(1)}$ point or non-$l_n^{(1)}$ point in $l_{(M)}$ whenever $x \in S(l_{(M)}(X_i))$ is a uniformly non-$l_n^{(1)}$ point or non-$l_n^{(1)}$ point.

3. Non-$l_n^{(1)}$ Point and Uniformly Non-$l_n^{(1)}$ Point Properties in $l_M(X_i)$

Similarly as the proof of Lemma 4.1 in [6], we have

Lemma 3.1 Suppose $N \in \delta_2, d > 0, l \in (1, +\infty)$ and $x_1 \in B_d(X) := \{x \in X : \|x\| \leq d\}$ with $\frac{x_1}{\|x_1\|}$ a uniformly non-$l_n^{(1)}$ point in X. Then there exists a $r \in (0, 1)$ such that, for any $x_2, \cdots, x_n \in B_d(X)$ and $k_1, \cdots, k_n \in (1, l)$, the inequality

$$\sum_{\pm} M\big(k_0(\|x_1 \pm \cdots \pm x_n\|)\big) \leq 2^{n-1} k_0 r \sum_{i=1}^{n} \frac{1}{k_i} M(k_i\|x_i\|)$$

holds, where $\frac{1}{k_0} = \frac{1}{k_1} + \frac{1}{k_2} + \cdots + \frac{1}{k_n}$, and $\sum\limits_{\pm}$ stands for the summation over all choice of signs.

Theorem 3.1 The point $x \in S(l_M(X_i))$ is a non-$l_n^{(1)}$ point of $l_M(X_i)$ if and only if for some $i_0 \in$supp$x$, $\frac{x(i_0)}{\|x(i_0)\|}$ is a non-$l_n^{(1)}$ point in X_{i_0}.

Proof (Necessity) Suppose that for any $i \in \text{supp}x$, $\frac{x(i)}{\|x(i)\|}$ is not a non-$l_n^{(1)}$ point in X_i. That is, for every $\frac{x(i)}{\|x(i)\|}$, there exist $v_2(i), \cdots, v_n(i) \in S(X_i)$ such that

$$\left\|\frac{x(i)}{\|x(i)\|} \pm v_2(i) \pm \cdots \pm v_n(i)\right\| = n$$

for any choice of signs. It shows that

$$\left\|\frac{x(i) \pm \|x(i)\|v_2(i) \pm \cdots \pm \|x(i)\|v_n(i)}{n}\right\| = \|x(i)\|$$

holds for any choice of signs.

Define $v'_m = (v'_m(1), v'_m(2), \cdots)$ with $v'_m(i) = \|x(i)\| \cdot v_m(i)$ for $i \in \mathbb{N}$ and $m = 2, \cdots, n$. Then $v'_m \in S(l_M(X_i))$ due to

$$\begin{aligned} \|v'_m\|_M &= \inf_{k>0} \frac{1}{k}\left(1 + \rho_M(kv'_m)\right) = \inf_{k>0} \frac{1}{k}\left(1 + \sum_{i=1}^{\infty} M(k\|v'_m(i)\|)\right) \\ &= \inf_{k>0} \frac{1}{k}\left(1 + \sum_{i=1}^{\infty} M(k\|x(i)\|\|v_m(i)\|)\right) = \inf_{k>0} \frac{1}{k}\left(1 + \sum_{i=1}^{\infty} M(k\|x(i)\|)\right) \end{aligned}$$

$$= \|x\|_M = 1.$$

Therefore

$$\begin{aligned}\left\|\frac{x+v_2'\pm\cdots\pm v_n'}{n}\right\| &= \inf_{k>0}\frac{1}{k}\left(1+\rho_M\left(k\frac{x\pm v_2'\pm\cdots+v_n'}{n}\right)\right)\\ &= \inf_{k>0}\frac{1}{k}\left(1+\sum_{i=1}^{\infty}M\left(k\left\|\frac{x(i)\pm v_2'(i)\pm\cdots+v_n'(i)}{n}\right\|\right)\right)\\ &= \inf_{k>0}\frac{1}{k}\left(1+\sum_{i=1}^{\infty}M\left(k\left\|\frac{x(i)\pm\|x(i)\|v_2(i)\pm\cdots\pm\|x(i)\|v_n(i)}{n}\right\|\right)\right)\\ &= \inf_{k>0}\frac{1}{k}\left(1+\sum_{i=1}^{\infty}M(k\|x(i)\|)\right) = \|x\|_M = 1,\end{aligned}$$

which contradicts with the fact that x is a non-$l_n^{(1)}$ point of $l_M(X_i)$.

(Sufficiency) Suppose that $x \in S(l_M(X_i))$, so there exists $k_1 > 1$ such that

$$1 = \|x\|_M = \frac{1}{k_1} + \frac{1}{k_1}\rho_M(k_1 x).$$

For any $v_2, \cdots, v_n \in S(l_M(X_i))$, there exist $k_2, \cdots, k_n \in (1, \infty)$ such that

$$1 = \|v_m\|_M = \frac{1}{k_m} + \frac{1}{k_m}\rho_M(k_m v_m) \quad (m = 2, \cdots, n).$$

Let $\frac{1}{k_0} = \frac{1}{k_1} + \cdots + \frac{1}{k_n}, A = \sum_{s=1}^{n}\prod_{j=1,j\neq s}^{n} k_j$. Suppose $\frac{x(i_0)}{\|x(i_0)\|}$ is the non-$l_n^{(1)}$ point of X_{i_0}. We may assume that

$$\min_{\pm}\{\|x(i_0)\pm v_2(i_0)\pm\cdots\pm v_n(i_0)\|\} = \|x(i_0)+\cdots+v_n(i_0)\|.$$

The remainder of the proof shall be given in two cases.

Case 1 $v_m(i_0) \neq 0$ for any $m \in \{2, \cdots, n\}$. By the convexity of M, the equality $\sum_{i=1}^{n}\frac{\prod_{j=1,j\neq i}^{n}k_j}{A} = 1$ and Lemma 2.1, one immediately gets

$$\begin{aligned}&M(k_0\|x(i_0)+\cdots+v_n(i_0)\|) < M(k_0(\|x(i_0)\|+\cdots+\|v_0(i_0)\|))\\ \le&\frac{\prod_{j=2}^{n}k_j}{A}M(k_1\|x(i_0)\|) + \sum_{m=2}^{n}\frac{\prod_{j=1,j\neq m}^{n}k_j}{A}M(k_m\|v_m(i_0)\|).\end{aligned}$$

Then

$$\begin{aligned}&\sum_{\pm}M(k_0\|x(i_0)\pm\cdots\pm v_n(i_0)\|) < 2^{n-1}M(k_0(\|x(i_0)\|+\cdots+\|v_0(i_0)\|))\\ \le&2^{n-1}\left(\frac{\prod_{j=2}^{n}k_j}{A}M(k_1\|x(i_0)\|) + \sum_{m=2}^{n}\frac{\prod_{j=1,j\neq m}^{n}k_j}{A}M(k_m\|v_m(i_0)\|)\right),\end{aligned}$$

and so

$$\begin{aligned}&\sum_{\pm}\left\|\frac{x\pm v_2\pm\cdots\pm v_n}{2^{n-1}}\right\|_M\\ \le&\frac{1}{2^{n-1}k_0}\left(1+\rho_M\left(2^{n-1}k_0\frac{x+\cdots+v_n}{2^{n-1}}\right)\right)+\cdots+\frac{1}{2^{n-1}k_0}\left(1+\rho_M\left(2^{n-1}k_0\frac{x-\cdots-v_n}{2^{n-1}}\right)\right)\\ =&\frac{1}{k_0}+\frac{1}{2^{n-1}k_0}\left(\sum_{\pm}\rho_M\left(k_0(x\pm v_2\pm\cdots\pm v_n)\right)\right)\\ =&\frac{1}{k_0}+\frac{1}{2^{n-1}k_0}\left(\sum_{i=1,i\neq i_0}^{\infty}\sum_{\pm}M(k_0\|x(i)\pm\cdots\pm v_n(i)\|)+\sum_{\pm}M(k_0\|x(i_0)\pm\cdots\pm v_n(i_0)\|)\right)\end{aligned}$$

$$<\frac{1}{k_0}+\frac{1}{2^{n-1}k_0}2^{n-1}\sum_{i=1}^{\infty}\left(\frac{\prod_{j=2}^{n}k_j}{A}M(k_1\|x(i)\|)+\sum_{m=2}^{n}\frac{\prod_{j=1,j\neq m}^{n}k_j}{A}M(k_m\|v_m(i)\|)\right)$$
$$=\frac{1}{k_0}+\frac{1}{k_1}\rho_M(k_1x)+\sum_{m=2}^{n}\frac{1}{k_m}\rho_M(k_mv_m)=n.$$

Certainly $\|x\pm v_2\pm\cdots\pm v_n\|_M<n$ holds for some choice of signs.

Case 2 $v_m(i_0)=0$ for some $m\in\{2,\cdots,n\}$. Without loss of generality, we may assume that for some $t\in\{1,2,\cdots,n\}$, $v_j(i_0)\neq 0$ for any $j\in\{1,2,\cdots,n-t\}$ and $v_j(i_0)=0$ for any $j\in\{n-t+1,\cdots,n\}$.

Let $\tilde{k}_0\in\mathbb{R}^+$ satisfy $\frac{1}{\tilde{k}_0}=\frac{1}{k_1}+\cdots+\frac{1}{k_{n-t}}$, then

$$\sum_{\pm}\left\|\frac{x\pm v_2\pm\cdots\pm v_n}{2^{n-1}}\right\|_M$$
$$\leq\frac{1}{2^{n-1}k_0}\left(1+\rho_M\left(2^{n-1}k_0\frac{x+\cdots+v_n}{2^{n-1}}\right)\right)+\cdots+\frac{1}{2^{n-1}k_0}\left(1+\rho_M\left(2^{n-1}k_0\frac{x-\cdots-v_n}{2^{n-1}}\right)\right)$$
$$=\frac{1}{k_0}+\frac{1}{2^{n-1}k_0}\left(\sum_{\pm}\rho_M\left(k_0(x\pm v_2\pm\cdots\pm v_n)\right)\right)$$
$$=\frac{1}{k_0}+\frac{1}{2^{n-1}k_0}\left(\sum_{\substack{i=1\\ i\neq i_0}}^{\infty}\sum_{\pm}M(k_0\|x(i)\pm\cdots\pm v_n(i)\|)+\sum_{\pm}M(\frac{k_0}{\tilde{k}_0}\tilde{k}_0\|x(i_0)\pm\cdots\pm v_{n-t}(i_0)\|)\right)$$
$$<\frac{1}{k_0}+\frac{1}{2^{n-1}k_0}\left(\sum_{\substack{i=1\\ i\neq i_0}}^{\infty}\sum_{\pm}M(k_0\|x(i)\pm\cdots\pm v_n(i)\|)+\sum_{\pm}\frac{k_0}{\tilde{k}_0}M(\tilde{k}_0\|x(i_0)\pm\cdots\pm v_{n-t}(i_0)\|)\right)$$
$$\leq\frac{1}{k_0}+\frac{1}{2^{n-1}k_0}2^{n-1}\sum_{i=1}^{\infty}\left(\frac{\prod_{j=2}^{n}k_j}{A}M(k_1\|x(i)\|)+\sum_{m=2}^{n}\frac{\prod_{j=1,j\neq m}^{n}k_j}{A}M(k_m\|v_m(i)\|)\right)$$
$$=\frac{1}{k_0}+\frac{1}{k_1}\rho_M(k_1x)+\sum_{m=2}^{n}\frac{1}{k_m}\rho_M(k_mv_m)=n.$$

Whence $\|x\pm v_2\pm\cdots\pm v_n\|_M<n$ holds for some choice of signs.

Combining the considerations from Case 1 and Case 2 we know that x is a non-$l_n^{(1)}$ point of $l_M(X_i)$.

For the special case $n=2$, we have

Corollary 3.1 $x\in S(l_M(X_i))$ is a non-square point if and only if for some $i_0\in$supp$x$, $\frac{x(i_0)}{\|x(i_0)\|}$ is a non-square point of X_{i_0}.

Theorem 3.2 The point $x\in S(l_M(X_i))$ is a uniformly non-$l_n^{(1)}$ point of $l_M(X_i)$ if and only if

(i) $N\in\delta_2$;

(ii) for some $i_0\in$supp$x$, $\frac{x(i_0)}{\|x(i_0)\|}$ is a uniformly non-$l_n^{(1)}$ point in X_{i_0}.

Proof(Necessity) The result (i) is an immediate consequence of Lemma 2.5 and Lemma 2.7. In the following we only need to show (ii).

Otherwise, suppose that $\frac{x(i)}{\|x(i)\|}$ is not a uniformly non-$l_n^{(1)}$ point for all $i\in$supp$x$, then for any given $i\in$supp$x$, and for any $j\in\mathbb{N}$, there exist $v_2^{(j)}(i),\cdots,v_n^{(j)}(i)\in S(X_i)$ such that

$$\min\left\{\left\|\frac{x(i)}{\|x(i)\|}\pm v_2^{(j)}(i)\pm\cdots\pm v_n^{(j)}(i)\right\|\right\}>n-\frac{1}{j}.$$

Define $x_m^{(j)}=\{x_m^{(j)}(i)\}_{i=1}^\infty$ with $x_m^{(j)}(i)=\|x(i)\|v_m^{(j)}(i)\ \ (m=2,\cdots,n)$, then

$$\begin{aligned}\|x_m^{(j)}\|_M&=\inf_{k>0}\frac{1}{k}(1+\rho_M(kx_m^{(j)}))=\inf_{k>0}\frac{1}{k}\left(1+\sum_{i=1}^\infty M(k\|x_m^{(j)}(i)\|)\right)\\&=\inf_{k>0}\frac{1}{k}\left(1+\sum_{i=1}^\infty M(k\|x(i)\|\cdot\|v_m^{(j)}(i)\|)\right)=\inf_{k>0}\frac{1}{k}\left(1+\sum_{i=1}^\infty M(k\|x(i)\|)\right)\\&=\|x\|_M=1,\end{aligned}$$

that is $x_m^{(j)}\in S(l_M(X_i))$. By the definition of the Orlicz norm, for any choice of signs, there holds

$$\begin{aligned}&\left\|x\pm x_2^{(j)}\pm\cdots\pm x_n^{(j)}\right\|_M=\left\|\left(x\pm x_2^{(j)}\pm\cdots\pm x_n^{(j)}\right)(\cdot)\right\|_M\\&=\sup_{\rho_N(\omega)\le1}\sum_{i=1}^\infty\left\|x(i)\pm\|x(i)\|v_2^{(j)}(i)\pm\cdots\pm\|x(i)\|v_n^{(j)}(i)\right\|\cdot\omega(i)\\&\ge\sup_{\rho_N(\omega)\le1}\sum_{i=1}^\infty\left(n-\frac{1}{j}\right)\|x(i)\|\cdot\omega(i)=\left(n-\frac{1}{j}\right)\|x\|_M\\&=n-\frac{1}{j},\end{aligned}$$

where $\omega\in L_N$. It follows that $\left\|x\pm x_2^{(j)}\pm\cdots\pm x_n^{(j)}\right\|_M\to n$ for any choice of signs when $j\to\infty$. A contradiction with x being a uniformly non-$l_n^{(1)}$ point.

(Sufficiency) Giving any $v_2,\cdots,v_n\in S(l_M(X_i))$, from Lemma 2.4, there exists $l>1$ such that $k_1,k_m\in(1,l)$ for any $k_1\in K(x)$ and any $k_m\in K(v_m)$ with $m\in\{2,\cdots,n\}$. Let $k_0\in\mathbb{R}^+$ satisfy $\frac{1}{k_0}=\frac{1}{k_1}+\cdots+\frac{1}{k_n}$, and denote $A=\sum_{s=1}^n\prod_{j=1,j\ne s}^n k_j$. Suppose $\frac{x(i_0)}{\|x(i_0)\|}$ is the uniformly non-$l_n^{(1)}$ point of X_{i_0}. By Lemma 3.1, there exists a $r\in(0,1)$ such that

$$\sum_\pm M\big(k_0(\|x(i_0)\pm\cdots\pm v_n(i_0)\|)\big)\le 2^{n-1}k_0r\left(\frac{1}{k_1}M(k_1\|x(i_0)\|)+\sum_{m=2}^n\frac{1}{k_m}M(k_m\|v_m(i_0)\|)\right).$$

Therefore,

$$\begin{aligned}&\sum_\pm\left\|\frac{x\pm v_2\pm\cdots\pm v_n}{2^{n-1}}\right\|_M\\&\le\frac{1}{2^{n-1}k_0}\left(1+\rho_M\left(2^{n-1}k_0\frac{x+\cdots+v_n}{2^{n-1}}\right)\right)+\cdots+\frac{1}{2^{n-1}k_0}\left(1+\rho_M\left(2^{n-1}k_0\frac{x-\cdots-v_n}{2^{n-1}}\right)\right)\\&=\frac{1}{k_0}+\frac{1}{2^{n-1}k_0}\left(\sum_\pm\rho_M\left(k_0(x\pm v_2\pm\cdots\pm v_n)\right)\right)\\&=\frac{1}{k_0}+\frac{1}{2^{n-1}k_0}\left(\sum_{i=1,i\ne i_0}^\infty\sum_\pm M(k_0\|x(i)\pm\cdots\pm v_n(i)\|)+\sum_\pm M(k_0\|x(i_0)\pm\cdots\pm v_{n-t}(i_0)\|)\right)\\&\le\frac{1}{k_0}+\frac{1}{2^{n-1}k_0}2^{n-1}k_0\sum_{i=1,i\ne i_0}^\infty\left(\frac{1}{k_1}M(k_1\|x(i)\|)+\sum_{m=2}^n\frac{1}{k_m}M(k_m\|v_m(i)\|)\right)\\&\quad+\frac{1}{2^{n-1}k_0}2^{n-1}k_0r\left(\frac{1}{k_1}M(k_1\|x(i_0)\|)+\sum_{m=2}^n\frac{1}{k_m}M(k_m\|v_m(i_0)\|)\right)\\&=n-(1-r)\left(\frac{1}{k_1}M(k_1\|x(i_0)\|)+\sum_{m=2}^n\frac{1}{k_m}M(k_m\|v_m(i_0)\|)\right)\end{aligned}$$

$$\leq n-(1-r)\frac{1}{k_1}M(k_1\|x(i_0)\|)\leq n-(1-r)M(\|x(i_0)\|).$$

Hence x is a uniformly non-$l_n^{(1)}$ point of $l_M(X_i)$ for $r\in(0,1)$ and

$$0<M(\|x(i_0)\|)\leq\rho_M(x)\leq\|x\|_{(M)}\leq\|x\|_M=1.$$

For the special case $n=2$, we have

Corollary 3.2 $x\in S(l_M(X_i))$ is a uniformly non-square point if and only if $N\in\delta_2$, and, for some $i_0\in$supp$x$, $\frac{x(i_0)}{\|x(i_0)\|}$ is a uniformly non-square point of X_{i_0}.

4. Non-$l_n^{(1)}$ Point and Uniformly Non-$l_n^{(1)}$ Point Properties in $l_{(M)}(X_i)$

Theorem 4.1 The point $x\in S(l_{(M)}(X_i))$ is a non-$l_n^{(1)}$ point of $l_{(M)}(X_i)$ if and only if

(i) $\theta(x)<1$;

(ii) for some $i_0\in$supp$x$, $\frac{x(i_0)}{\|x(i_0)\|}$ is a non-$l_n^{(1)}$ point in X_{i_0}.

Proof(Necessity) The result (i) can be obtained by Lemma 2.6 and Lemma 2.7 immediately and, (ii) can be proved by the same way as the proof in Theorem 3.1.

(Sufficiency) By $\theta(x)<1$, we see $\rho_M(\lambda x)<+\infty$ for some $\lambda>1$. For any $v_2,v_3,\cdots,v_n\in S(l_{(M)}(X_i))$, take $\lambda_j\in(1,\lambda)$, such that $\lambda_j\searrow 1$. For any given $i\in\mathbb{N}$, we have

$$\left|\left\|\lambda_j x(i)+\sum_{s=2}^{n}(-1)^{k_s}v_s(i)\right\|-\left\|x(i)+\sum_{s=2}^{n}(-1)^{k_s}v_s(i)\right\|\right|\leq(\lambda_j-1)\|x(i)\|,$$

where $k_s\in\{0,1\}$, which yields

$$\lim_{j\to\infty}\left\|\lambda_j x(i)+\sum_{s=2}^{n}(-1)^{k_s}v_s(i)\right\|=\left\|x(i)+\sum_{s=2}^{n}(-1)^{k_s}v_s(i)\right\|,$$

and, by the continuity of M,

$$\lim_{j\to\infty}M\left(\frac{\|\lambda_j x(i)+\sum_{s=2}^{n}(-1)^{k_s}v_s(i)\|}{n}\right)=M\left(\frac{\|x(i)+\sum_{s=2}^{n}(-1)^{k_s}v_s(i)\|}{n}\right).$$

Noticing that

$$M\left(\frac{\|\lambda_j x(i)+\sum_{s=2}^{n}(-1)^{k_s}v_s(i)\|}{n}\right)\leq\frac{M(\lambda_j\|x(i)\|)+\sum_{s=2}^{n}M(\|v_s(i)\|)}{n}$$

and

$$\sum_{i=1}^{\infty}\frac{M(\|\lambda x(i)\|)+\sum_{s=2}^{n}M(\|v_s(i)\|)}{n}=\frac{1}{n}\rho_M(\lambda x)+\frac{1}{n}\sum_{s=2}^{n}\rho_M(v_s)<\infty,$$

by the dominated convergence, we can get

$$\lim_{j\to\infty}\sum_{i=1}^{\infty}M\left(\frac{\|\lambda_j x(i)+\sum_{s=2}^{n}(-1)^{k_s}v_s(i)\|}{n}\right)=\sum_{i=1}^{\infty}M\left(\frac{\|x(i)+\sum_{s=2}^{n}(-1)^{k_s}v_s(i)\|}{n}\right).\quad(4.1)$$

Next we will divide the proof into two cases.

Case 1 $v_m(i_0)\neq 0$ for any $m\in\{2,\cdots,n\}$. From (ii) and Lemma 2.1, the inequality

$$\left\|\frac{x(i_0)}{\|x(i_0)\|}\pm\frac{v_2(i_0)}{\|x(i_0)\|}\pm\cdots\pm\frac{v_n(i_0)}{\|x(i_0)\|}\right\|<\left\|\frac{x(i_0)}{\|x(i_0)\|}\right\|+\left\|\frac{v_2(i_0)}{\|x(i_0)\|}\right\|+\cdots+\left\|\frac{v_n(i_0)}{\|x(i_0)\|}\right\|$$

holds for some choice of signs, that is,

$$\|x(i_0)\pm v_2(i_0)\pm\cdots\pm v_n(i_0)\|<\|x(i_0)\|+\|v_2(i_0)\|+\cdots+\|v_n(i_0)\|$$

holds for some choice of signs. Withous loss of generality, we may assume

$$\min_{\pm}\{\|x(i_0)\pm v_2(i_0)\pm\cdots\pm v_n(i_0)\|\}=\|x(i_0)+v_2(i_0)+\cdots+v_n(i_o)\|.$$

Since M is increasing on $[0,\infty)$, we get

$$\begin{aligned}M\left(\frac{\|x(i_0)+v_2(i_0)+\cdots+v_n(i_0)\|}{n}\right) <&M\left(\frac{\|x(i_0)\|+\|v_2(i_0)\|+\cdots+\|v_n(i_0)\|}{n}\right)\\ \le&\frac{M(\|x(i_0)\|)+M(\|v_2(i_0)\|)+\cdots+M(\|v_n(i_0)\|)}{n},\end{aligned}$$

and

$$\begin{aligned}M\left(\frac{\|x(i_0)\pm v_2(i_0)\pm\cdots\pm v_n(i_0)\|}{n}\right) \le&M\left(\frac{\|x(i_0)\|+\|v_2(i_0)\|+\cdots+\|v_n(i_0)\|}{n}\right)\\ \le&\frac{M(\|x(i_0)\|)+M(\|v_2(i_0)\|)+\cdots+M(\|v_n(i_0)\|)}{n}\end{aligned}$$

for any other choice of signs. Consequently,

$$\sum_{\pm}M\left(\frac{\|x(i_0)\pm v_2(i_0)\pm\cdots\pm v_n(i_0)\|}{n}\right)<\frac{2^{n-1}}{n}\left(M(\|x(i_0)\|)+\sum_{s=2}^{n}M(\|v_s(i_0)\|)\right), \quad (4.2)$$

where $\sum\limits_{\pm}$ stands for the summation over all choice of signs. Combining (4.1) with (4.2), we have

$$\begin{aligned}&\lim_{j\to\infty}\sum_{\pm}\rho_M\left(\frac{\lambda_j x\pm v_2\pm\cdots\pm v_n}{n}\right)\\ =&\sum_{\pm}\sum_{\substack{i=1\\ i\neq i_0}}^{\infty}M\left(\frac{\|x(i)\pm v_2(i)\pm\cdots\pm v_n(i)\|}{n}\right)+\sum_{\pm}M\left(\frac{\|x(i_0)\pm v_2(i_0)\pm\cdots\pm v_n(i_0)\|}{n}\right)\\ <&2^{n-1}\sum_{\substack{i=1\\ i\neq i_0}}^{\infty}\frac{M(\|x(i)\|)+\sum_{s=2}^{n}M(\|v_s(i)\|)}{n}+\frac{2^{n-1}}{n}\left(M(\|x(i_0)\|)+\sum_{s=2}^{n}M(\|v_s(i_0)\|)\right)\\ =&\frac{2^{n-1}}{n}\sum_{i=1}^{\infty}\left(M(\|x(i)\|)+\sum_{s=2}^{n}M(\|v_s(i)\|)\right)\\ \le&2^{n-1}.\end{aligned}$$

Case 2 $\quad v_m(i_0)=0$ for some $m\in\{2,\cdots,n\}$. Without loss of generality, we may assume that for some $k\in\{1,2,\cdots,n\}$, $v_j(i_0)\neq 0$ for any $j\in\{1,2,\cdots,n-k\}$ and $v_j(i_0)=0$ for any $j\in\{n-k+1,\cdots,n\}$. Similarly as Case 1, in view of $M(\alpha u)<\alpha M(u)$ whenever $\alpha\in(0,1)$ and $u\neq 0$, we obtain

$$\begin{aligned}&\lim_{j\to\infty}\sum_{\pm}\rho_M\left(\frac{\lambda_j x\pm v_2\pm\cdots\pm v_n}{n}\right)\\ =&\sum_{\pm}\sum_{\substack{i=1\\ i\neq i_0}}^{\infty}M\left(\frac{\|x(i)\pm v_2(i)\pm\cdots\pm v_n(i)\|}{n}\right)+\sum_{\pm}M\left(\frac{\|x(i_0)\pm v_2(i_0)\pm\cdots\pm v_{n-k}(i_0)\|}{n}\right)\\ \le&2^{n-1}\sum_{\substack{i=1\\ i\neq i_0}}^{\infty}\frac{M(\|x(i)\|)+\sum_{s=2}^{n}M(\|v_s(i)\|)}{n}+2^{n-1}M\left(\left(1-\frac{k}{n}\right)\frac{\|x(i_0)\|+\sum_{s=2}^{n-k}(\|v_s(i)\|)}{n-k}\right)\\ <&2^{n-1}\sum_{\substack{i=1\\ i\neq i_0}}^{\infty}\frac{M(\|x(i)\|)+\sum_{s=2}^{n}M(\|v_s(i)\|)}{n}+\left(1-\frac{k}{n}\right)2^{n-1}M\left(\frac{\|x(i_0)\|+\sum_{s=2}^{n-k}(\|v_s(i)\|)}{n-k}\right)\\ \le&2^{n-1}\sum_{\substack{i=1\\ i\neq i_0}}^{\infty}\frac{M(\|x(i)\|)+\sum_{s=2}^{n}M(\|v_s(i)\|)}{n}+\frac{2^{n-1}}{n}\left(M(\|x(i_0)\|)+\sum_{s=2}^{n-k}M(\|v_s(i_0)\|)\right)\end{aligned}$$

$$=\frac{2^{n-1}}{n}\sum_{i=1}^{\infty}\left(M(\|x(i)\|)+\sum_{s=2}^{n}M(\|v_s(i)\|)\right)$$
$$\leq 2^{n-1}.$$

Combining the considerations from Case 1 and Case 2, there exists λ_{j_0} such that

$$\sum_{\pm}\rho_M\left(\frac{\lambda_{j_0}x\pm v_2\pm\cdots\pm v_n}{n}\right)<2^{n-1}.$$

Therefore

$$\min_{\pm}\left\{\rho_M\left(\frac{\lambda_{j_0}x\pm v_2\pm\cdots\pm v_n}{n}\right)\right\}<1.$$

We may assume that $\rho_M\left(\frac{\lambda_{j_0}x+v_2+\cdots+v_n}{n}\right)<1$, then $\left\|\frac{\lambda_{j_0}x+v_2+\cdots+v_n}{n}\right\|_{(M)}\leq 1$, and so

$$\left\|\frac{x+\frac{v_2}{\lambda_{j_0}}+\cdots+\frac{v_n}{\lambda_{j_0}}}{n}\right\|_{(M)}\leq\frac{1}{\lambda_{j_0}}.$$

It follows that

$$\begin{aligned}&\left\|\frac{x+v_2+\cdots+v_n}{n}\right\|_{(M)}\\ \leq&\left\|\frac{x+v_2+\cdots+v_n}{n}-\frac{x+\frac{v_2}{\lambda_{j_0}}+\cdots+\frac{v_n}{\lambda_{j_0}}}{n}\right\|_{(M)}+\left\|\frac{x+\frac{v_2}{\lambda_{j_0}}+\cdots+\frac{v_n}{\lambda_{j_0}}}{n}\right\|_{(M)}\\ \leq&\frac{1}{n}(1-\frac{1}{\lambda_{j_0}})(n-1)+\frac{1}{\lambda_{j_0}}=1-\frac{\lambda_{j_0}-1}{n\lambda_{j_0}}\\ <&1,\end{aligned}$$

that is, $\min_{\pm}\{\|x\pm v_2\pm\cdots\pm v_n\|\}<n$, and thus x is a non-$l_n^{(1)}$ point.

Theorem 4.2 The point $x\in S(l_{(M)}(X_i))$ is a uniformly non-$l_n^{(1)}$ point of $l_{(M)}(X_i)$ if and only if

(i) $\theta(x)<1$;

(ii) For some $i_0\in\operatorname{supp}x$, $\frac{x(i_0)}{\|x(i_0)\|}$ is a uniformly non-$l_n^{(1)}$ point in X_{i_0}.

Proof (Necessity) By Theorem 4.1 and a uniformly non-$l_n^{(1)}$ point being a non-$l_n^{(1)}$ point, we can prove (i) immediately. Similarly as the proof of Theorem 3.2 we can get (ii).

(Sufficiency) The condition $\theta(x)<1$ follows $\rho_M(x)=1$ and $\rho_M(\lambda x)<\infty$ for some $\lambda>1$. From Lemma 2.2 and (ii), there exists $\delta\in(0,1)$ such that, for any $\lambda_0\in(1,\lambda)$ and any $x_2,\cdots,x_n\in X\backslash\{0\}$, the inequality

$$\begin{aligned}&\|\lambda_0x(i_0)\pm x_2\pm\cdots\pm x_n\|\\ \leq&\left(1-\frac{n\delta\min\{\|\lambda_0x(i_0)\|,\|x_2\|,\cdots,\|x_n\|\}}{\|\lambda_0x(i_0)\|+\|x_2\|+\cdots+\|x_n\|}\right)(\|\lambda_0x(i_0)\|+\|x_2\|+\cdots+\|x_n\|)\end{aligned}\tag{4.3}$$

holds for some choice of signs. By the continuity of the Orlicz function, we see

$$f_0:=\sup\left\{\frac{M(\frac{n}{n+1}s)}{\frac{n}{n+1}M(s)}:\frac{\|x(i_0)\|}{n-1}\leq s\leq\lambda M^{-1}(1)\right\}\in(0,1).$$

Set

$$\alpha=\frac{1}{2}\min\left\{\frac{1}{n^2-1},\frac{1}{f_0}-1\right\},$$

then $\alpha \in (0,1)$, and $\frac{(1+\alpha)(n-1)}{n} < \frac{n}{n+1}$.

We will show that there exists $\sigma \in (0,1)$ such that

$$\sum_{\pm} M\left(\frac{\|\lambda_0 x(i_0) \pm v_2(i_0) \pm \cdots \pm v_n(i_0)\|}{n}\right) \le \frac{2^{n-1}}{n}\sigma\left(M(\|\lambda_0 x(i_0)\|) + \sum_{s=2}^{n} M(\|v_s(i_0)\|)\right).$$

The proof will be divided into two parts.

Part 1 $\|\lambda_0 x(i_0)\| = \min\{\|\lambda_0 x(i_0)\|, \|v_2(i_0)\|, \cdots, \|v_n(i_0)\|\}$. Clearly $v_s(i_0) \neq 0$ for $s \in \{2,3,\cdots,n\}$. If $\frac{\|\lambda_0 x(i_0)\|}{\alpha} \ge \|v_2(i_0)\| + \cdots + \|v_n(i_0)\|$, thanks to (4.3), we obtain

$$\begin{aligned}&\min_{\pm}\{\|\lambda_0 x(i_0) \pm v_2(i_0) \pm \cdots \pm v_n(i_0)\|\}\\ \le&\left(1 - \frac{n\delta\|\lambda_0 x(i_0)\|}{\|\lambda_0 x(i_0)\| + \cdots + \|v_n(i_0)\|}\right)(\|\lambda_0 x(i_0)\| + \cdots + \|v_n(i_0)\|).\end{aligned}$$

Therefore by the convexity of M and the equality above we have

$$\begin{aligned}&\sum_{\pm} M\left(\frac{\|\lambda_0 x(i_0) \pm v_2(i_0) \pm \cdots \pm v_n(i_0)\|}{n}\right)\\
\le&\left(1 - \frac{n\delta\|\lambda_0 x(i_0)\|}{\|\lambda_0 x(i_0)\| + \sum_{s=2}^{n}\|v_s(i_0)\|}\right) M\left(\frac{\|\lambda_0 x(i_0)\| + \sum_{s=2}^{n}\|v_s(i_0)\|}{n}\right)\\
&+ (2^{n-1}-1) M\left(\frac{\|\lambda_0 x(i_0)\| + \sum_{s=2}^{n}\|v_s(i_0)\|}{n}\right)\\
=&n\frac{2^{n-1}}{n}\left(1 - \frac{n}{2^{n-1}}\frac{\delta\|\lambda_0 x(i_0)\|}{\|\lambda_0 x(i_0)\| + \sum_{s=2}^{n}\|v_s(i_0)\|}\right) M\left(\frac{\|\lambda_0 x(i_0)\| + \sum_{s=2}^{n}\|v_s(i_0)\|}{n}\right)\\
\le&\frac{2^{n-1}}{n}\left(1 - \frac{n}{2^{n-1}}\frac{\delta\|\lambda_0 x(i_0)\|}{\|\lambda_0 x(i_0)\| + \sum_{s=2}^{n}\|v_s(i_0)\|}\right)\left(M(\|\lambda_0 x(i_0)\|) + \sum_{s=2}^{n} M(\|v_s(i_0)\|)\right)\\
\le&\frac{2^{n-1}}{n}\left(1 - \frac{n}{2^{n-1}}\frac{\delta\alpha}{\alpha+n-1}\right)\left(M(\|\lambda_0 x(i_0)\|) + \sum_{s=2}^{n} M(\|v_s(i_0)\|)\right).\end{aligned}$$

If $\frac{\|\lambda_0 x(i_0)\|}{\alpha} < \|v_2(i_0)\| + \cdots + \|v_n(i_0)\|$, owing to

$$\|x(i_0)\| < \frac{\|\lambda_0 x(i_0)\|}{\alpha} < \|v_2(i_0)\| + \cdots + \|v_n(i_0)\| \le (n-1)M^{-1}(1) < \lambda(n-1)M^{-1}(1),$$

there holds

$$\begin{aligned}&\sum_{\pm} M\left(\frac{\|\lambda_0 x(i_0) \pm v_2(i_0) \pm \cdots \pm v_n(i_0)\|}{n}\right)\\
\le&2^{n-1} M\left(\frac{\|\lambda_0 x(i_0)\| + \sum_{s=2}^{n}\|v_s(i_0)\|}{n}\right) \le 2^{n-1} M\left(\frac{1+\alpha}{n}\sum_{s=2}^{n}\|v_s(i_0)\|\right)\\
=&2^{n-1}\frac{M\left(\frac{(1+\alpha)(n-1)}{n}\frac{\sum_{s=2}^{n}\|v_s(i_0)\|}{n-1}\right)}{\frac{(1+\alpha)(n-1)}{n}M\left(\frac{\sum_{s=2}^{n}\|v_s(i_0)\|}{n-1}\right)}\frac{(1+\alpha)(n-1)}{n} M\left(\frac{\sum_{s=2}^{n}\|v_s(i_0)\|}{n-1}\right)\\
\le&2^{n-1} f\left(\frac{(1+\alpha)(n-1)}{n}\right)\frac{(1+\alpha)(n-1)}{n} M\left(\frac{\sum_{s=2}^{n}\|v_s(i_0)\|}{n-1}\right)\\
\le&\frac{2^{n-1}}{n}(n-1)(1+\alpha) f_0 M\left(\frac{\sum_{s=2}^{n}\|v_s(i_0)\|}{n-1}\right)\\
\le&\frac{2^{n-1}}{n}(n-1)\left(1 + \frac{1}{2}\left(\frac{1}{f_0} - 1\right)\right) f_0 M\left(\frac{\sum_{s=2}^{n}\|v_s(i_0)\|}{n-1}\right)\end{aligned}$$

$$=\frac{2^{n-1}}{n}(n-1)\frac{1+f_0}{2}M\left(\frac{\sum_{s=2}^n\|v_s(i_0)\|}{n-1}\right)\leq\frac{2^{n-1}}{n}(n-1)\frac{1+f_0}{2}\frac{1}{n-1}\left(\sum_{s=2}^n M(\|v_s(i_0)\|)\right)$$
$$\leq\frac{2^{n-1}}{n}\frac{1+f_0}{2}\left(M(\|\lambda_0x(i_0)\|)+\sum_{s=2}^n M(\|v_s(i_0)\|)\right).$$

From above we know that $\sigma:=\max\left\{1-\frac{n}{2^{n-1}}\frac{\delta\alpha}{\alpha+n-1},\frac{1+f_0}{2}\right\}\in(0,1)$ satisfies

$$\sum_{\pm}M\left(\frac{\|\lambda_0x(i_0)\pm v_2(i_0)\pm\cdots\pm v_n(i_0)\|}{n}\right)\leq\frac{2^{n-1}}{n}\sigma\left(M(\|\lambda_0x(i_0)\|)+\sum_{s=2}^n M(\|v_s(i_0)\|)\right).$$

Part 2 $\|\lambda_0x(i_0)\|>\min\{\|\lambda_0x(i_0)\|,\|v_2(i_0)\|,\cdots,\|v_n(i_0)\|\}$. Without loss of generality, we may assume $\|v_n(i_0)\|=\min\{\|\lambda_0x(i_0)\|,\|v_2(i_0)\|,\cdots,\|v_n(i_0)\|\}$. Using the same method as in Part 1 we can get the result.

Combining Part 1 with Part 2, there exists $\sigma\in(0,1)$ such that, for any $\lambda_0\in(1,\lambda)$, and any $v_2,\cdots,v_n\in S(l_{(M)}(X_i))$, there holds

$$\sum_{\pm}M\left(\frac{\|\lambda_0x(i_0)\pm v_2(i_0)\pm\cdots\pm v_n(i_0)\|}{n}\right)\leq\frac{2^{n-1}}{n}\sigma\left(M(\|\lambda_0x(i_0)\|)+\sum_{s=2}^n M(\|v_s(i_0)\|)\right).$$

Therefore, for any $\lambda_0\in(1,\lambda)$ and any $v_2,\cdots,v_n\in S(l_{(M)}(X))$, we see

$$\sum_{\pm}\rho_M\left(\frac{\lambda_0x\pm v_2\pm\cdots\pm v_n}{n}\right)$$
$$\leq\sum_{\substack{i=1\\i\neq i_0}}^{\infty}2^{n-1}M\left(\frac{\|\lambda_0x(i)\|+\sum_{s=2}^n M(\|v_s(i_0)\|)}{n}\right)+\frac{2^{n-1}}{n}\sigma\left(M(\|\lambda_0x(i_0)\|)+\sum_{s=2}^n M(\|v_s(i_0)\|)\right)$$
$$\leq\frac{2^{n-1}}{n}\sum_{\substack{i=1\\i\neq i_0}}^{\infty}\left(M(\|\lambda_0x(i)\|)+\sum_{s=2}^n M(\|v_s(i_0)\|)\right)+\frac{2^{n-1}}{n}\sigma\left(M(\|\lambda_0x(i_0)\|)+\sum_{s=2}^n M(\|v_s(i_0)\|)\right)$$
$$=\frac{2^{n-1}}{n}\left(\rho_M(\lambda_0x)+\cdots+\rho_M(v_n)-(1-\sigma)\big(M(\|\lambda_0x(i_0)\|)+\sum_{s=2}^n M(\|v_s(i_0)\|)\big)\right)$$
$$\leq\frac{2^{n-1}}{n}\Big(\rho_M(\lambda_0x)+n-1-(1-\sigma)M(\|x(i_0)\|)\Big).$$

In view of $\rho_M(x)=1$, we obtain

$$\begin{aligned}&\lim_{\lambda_0\to1^+}\sup_{v_2,\cdots,v_n\in S(l_{(M)}(X))}\sum_{\pm}\rho_M\left(\frac{\lambda_0x\pm v_2\pm\cdots\pm v_n}{n}\right)\\&\leq\lim_{\lambda_0\to1^+}(\rho_M(\lambda_0x)+n-1-(1-\sigma)M(\|x(i_0)\|))\\&=\frac{2^{n-1}}{n}(n-(1-\sigma)M(\|x(i_0)\|))<2^{n-1}.\end{aligned}$$

And so for some $\lambda_0'\in(1,\lambda)$ there holds

$$\sup_{v_2,\cdots,v_n\in S(l_{(M)}(X))}\sum_{\pm}\rho_M\left(\frac{\lambda_0x\pm v_2\pm\cdots\pm v_n}{n}\right)<2^{n-1}.$$

Consequently for any $v_2,\cdots,v_n\in S(l_{(M)}(X))$ we have

$$\min_{\pm}\left\{\rho_M\left(\frac{\lambda_0x\pm v_2\pm\cdots\pm v_n}{n}\right)\right\}<1.$$

This means $\min\limits_{\pm}\left\{\left\|\frac{\lambda_0'x\pm v_2\pm\cdots\pm v_n}{n}\right\|_{(M)}\right\}\leq 1$, without loss of generality, we may assume

$$\left\|\frac{\lambda_0'x+v_2+\cdots+v_n}{n}\right\|_{(M)}\leq 1.$$

Hence,

$$\begin{aligned}&\left\|\frac{x+v_2+\cdots+v_n}{n}\right\|_{(M)}\\ \leq&\left\|\frac{x+v_2+\cdots+v_n}{n}-\frac{x+\frac{v_2}{\lambda_0'}+\cdots+\frac{v_n}{\lambda_0'}}{n}\right\|_{(M)}+\left\|\frac{x+\frac{v_2}{\lambda_0'}+\cdots+\frac{v_n}{\lambda_0'}}{n}\right\|_{(M)}\\ \leq&\frac{1}{n}(1-\frac{1}{\lambda_0'})(n-1)+\frac{1}{\lambda_0'}=1-\frac{\lambda_0'-1}{n\lambda_0'}.\end{aligned}$$

Thus complete the proof.

For the special case $n=2$, we have

Corollary 4.1 $x\in S(l_{(M)}(X_i))$ is a non-square point if and only if

(i) $\theta(x)<1$;

(ii) For some $i_0\in\text{supp}x$, $\frac{x(i_0)}{\|x(i_0)\|}$ is a non-square point of X_{i_0}.

Corollary 4.2 $x\in S(l_{(M)}(X_i))$ is a uniformly non-square point if and only if

(i) $\theta(x)<1$;

(ii) For some $i_0\in\text{supp}x$, $\frac{x(i_0)}{\|x(i_0)\|}$ is a uniformly non-square point of X_{i_0}.

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Orlicz-Bochner序列空间中的非$l_n^{(1)}$点与一致非$l_n^{(1)}$点

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摘要： 本文给出Banach空间中非$l_n^{(1)}$点和一致非$l_n^{(1)}$点的部分特征，在此基础上得到了Orlicz-Bochner序列空间中非$l_n^{(1)}$点和一致非$l_n^{(1)}$点的判据.

关键词： 非$l_n^{(1)}$点; 一致非$l_n^{(1)}$点; Orlicz-Bochner序列空间; Orlicz范数; Luxemburg范数