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分数阶Schrödinger-Poisson系统规范化解的存在性

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摘要：本文研究分数阶Schrödinger-Poisson系统规范化解的存在性, 首先在变分框架下将其规范化解转化为约束极小化问题的极小元, 然后利用集中紧性原理证明了极小元的存在性与不存在性.

关键词：分数阶Schrödinger-Poisson系统; 变分法; 集中紧性原理; 规范化解
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1. 引言

本文研究如下分数阶Schrödinger-Poisson系统

$$\begin{cases} (-\Delta)^s u + \ell_M u + \phi u = |u|^{p-2}u, & x \in \mathbb{R}^3, \\ (-\Delta)^s \phi = \frac{1}{c_s} u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

规范化解的存在性, 其中$s \in (\frac{3}{4}, 1)$, $p \in [2, \frac{4}{3}s+2]$, $\ell_M \in \mathbb{R}$, $c_s = \pi^{-\frac{3}{2}} 2^{-2s} \frac{\Gamma(\frac{3-2s}{2})}{\Gamma(s)}$.

过去的几十年里, 像(1.1)这样的系统被广泛研究, 研究者主要关注的是该系统所对应能量泛函的极小元的存在性, 读者可参见文[5, 12, 14, 17–19]. 而最近几年, 由于规范化解在物理学上的重要意义, 这促使许多数学家致力于规范化解的研究[1,10−11,20].

系统(1.1)等价于下面的方程

$$(-\Delta)^s u + \ell_M u + (|u|^2 * |x|^{2s-3})u - |u|^{p-2}u = 0, \tag{1.2}$$

该方程的规范化解即为约束极小化问题

$$I_M := \inf\{E(u) : u \in \Sigma_M\} \tag{1.3}$$

的极小元, 其中

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 \,\mathrm{d}x + \frac{1}{4}\int\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|^{3-2s}} \,\mathrm{d}x\,\mathrm{d}y - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p \,\mathrm{d}x,$$

$$\Sigma_M := \{u \in H^s(\mathbb{R}^3) : |u|_2^2 = M\}.$$

通过伸缩变换(见下文引理2.3), 可知$p = \frac{4}{3}s + 2$是I_M的L^2-临界指数. 对任意$M > 0$, 当$p \in [2, \frac{4}{3}s+2)$时$I_M > -\infty$, 当$p \in (\frac{4}{3}s+2, \frac{6}{3-2s}]$时$I_M = -\infty$, 因此本文主要在$2 \le p \le \frac{4}{3}s+2$上考虑约束极小化问题$I_M$.

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对于规范化解, 学者们最初研究的是经典的Schrödinger方程, 读者可见文[4, 6–8, 13, 15]. Schrödinger方程所对应的约束极小化问题可写为

$$i_M := \inf\Big\{\frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2+V(x)u^2)\,\mathrm{d}x-\frac{1}{p}\int_{\mathbb{R}^3}|u|^p\,\mathrm{d}x : u\in S_M\Big\},$$

其中$S_M := \{u\in H^1(\mathbb{R}^3) : |u|_2^2 = M\}$. 在文[13]中, Maeda证明了当$M$充分小时$i_M$有唯一极小元且当$M$充分大时极小元有集中行为. 在文[7]中, GUO等人证明了当$M>0$适当小时i_M有一个极小元且该极小元是径向对称的, 此外, 他们还证明了当$M\nearrow M_0$时i_M的任意极小元都集中在位势$V(x)$的极小值处且极小元是非径向对称的.

关于如下Schrödinger-Poisson系统

$$\begin{cases} -\Delta u+\ell_M u+\phi u = |u|^{p-2}u, & x\in\mathbb{R}^3,\\ -\Delta\phi = 4\pi u^2, & x\in\mathbb{R}^3\end{cases} \tag{1.4}$$

规范化解的研究, 近几年有许多重要和有意义的结果, 例如: 记对应于系统(1.4)的极小化问题为

$$\widetilde{i_M} := \inf\Big\{\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x+\frac{1}{4}\int\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{u^2(x)u^2(y)}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y-\frac{1}{p}\int_{\mathbb{R}^3}|u|^p\,\mathrm{d}x : u\in S_M\Big\}.$$

Sánchez和Soler在文[16]中证明了当$p=2$时$\widetilde{i_M}$不存在极小元. 在文[3]中, Bellazzini和Siciliano研究了当$p\in(2,3)$且$M>0$充分小时极小元的存在性. Jeanjean和LUO在文[9]中考虑了$p\in[3,\frac{10}{3}]$的情况, 他们给出了$M>0$的一个临界值来划分$\widetilde{i_M}$ 极小元的存在性和不存在性. Bellazzini等人在文[2]中考虑了$p\in(\frac{10}{3},6)$的情况, 他们采用在S_M上发展山路定理的方法, 证明了当$M>0$ 充分小时$\widetilde{i_M}$存在极小元.

一个自然的问题, 系统(1.1)的规范化解在p的哪些范围内存在，在哪些范围内不存在？这是本文主要研究的问题.

我们的主要结果如下:

定理1.1　1) 当 $p\in[2,3)$, 对任意 $M>0$, $I_M<0$.

2) 当 $p\in[3,\frac{4}{3}s+2]$, 对任意 $M>0$, $I_M=0$ 当且仅当 $M^{\frac{2s(p-3)}{4s-3}}\le V_c(p)$. 相反的, 若$M^{\frac{2s(p-3)}{4s-3}}>V_c(p)$, $I_M<0$. 这里

$$V_c(p) := \frac{p(4s-3)}{\mathrm{C_p}}\Big(\frac{1}{4s+3p-12}\Big)^{\frac{4s+3p-12}{8s-6}}\Big(\frac{1}{2(4s-3p+6)}\Big)^{\frac{4s-3p+6}{8s-6}}.$$

定理1.2　1) 当 $p=2$ 或 $p=\frac{4}{3}s+2$, 对任意 $M>0$, I_M 无极小元.

2) 当 $p\in(2,3)$, $M>0$ 充分小时, 严格不等式

$$I_M<I_{M'}+I_{M-M'},\quad \forall\, 0<M'<M, \tag{1.5}$$

成立且I_M可达.

3) 当 $p=3$, 对任意$M>0$, I_M都不可达.

4) 当 $p\in(3,\frac{4}{3}s+2)$, 若$M=M_c$, I_M有一个极小元; 若$M>M_c$, 严格不等式(1.5)成立且I_M有一个极小元. 这里

$$M_c := V_c^{\frac{4s-3}{2s(p-3)}}.$$

2. 估计及一些重要的结果

后续, 我们将会用到以下符号.

$\mathrm{D}[u]$:	$\mathrm{D}[u]=\int\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{u^2(x)u^2(y)}{\lvert x-y\rvert^{3-2s}}\,\mathrm{d}x\,\mathrm{d}y$
$\lVert\cdot\rVert_{H^s}$:	空间$H^s(\mathbb{R}^3)$上的范数, $\lVert u\rVert_{H^s}=\Big(\int_{\mathbb{R}^3}(\lvert(-\Delta)^{\frac{s}{2}}u\rvert^2+u^2)\mathrm{d}x\Big)^{1/2}$
$\lvert u\rvert_p, 1\le p\le+\infty$:	Lebesgue空间$L^p(\mathbb{R}^3)$上的范数
$B_\rho(y), \forall y\in\mathbb{R}^3$:	以y为中心ρ为半径的开球.

我们首先回顾 Gagliardo-Nirenberg 不等式

$$|u|_p^p \le \mathrm{C_{GN}}(p)|(-\Delta)^{\frac{s}{2}}u|_2^{\frac{3}{s}(\frac{p}{2}-1)}|u|_2^{\frac{3}{s}(1-\frac{3-2s}{6}p)}, \tag{2.1}$$

其中 $\mathrm{C_{GN}}(p)$ 是最优常数, 仅取决于 $p\in[2,\frac{6}{3-2s}]$.

引理2.1 对任意 $p\in[2,3]$, 存在正常数 K_p, 使得 $\forall u\in H^s(\mathbb{R}^3)$, 有

$$|u|_p^p \le \mathrm{K}_p|u|_2^{6-2p}\mathrm{D}[u]^{\frac{p}{2}-1}|(-\Delta)^{\frac{s}{2}}u|_2^{p-2}. \tag{2.2}$$

对任意 $p\in[3,\frac{4}{3}s+2]$, 存在正常数 C_p, 使得 $\forall u\in H^s(\mathbb{R}^3)$, 有

$$|u|_p^p \le \mathrm{C}_p|u|_2^{\frac{4s(p-3)}{4s-3}}\mathrm{D}[u]^{\frac{4s-3p+6}{8s-6}}|(-\Delta)^{\frac{s}{2}}u|_2^{\frac{4s+3p-12}{4s-3}}. \tag{2.3}$$

当 $p=3$, (2.2) 和 (2.3) 是一致的, 其中 $\mathrm{K}_3=\mathrm{C}_3$. 当 $p=\frac{4}{3}s+2$, (2.3) 和 (2.1) 是一致的, 其中 $\mathrm{C}_{\frac{4}{3}s+2}=\mathrm{C_{GN}}(\frac{4}{3}s+2)$.

证 回顾 $\mathrm{D}[u]=\frac{1}{c_s}\int_{\mathbb{R}^3}u^2(-\Delta)^{-s}u^2\,\mathrm{d}x$. 通过平方展开和分部积分, 可得

$$\begin{aligned}0&\le\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u-a(-\Delta)^{\frac{s}{2}}(-\Delta)^{-s}u^2|^2\,\mathrm{d}x\\&=\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\,\mathrm{d}x+a^2\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}(-\Delta)^{-s}u^2|^2\,\mathrm{d}x-2a\int_{\mathbb{R}^3}(-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}(-\Delta)^{-s}u^2\,\mathrm{d}x,\end{aligned}$$

其中

$$\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}(-\Delta)^{-s}u^2|^2\,\mathrm{d}x=\int_{\mathbb{R}^3}(-\Delta)^{s}(-\Delta)^{-s}u^2(-\Delta)^{-s}u^2\,\mathrm{d}x=\int_{\mathbb{R}^3}u^2(-\Delta)^{-s}u^2\,\mathrm{d}x,$$

$$\int_{\mathbb{R}^3}(-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}(-\Delta)^{-s}u^2\,\mathrm{d}x=\int_{\mathbb{R}^3}u(-\Delta)^{s}(-\Delta)^{-s}u^2\,\mathrm{d}x=\int_{\mathbb{R}^3}u^3\,\mathrm{d}x.$$

因此

$$0\le\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\,\mathrm{d}x+a^2c_s\mathrm{D}[u]-2a|u|_3^3,$$

即对任意正参数 a,

$$|u|_3^3\le\frac{1}{2a}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\,\mathrm{d}x+\frac{ac_s}{2}\mathrm{D}[u].$$

令

$$J(a)=\frac{1}{2a}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\,\mathrm{d}x+\frac{ac_s}{2}\mathrm{D}[u],$$

关于 a 求导得

$$J'(a)=-\frac{1}{2a^2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\,\mathrm{d}x+\frac{c_s}{2}\mathrm{D}[u].$$

令 $J'(a)=0$, 即得 $a_0=c_s^{-\frac{1}{2}}|(-\Delta)^{\frac{s}{2}}u|_2\mathrm{D}[u]^{-\frac{1}{2}}$; 当 $0<a<a_0$, $J'(a)<0$; 当 $a>a_0$, $J'(a)>0$. 因此 $\min\limits_{a>0}J(a)=J(a_0)=c_s^{\frac{1}{2}}|(-\Delta)^{\frac{s}{2}}u|_2\mathrm{D}[u]^{\frac{1}{2}}$, 从而

$$|u|_3^3\le c_s^{\frac{1}{2}}|(-\Delta)^{\frac{s}{2}}u|_2\mathrm{D}[u]^{\frac{1}{2}}. \tag{2.4}$$

这就证明了(2.2)和(2.3)中$p=3$的情况.

当 $p\in[2,3]$, 利用Hölder不等式 $|u|_p^p\le|u|_2^{2(3-p)}|u|_3^{3(p-2)}$和(2.4)推导出

$$|u|_p^p\le|u|_2^{2(3-p)}\big(c_s^{\frac{1}{2}}|(-\Delta)^{\frac{s}{2}}u|_2\mathrm{D}[u]^{\frac{1}{2}}\big)^{p-2}=c_s^{\frac{p}{2}-1}|u|_2^{6-2p}\mathrm{D}[u]^{\frac{p}{2}-1}|(-\Delta)^{\frac{s}{2}}u|_2^{p-2}.$$

当 $p=\frac{4}{3}s+2$, (2.3)和(2.1)一致, 即

$$|u|_{\frac{4}{3}s+2}^{\frac{4}{3}s+2}\le\mathrm{C_{GN}}(\frac{4}{3}s+2)|(-\Delta)^{\frac{s}{2}}u|_2^2|u|_2^{\frac{4}{3}s}. \tag{2.5}$$

当 $p\in[3,\frac{4}{3}s+2]$, 利用Hölder不等式 $|u|_p^p\le|u|_3^{\frac{3(4s-3p+6)}{4s-3}}|u|_{\frac{4}{3}s+2}^{\frac{(4s+6)(p-3)}{4s-3}}$ 及(2.4)和(2.5), 可得

$$\begin{aligned}|u|_p^p&\le\left(c_s^{\frac{1}{2}}|(-\Delta)^{\frac{s}{2}}u|_2\mathrm{D}[u]^{\frac{1}{2}}\right)^{\frac{4s-3p+6}{4s-3}}\left(\mathrm{C_{GN}}(\frac{4}{3}s+2)|(-\Delta)^{\frac{s}{2}}u|_2^2|u|_2^{\frac{4}{3}s}\right)^{\frac{3(p-3)}{4s-3}}\\&=c_s^{\frac{4s-3p+6}{8s-6}}\mathrm{C_{GN}}(\frac{4}{3}s+2)^{\frac{3(p-3)}{4s-3}}|u|_2^{\frac{4s(p-3)}{4s-3}}\mathrm{D}[u]^{\frac{4s-3p+6}{8s-6}}|(-\Delta)^{\frac{s}{2}}u|_2^{\frac{4s+3p-12}{4s-3}}.\end{aligned}$$

这就完成了引理2.1的证明.

由(2.4)以及c_s的定义可得

$$\mathrm{C}_3=c_s^{\frac{1}{2}}<1. \tag{2.6}$$

引理2.2　若 $p\in[2,\frac{4}{3}s+2)$ 或 $p=\frac{4}{3}s+2$ 且 $\mathrm{C_{GN}}(\frac{4}{3}s+2)M^{\frac{2s}{3}}\le\frac{2s+3}{3}$ 时, 能量泛函 E 在 Σ_M 上有下界. 此外, 当 $p\in[2,\frac{4}{3}s+2)$ 或 $p=\frac{4}{3}s+2$ 且 $\mathrm{C_{GN}}(\frac{4}{3}s+2)M^{\frac{2s}{3}}<\frac{2s+3}{3}$ 时, I_M 的任意极小化序列在 $H^s(\mathbb{R}^3)$ 上有界.

证　$\forall u\in\Sigma_M$, 利用 (2.1) 直接可得

$$E(u)\ge\frac{1}{2}|(-\Delta)^{\frac{s}{2}}u|_2^2-\frac{\mathrm{C_{GN}}(p)}{p}M^{\frac{3}{2s}(1-\frac{3-2s}{6}p)}|(-\Delta)^{\frac{s}{2}}u|_2^{\frac{3}{s}(\frac{p}{2}-1)}.$$

显然, 当$p\in[2,\frac{4}{3}s+2)$时E有下界. 当$p=\frac{4}{3}s+2$且$\mathrm{C_{GN}}(\frac{4}{3}s+2)M^{\frac{2s}{3}}\le\frac{2s+3}{3}$时, $E(u)\ge0$.

引理2.3　定义$u_\lambda^{m,n}(x):=\lambda^m u(\lambda^n x)$, 其中$\lambda>0$, m和n是实数. 那么通过计算可得

$$\begin{aligned}\int_{\mathbb{R}^3}|u_\lambda^{m,n}(x)|^2\mathrm{d}x&=\lambda^{2m}\int_{\mathbb{R}^3}|u(\lambda^n x)|^2\mathrm{d}x=\lambda^{2m}\int_{\mathbb{R}^3}|u(y)|^2\lambda^{-3n}\mathrm{d}y\\&=\lambda^{2m-3n}\int_{\mathbb{R}^3}|u(x)|^2\mathrm{d}x,\end{aligned}$$

$$\begin{aligned}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_\lambda^{m,n}(x)|^2\,\mathrm{d}x&=\lambda^{2m+2sn}\int_{\mathbb{R}^3}|((-\Delta)^{\frac{s}{2}}u)(\lambda^n x)|^2\,\mathrm{d}x\\&=\lambda^{2m+2sn}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u(y)|^2\lambda^{-3n}\mathrm{d}y\\&=\lambda^{2m+(2s-3)n}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u(x)|^2\,\mathrm{d}x,\end{aligned}$$

$$\begin{aligned}\mathrm{D}[u_\lambda^{m,n}]&=\lambda^{4m}\int\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|u(\lambda^n x)|^2|u(\lambda^n y)|^2}{|x-y|^{3-2s}}\,\mathrm{d}x\,\mathrm{d}y\\&=\lambda^{4m}\int\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|u(x')|^2|u(y')|^2}{|\lambda^{-n}x'-\lambda^{-n}y'|^{3-2s}}\lambda^{-3n}\lambda^{-3n}\,\mathrm{d}x'\,\mathrm{d}y'\\&=\lambda^{4m-(2s+3)n}\int\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|u(x)|^2|u(y)|^2}{|x-y|^{3-2s}}\,\mathrm{d}x\,\mathrm{d}y,\end{aligned}$$

$$\begin{aligned}\int_{\mathbb{R}^3}|u_\lambda^{m,n}(x)|^p\,\mathrm{d}x&=\lambda^{mp}\int_{\mathbb{R}^3}|u(\lambda^n x)|^p\,\mathrm{d}x=\lambda^{mp}\int_{\mathbb{R}^3}|u(y)|^p\lambda^{-3n}\,\mathrm{d}y\\&=\lambda^{mp-3n}\int_{\mathbb{R}^3}|u|^p\,\mathrm{d}x,\end{aligned}$$

则

$$E(u_\lambda^{m,n})=\frac{1}{2}\lambda^{2m+(2s-3)n}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\,\mathrm{d}x+\frac{1}{4}\lambda^{4m-(2s+3)n}\mathrm{D}[u]-\frac{1}{p}\lambda^{mp-3n}\int_{\mathbb{R}^3}|u|^p\,\mathrm{d}x.$$

取$u_\lambda(x):=\lambda^{\frac{3}{2}}u(\lambda x)$使得$|u_\lambda|_2^2=M$, 则$M\to I_M$是非增的且

$$I_M\le0,\quad\forall M\ge0.$$

另外, 当$p>\frac{4}{3}s+2$时, 对于任意$M>0$, $I_M=-\infty$.

证 对任意$u\in\Sigma_M$, 有

$$I_M\le E(u_\lambda)=\frac{\lambda^{2s}}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\,\mathrm{d}x+\frac{\lambda^{3-2s}}{4}\mathrm{D}[u]-\frac{\lambda^{\frac{3}{2}p-3}}{p}\int_{\mathbb{R}^3}|u|^p\,\mathrm{d}x.$$

令$\lambda\to 0^+$可得$I_M\le 0$. 鉴于$I_M\le 0$, 则$I_M\le I_{M'}+I_{M-M'}\le I_{M'}$(或$I_{M-M'}$), 从而函数$M\to I_M$是非增的. 当$p>\frac{4}{3}s+2$时, 令$\lambda\to+\infty$可得$I_M=-\infty$.

引理2.4 $\forall M>0$且$p\in[4-\frac{4}{3}s,\frac{4}{3}s+2]$, 能量泛函$E$在$\Sigma_M$上可达负值当且仅当如下泛函

$$u\to\Big(\frac{1}{4s+3p-12}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\,\mathrm{d}x\Big)^{\frac{4s+3p-12}{8s-6}}\Big(\frac{\mathrm{D}[u]}{2(4s-3p+6)}\Big)^{\frac{4s-3p+6}{8s-6}}-\frac{1}{p(4s-3)}\int_{\mathbb{R}^3}|u|^p\,\mathrm{d}x \tag{2.7}$$

在Σ_M上可达负值.

我们约定当$x=0$时$x^x=1$.

证 令$u\in\Sigma_M$, 考虑与u相关的序列$\{u_\lambda\}_{\lambda>0}$满足$|u_\lambda|_2^2=M$, $\forall\lambda>0$, 正如引理2.3. 我们关注的是如下泛函的符号

$$\frac{E(u_\lambda)}{\lambda^{3-2s}}=\frac{\lambda^{4s-3}}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\,\mathrm{d}x+\frac{1}{4}\mathrm{D}[u]-\frac{\lambda^{\frac{4s+3p-12}{2}}}{p}\int_{\mathbb{R}^3}|u|^p\,\mathrm{d}x. \tag{2.8}$$

当$p=4-\frac{4}{3}s$, 令λ充分小可得E可达负值当且仅当

$$\frac{1}{4}\mathrm{D}[u]-\frac{3}{12-4s}\int_{\mathbb{R}^3}|u|^{4-\frac{4}{3}s}\,\mathrm{d}x<0.$$

当$p\in(4-\frac{4}{3}s,\frac{4}{3}s+2)$, 设

$$f(\lambda):=\frac{\lambda^{4s-3}}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\,\mathrm{d}x+\frac{1}{4}\mathrm{D}[u]-\frac{\lambda^{\frac{4s+3p-12}{2}}}{p}\int_{\mathbb{R}^3}|u|^p\,\mathrm{d}x,$$

关于λ求导可得

$$f'(\lambda)=\frac{4s-3}{2}\lambda^{4s-4}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\,\mathrm{d}x-\frac{4s+3p-12}{2p}\lambda^{\frac{4s+3p-14}{2}}\int_{\mathbb{R}^3}|u|^p\,\mathrm{d}x.$$

令$f'(\lambda)=0$得$\lambda_0=\Big(\frac{4s+3p-12}{p(4s-3)}\cdot\frac{\int_{\mathbb{R}^3}|u|^p\,\mathrm{d}x}{\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\,\mathrm{d}x}\Big)^{\frac{2}{4s-3p+6}}$; 当$0<\lambda<\lambda_0$, $f'(\lambda)<0$; 当$\lambda>\lambda_0$, $f'(\lambda)>0$. 因此

$$\begin{aligned}
&\min_{\lambda>0}f(\lambda)=f(\lambda_0)\\
=&\frac{1}{2}\Big(\frac{4s+3p-12}{p(4s-3)}\cdot\frac{\int_{\mathbb{R}^3}|u|^p\,\mathrm{d}x}{\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\,\mathrm{d}x}\Big)^{\frac{8s-6}{4s-3p+6}}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\,\mathrm{d}x\\
&-\frac{1}{p}\Big(\frac{4s+3p-12}{p(4s-3)}\cdot\frac{\int_{\mathbb{R}^3}|u|^p\,\mathrm{d}x}{\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\,\mathrm{d}x}\Big)^{\frac{4s+3p-12}{4s-3p+6}}\int_{\mathbb{R}^3}|u|^p\,\mathrm{d}x+\frac{1}{4}\mathrm{D}[u]\\
=&\frac{1}{2}\Big(\frac{4s+3p-12}{p(4s-3)}\Big)^{\frac{8s-6}{4s-3p+6}}\Big(\int_{\mathbb{R}^3}|u|^p\,\mathrm{d}x\Big)^{\frac{8s-6}{4s-3p+6}}\Big(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\,\mathrm{d}x\Big)^{\frac{12-4s-3p}{4s-3p+6}}\\
&-\frac{1}{p}\Big(\frac{4s+3p-12}{p(4s-3)}\Big)^{\frac{4s+3p-12}{4s-3p+6}}\Big(\int_{\mathbb{R}^3}|u|^p\,\mathrm{d}x\Big)^{\frac{8s-6}{4s-3p+6}}\Big(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\,\mathrm{d}x\Big)^{\frac{12-4s-3p}{4s-3p+6}}+\frac{1}{4}\mathrm{D}[u]\\
=&\Big[\frac{1}{2}\Big(\frac{4s+3p-12}{p(4s-3)}\Big)^{\frac{8s-6}{4s-3p+6}}-\frac{1}{p}\Big(\frac{4s+3p-12}{p(4s-3)}\Big)^{\frac{4s+3p-12}{4s-3p+6}}\Big].\\
&\Big(\int_{\mathbb{R}^3}|u|^p\,\mathrm{d}x\Big)^{\frac{8s-6}{4s-3p+6}}\Big(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\,\mathrm{d}x\Big)^{\frac{12-4s-3p}{4s-3p+6}}+\frac{1}{4}\mathrm{D}[u]\\
=&-\frac{4s-3p+6}{2}\Big(\frac{1}{p(4s-3)}\int_{\mathbb{R}^3}|u|^p\,\mathrm{d}x\Big)^{\frac{8s-6}{4s-3p+6}}\Big(\frac{1}{4s+3p-12}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\,\mathrm{d}x\Big)^{\frac{12-4s-3p}{4s-3p+6}}+\frac{1}{4}\mathrm{D}[u].
\end{aligned}$$

显然, $E(u_\lambda)$可达负值当

$$\frac{1}{4}\mathrm{D}[u] < \frac{4s-3p+6}{2}\Big(\frac{1}{p(4s-3)}\int_{\mathbb{R}^3}|u|^p\,\mathrm{d}x\Big)^{\frac{8s-6}{4s-3p+6}}\Big(\frac{1}{4s+3p-12}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\,\mathrm{d}x\Big)^{\frac{12-4s-3p}{4s-3p+6}},$$

这意味着

$$\Big(\frac{1}{4s+3p-12}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\,\mathrm{d}x\Big)^{\frac{4s+3p-12}{8s-6}}\Big(\frac{\mathrm{D}[u]}{2(4s-3p+6)}\Big)^{\frac{4s-3p+6}{8s-6}} - \frac{1}{p(4s-3)}\int_{\mathbb{R}^3}|u|^p\,\mathrm{d}x < 0.$$

当$p=\frac{4}{3}s+2$,

$$E(u_\lambda) = \lambda^{2s}\Big(\frac{1}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\,\mathrm{d}x - \frac{3}{4s+6}\int_{\mathbb{R}^3}|u|^{\frac{4}{3}s+2}\,\mathrm{d}x\Big) + \frac{\lambda^{3-2s}}{4}\mathrm{D}[u]. \tag{2.9}$$

(2.9)可达负值当且仅当λ^{2s}的系数小于0, 即

$$\frac{1}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\,\mathrm{d}x - \frac{3}{4s+6}\int_{\mathbb{R}^3}|u|^{\frac{4}{3}s+2}\,\mathrm{d}x < 0.$$

注2.1　当$p=\frac{4}{3}s+2$且λ^{2s}的系数小于0时, 泛函(2.9)在Σ_M上无下界, 因此引理2.2中关于$p=\frac{4}{3}s+2$ 时的附加条件$\mathrm{C}_{\mathrm{GN}}(\frac{4}{3}s+2)M^{\frac{2s}{3}} \le \frac{2s+3}{3}$是必要的且最佳的.

推论2.1　令$p\in(3,\frac{4}{3}s+2)$, 则$\forall u\in\Sigma_M$, 有

$$|u|_p^p \le \mathrm{C}_3^{4-p}\mathrm{C}_{\mathrm{GN}}(4)^{p-3}M^{\frac{(4s-3)(p-3)}{2s}}|(-\Delta)^{\frac{s}{2}}u|_2^{\frac{(3-s)p+4s-9}{s}}\mathrm{D}[u]^{2-\frac{p}{2}}.$$

证　由$p\in(3,\frac{4}{3}s+2)$得$3<p<4$. 那么$\forall u\in\Sigma_M$, 利用插值不等式可得

$$|u|_p^p \le |u|_3^{3(4-p)}|u|_4^{4(p-3)}.$$

在(2.3)中取$p=3$, 即为

$$|u|_3^3 \le \mathrm{C}_3\mathrm{D}[u]^{\frac{1}{2}}|(-\Delta)^{\frac{s}{2}}u|_2.$$

在(2.1)中取$p=4$, 即为

$$|u|_4^4 \le \mathrm{C}_{\mathrm{GN}}(4)|(-\Delta)^{\frac{s}{2}}u|_2^{\frac{3}{s}}M^{2-\frac{3}{2s}}.$$

结合以上不等式即可得出结论.

命题2.1　当$p\in[2,3)$, $\forall M>0$, 能量泛函E在Σ_M上可达负值且因此$I_M<0$.

证　当$p\in[2,4-\frac{4}{3}s)$, $\forall u_\lambda\in\Sigma_M$, 令$\lambda\to0^+$, 如下泛函取负值

$$\frac{E(u_\lambda)}{\lambda^{\frac{3}{2}p-3}} = \frac{1}{2}\lambda^{2s-\frac{3}{2}p+3}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\,\mathrm{d}x + \frac{1}{4}\lambda^{6-2s-\frac{3}{2}p}\mathrm{D}[u] - \frac{1}{p}\int_{\mathbb{R}^3}|u|^p\,\mathrm{d}x.$$

当$p\in[4-\frac{4}{3}s,3)$, $\forall M>0$, 我们需要寻找一个试验函数$u\in\Sigma_M$具有负的能量泛函. 考虑$\eta\in\Sigma_M$使得$\mathrm{supp}\eta\subset B_1(0)$. 对于所有的正整数$n$, 定义$\eta_n(x):=\eta(n^{\frac{1}{3}}x)$, 那么$\mathrm{supp}\eta_n\subset B_1(0)$且通过计算可得

$$|\eta_n|_2^2 = \frac{1}{n}|\eta|_2^2 = \frac{M}{n}, \quad \mathrm{D}[\eta_n] = \frac{1}{n^{\frac{2}{3}s+1}}\mathrm{D}[\eta],$$

$$|(-\Delta)^{\frac{s}{2}}\eta_n|_2^2 = \frac{1}{n^{1-\frac{2}{3}s}}|(-\Delta)^{\frac{s}{2}}\eta|_2^2, \quad |\eta_n|_p^p = \frac{1}{n}|\eta|_p^p.$$

令n是大于1的整数, 考虑试验函数$u(x):=\sum_{i=1}^n\eta_n(x-x_i)$, 其中选取的点$x_i\in\mathbb{R}^3$, $i=1,2,...,n$, 满足

$$|x_i-x_j| \ge \frac{M^2}{\mathrm{D}[\eta]}n^{\frac{2}{3}s}+2, \quad \forall i\ne j.$$

因此, $\mathrm{supp}\eta_n(x-x_i)\cap\mathrm{supp}\eta_n(x-x_j)=\varnothing$, $\forall i\ne j$, 此外利用试验函数u的定义可得

$$|u|_2^2 = \int_{\mathbb{R}^3}\Big|\sum_{i=1}^n\eta_n(x-x_i)\Big|^2\,\mathrm{d}x$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \int_{\mathbb{R}^3} |\eta_n(x-x_i)|^2 \, \mathrm{d}x + \sum_{i\neq j} \int_{\mathbb{R}^3} \eta_n(x-x_i)\eta_n(x-x_j) \, \mathrm{d}x \\
&= n \int_{\mathbb{R}^3} |\eta_n(x)|^2 \, \mathrm{d}x = |\eta|_2^2 = M,
\end{aligned}
$$

$$
\begin{aligned}
&\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 \, \mathrm{d}x \\
=&\frac{1}{2} \int\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}} \, \mathrm{d}x \, \mathrm{d}y \\
=&\frac{1}{2} \int\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\left| \sum_{i=1}^{n} \eta_n(x-x_i) - \sum_{i=1}^{n} \eta_n(y-x_i) \right|^2}{|x-y|^{3+2s}} \, \mathrm{d}x \, \mathrm{d}y \\
=&\frac{1}{2} \int\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\left| \sum_{i=1}^{n} (\eta_n(x-x_i) - \eta_n(y-x_i)) \right|^2}{|x-y|^{3+2s}} \, \mathrm{d}x \, \mathrm{d}y \\
=&\frac{1}{2} \int\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\sum_{i=1}^{n} (\eta_n(x-x_i) - \eta_n(y-x_i))^2 - \sum_{i\neq j} \eta_n(x-x_i)\eta_n(y-x_j)}{|x-y|^{3+2s}} \, \mathrm{d}x \, \mathrm{d}y \\
=&\frac{1}{2} \sum_{i=1}^{n} \int\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|\eta_n(x-x_i) - \eta_n(y-x_i)|^2}{|x-y|^{3+2s}} \, \mathrm{d}x \, \mathrm{d}y - \frac{1}{2} \sum_{i\neq j} \int\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\eta_n(x-x_i)\eta_n(y-x_j)}{|x-y|^{3+2s}} \, \mathrm{d}x \, \mathrm{d}y \\
=&n \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} \eta_n|^2 \, \mathrm{d}x + o(1) = n^{\frac{2}{3}s} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} \eta|^2 \, \mathrm{d}x + o(1),
\end{aligned}
$$

其中

$$
\begin{aligned}
&\int\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\eta_n(x-x_i)\eta_n(y-x_j)}{|x-y|^{3+2s}} \, \mathrm{d}x \, \mathrm{d}y \\
=& \int\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\eta(n^{\frac{1}{3}}(x-x_i))\eta(n^{\frac{1}{3}}(y-x_j))}{|x-y|^{3+2s}} \, \mathrm{d}x \, \mathrm{d}y \\
=& \frac{1}{n^2} \int\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\eta(z_1)\eta(z_2)}{|\frac{z_1}{n^{\frac{1}{3}}} + x_i - \frac{z_2}{n^{\frac{1}{3}}} - x_j|^{3+2s}} \, \mathrm{d}z_1 \, \mathrm{d}z_2 \\
=& \frac{1}{n^{1-\frac{2}{3}s}} \int\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\eta(z_1)\eta(z_2)}{|z_1 - z_2 + n^{\frac{1}{3}}(x_i - x_j)|^{3+2s}} \, \mathrm{d}z_1 \, \mathrm{d}z_2 \\
=& \frac{1}{n^{1-\frac{2}{3}s}} \int\int_{\mathrm{supp}\eta\times\mathrm{supp}\eta} \frac{\eta(z_1)\eta(z_2)}{|z_1 - z_2 + n^{\frac{1}{3}}(x_i - x_j)|^{3+2s}} \, \mathrm{d}z_1 \, \mathrm{d}z_2 \\
\leq& \frac{1}{n^{1-\frac{2}{3}s}} \int\int_{\mathrm{supp}\eta\times\mathrm{supp}\eta} \frac{\eta(z_1)\eta(z_2)}{|n^{\frac{1}{3}}(x_i - x_j)|^{3+2s} - |z_1 - z_2|^{3+2s}} \, \mathrm{d}z_1 \, \mathrm{d}z_2 \\
\leq& \frac{1}{n^{1-\frac{2}{3}s}} \int\int_{\mathrm{supp}\eta\times\mathrm{supp}\eta} \frac{\eta(z_1)\eta(z_2)}{n^{1+\frac{2}{3}s}|\frac{M^2}{\mathrm{D}[\eta]} n^{\frac{2}{3}s} + 2|^{3+2s} - 2^{3+2s}} \, \mathrm{d}z_1 \, \mathrm{d}z_2 \to 0, \qquad \text{当 } n \to \infty,
\end{aligned}
$$

$$
\begin{aligned}
|u|_p^p &= \int_{\mathbb{R}^3} \Big| \sum_{i=1}^{n} \eta_n(x-x_i) \Big|^p \, \mathrm{d}x \\
&= \sum_{i=1}^{n} \int_{\mathbb{R}^3} |\eta_n(x-x_i)|^p \, \mathrm{d}x + \sum_{i\neq j, k=1}^{n} C_p \int_{\mathbb{R}^3} \eta_n^k(x-x_i)\eta_n^{p-k}(x-x_j) \, \mathrm{d}x \\
&= n \int_{\mathbb{R}^3} |\eta_n(x)|^p \, \mathrm{d}x = |\eta|_p^p,
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{D}[u]=&\int\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{\left|\sum_{i=1}^n\eta_n(x-x_i)\right|^2\left|\sum_{i=1}^n\eta_n(y-x_i)\right|^2}{|x-y|^{3-2s}}\,\mathrm{d}x\,\mathrm{d}y\\
=&\int\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{\sum_{i=1}^n|\eta_n(x-x_i)|^2\sum_{i=1}^n|\eta_n(y-x_i)|^2}{|x-y|^{3-2s}}\,\mathrm{d}x\,\mathrm{d}y\\
=&\sum_{i=1}^n\int\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|\eta_n(x-x_i)|^2|\eta_n(y-x_i)|^2}{|x-y|^{3-2s}}\,\mathrm{d}x\,\mathrm{d}y\\
&+\sum_{i\neq j}\int\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|\eta_n(x-x_i)|^2|\eta_n(y-x_j)|^2}{|x-y|^{3-2s}}\,\mathrm{d}x\,\mathrm{d}y\\
=&n\mathrm{D}[\eta_n]+\sum_{i\neq j}\int\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|\eta_n(x')|^2|\eta_n(y')|^2}{|x'+x_i-y'-x_j|^{3-2s}}\,\mathrm{d}x'\,\mathrm{d}y',\qquad 3-2s\in(1,\frac{3}{2}),\\
\leq&\frac{\mathrm{D}[\eta]}{n^{\frac{2}{3}s}}+\sum_{i\neq j}\int\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|\eta_n(x')|^2|\eta_n(y')|^2}{|x_i-x_j|-2}\,\mathrm{d}x'\,\mathrm{d}y'\\
\leq&\frac{\mathrm{D}[\eta]}{n^{\frac{2}{3}s}}+\frac{\mathrm{D}[\eta]}{M^2n^{\frac{2}{3}s}}\sum_{i\neq j}\int\int_{\mathbb{R}^3\times\mathbb{R}^3}|\eta_n(x)|^2|\eta_n(y)|^2\,\mathrm{d}x\,\mathrm{d}y\\
=&\frac{\mathrm{D}[\eta]}{n^{\frac{2}{3}s}}+\frac{\mathrm{D}[\eta]}{M^2n^{\frac{2}{3}s}}\cdot n(n-1)\cdot\frac{M^2}{n^2}\\
\leq&\frac{2\mathrm{D}[\eta]}{n^{\frac{2}{3}s}}.
\end{aligned}
$$

结合引理2.4与以上估计可得

$$
\begin{aligned}
&\Big(\frac{1}{4s+3p-12}|(-\Delta)^{\frac{s}{2}}u|_2^2\Big)^{\frac{4s+3p-12}{8s-6}}\Big(\frac{\mathrm{D}[u]}{2(4s-3p+6)}\Big)^{\frac{4s-3p+6}{8s-6}}-\frac{1}{p(4s-3)}|u|_p^p\\
\leq&\Big(\frac{n^{\frac{2}{3}s}}{4s+3p-12}|(-\Delta)^{\frac{s}{2}}\eta|_2^2\Big)^{\frac{4s+3p-12}{8s-6}}\Big(\frac{2\mathrm{D}[\eta]}{2n^{\frac{2}{3}s}(4s-3p+6)}\Big)^{\frac{4s-3p+6}{8s-6}}-\frac{1}{p(4s-3)}|\eta|_p^p\\
=&n^{\frac{2s(p-3)}{4s-3}}\Big(\frac{1}{4s+3p-12}|(-\Delta)^{\frac{s}{2}}\eta|_2^2\Big)^{\frac{4s+3p-12}{8s-6}}\Big(\frac{\mathrm{D}[\eta]}{4s-3p+6}\Big)^{\frac{4s-3p+6}{8s-6}}-\frac{1}{p(4s-3)}|\eta|_p^p,
\end{aligned}
$$

由于$p-3<0$, 所以当$n\to\infty$时上式取负值.

命题2.2　当$p\in[3,\frac{4}{3}s+2]$, $I_M=0$当且仅当

$$M^{\frac{2s(p-3)}{4s-3}}\leq V_c(p)\tag{2.10}$$

成立. 相反的, 若$M^{\frac{2s(p-3)}{4s-3}}>V_c(p)$, $I_M<0$.

证　当$p\in[3,\frac{4}{3}s+2]$, $\forall u\in\Sigma_M$, 根据引理2.4, $I_M=0$当且仅当

$$|u|_p^p\leq p(4s-3)\Big(\frac{|(-\Delta)^{\frac{s}{2}}u|_2^2}{4s+3p-12}\Big)^{\frac{4s+3p-12}{8s-6}}\Big(\frac{\mathrm{D}[u]}{2(4s-3p+6)}\Big)^{\frac{4s-3p+6}{8s-6}}.$$

再结合(2.3)可以得到$I_M=0$当且仅当(2.10)成立. 根据引理2.3中$M\to I_M$的单调性知, 当$M^{\frac{2s(p-3)}{4s-3}}>V_c(p)$时, $I_M<0$.

引理2.5　当$2<p<\frac{4}{3}s+2$, I_M的任意极小元u_M都满足

$$s\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_M|^2\,\mathrm{d}x+\frac{3-2s}{4}\mathrm{D}[u_M]-\frac{3p-6}{2p}\int_{\mathbb{R}^3}|u_M|^p\,\mathrm{d}x=0.\tag{2.11}$$

证　假设$u_M\in\Sigma_M$是I_M的极小元, 根据引理2.3, $\forall\lambda>0$, 伸缩函数$u_{M,\lambda}(x)=\lambda^{\frac{3}{2}}u_M(\lambda x)\in\Sigma_M$. 回顾

$$E(u_{M,\lambda})=\frac{\lambda^{2s}}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_M|^2\,\mathrm{d}x+\frac{\lambda^{3-2s}}{4}\mathrm{D}[u_M]-\frac{\lambda^{\frac{3}{2}p-3}}{p}\int_{\mathbb{R}^3}|u_M|^p\,\mathrm{d}x,$$

由于u_M是I_M的极小元, 因此$E(u_{M,\lambda})$在$\lambda=1$时取到极小值. 所以$E(u_{M,\lambda})$关于$\lambda=1$时的导数为0, 即为(2.11).

引理2.6 当$2<p<\frac{4}{3}s+2$, I_M的任意极小元u_M都满足(1.2)和

$$\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_M|^2\,\mathrm{d}x+\mathrm{D}[u_M]-\int_{\mathbb{R}^3}|u_M|^p\,\mathrm{d}x+\ell_M M=0. \tag{2.12}$$

特别地, 当$p\in(2,\frac{12}{2s+3}]\cup(3,\frac{4}{3}s+2)$时, $\ell_M>0$. 当$p=3$时, $\ell_M=-\frac{6(1-2s)}{M(3-4s)}I_M\geq 0$.

证 显然, I_M的任意极小元u_M都满足(1.2), 在Euler-Lagrange方程(1.2)的两边同乘u_M在$\mathbb{R}^3$下进行分部积分可得(2.12). 利用u_M是I_M的极小元以及$I_M\leq 0$可知

$$E(u_M)=\frac{1}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_M|^2\,\mathrm{d}x+\frac{1}{4}\mathrm{D}[u_M]-\frac{1}{p}\int_{\mathbb{R}^3}|u_M|^p\,\mathrm{d}x=-|I_M|. \tag{2.13}$$

结合(2.11)和(2.13)得到

$$\mathrm{D}[u_M]=\frac{8}{12-4s-3p}\Big(\frac{3p-6}{2}|I_M|-\frac{4s-3p+6}{4}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_M|^2\,\mathrm{d}x\Big),$$

$$\int_{\mathbb{R}^3}|u_M|^p\,\mathrm{d}x=\frac{2p}{12-4s-3p}\Big((3-2s)|I_M|-\frac{4s-3}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_M|^2\,\mathrm{d}x\Big).$$

利用以上两个等式和(2.12)可得

$$\ell_M=\frac{2}{M}\Big(\frac{2s(3-p)}{12-4s-3p}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_M|^2\,\mathrm{d}x+\frac{12-p(2s+3)}{12-4s-3p}|I_M|\Big),$$

其中$\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_M|^2\,\mathrm{d}x>0$且$|I_M|\geq 0$. 因此通过计算可得, 当$p\in(2,\frac{12}{2s+3}]\cup(3,\frac{4}{3}s+2)$时, $\ell_M>0$. 当$p=3$时, $\ell_M=-\frac{6(1-2s)}{M(3-4s)}I_M\geq 0$.

推论2.2 假设$p\in(2,3)\cup(3,\frac{4}{3}s+2)$, 则$I_M$的任意极小元$u_M$都满足

$$\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_M|^2\,\mathrm{d}x=\frac{4s+3p-12}{4s}\varepsilon_M-\frac{p(2s+3)-12}{2s}\eta_M,$$

$$\mathrm{D}[u_M]=\frac{4s-3p+6}{2s}\varepsilon_M-\frac{p(2s-3)+6}{s}\eta_M,$$

$$\frac{1}{p}\int_{\mathbb{R}^3}|u_M|^p\,\mathrm{d}x=\frac{4s-3}{4s}\varepsilon_M-\frac{6s-3}{2s}\eta_M,$$

其中

$$\varepsilon_M=\frac{M\ell_M}{p-3}\qquad \eta_M=\frac{I_M}{3-p}.$$

证 利用$E(u_M)=I_M$, (2.11)和(2.12)直接计算可得出结论.

伸缩问题 我们证明极小元存在性的主要工具是验证严格不等式(1.5)成立, 为此我们将会把I_M当作一个关于M的函数来对其进行研究. 我们固定一个函数$u_1\in\varSigma_1$且利用引理2.3中的伸缩性质使得$2m-3n=1$, $\lambda=M$, 记$u_{M,m}$为相对应的伸缩函数. 那么根据引理2.3可知, 对任意实数m, 都有$u_{M,m}\in\varSigma_M$且

$$E(u_{M,m})=\frac{1}{2}M^{\frac{2s(2m-1)+3}{3}}|(-\Delta)^{\frac{s}{2}}u_1|_2^2+\frac{1}{4}M^{\frac{2s(1-2m)+6m+3}{3}}\mathrm{D}[u_1]-\frac{1}{p}M^{m(p-2)+1}|u_1|_p^p. \tag{2.14}$$

3. 规范化解的存在性和不存在性

这一节主要分析约束极小化问题(1.3)极小元的存在性.

情形 1 $p=2$ 或 $p=\frac{4}{3}s+2$

命题3.1 当$p=2$或$p=\frac{4}{3}s+2$时, $\forall M>0$, I_M都不存在极小元.

证 当$p=2$时, 约束极小化问题退化为

$$\begin{aligned}I_M&=\inf\Big\{\frac{1}{2}|(-\Delta)^{\frac{s}{2}}u|_2^2+\frac{1}{4}\mathrm{D}[u]-\frac{1}{2}|u|_2^2:u\in\varSigma_M\Big\}\\&=\inf\Big\{\frac{1}{2}|(-\Delta)^{\frac{s}{2}}u|_2^2+\frac{1}{4}\mathrm{D}[u]:u\in\varSigma_M\Big\}-\frac{1}{2}M=-\frac{1}{2}M,\end{aligned}$$

这里我们利用引理2.3中的伸缩变换可得

$$0 \le \inf\left\{\frac{1}{2}|(-\Delta)^{\frac{s}{2}}u|_2^2+\frac{1}{4}\mathrm{D}[u]: u\in\Sigma_M\right\} \le \frac{\lambda^{2s}}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\,\mathrm{d}x+\frac{\lambda^{3-2s}}{4}\mathrm{D}[u]\to 0,\quad 当\lambda\to 0^+.$$

因此, I_M的任意可能的极小元都为0, 所以$\forall M>0$, I_M都不可达.

当$p=\frac{4}{3}s+2$时, $I_M=0$或$I_M=-\infty$, 且在这两种情况下I_M都不存在极小元. 事实上, 根据引理2.4, $I_M=0$(或$-\infty$)当且仅当

$$\frac{1}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\,\mathrm{d}x-\frac{3}{4s+6}\int_{\mathbb{R}^3}|u|^{\frac{4}{3}s+2}\,\mathrm{d}x\ge(或<)0.$$

若I_M可达, 则存在$u\in\Sigma_M$满足$E(u)=I_M$, 即

$$\frac{1}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\,\mathrm{d}x+\frac{1}{4}\mathrm{D}[u]-\frac{3}{4s+6}\int_{\mathbb{R}^3}|u|^{\frac{4}{3}s+2}\,\mathrm{d}x=I_M.$$

显然, 当$I_M=0$时$\mathrm{D}[u]\le 0$, 当$I_M=-\infty$时$\mathrm{D}[u]<0$, 这都是不可能的, 因此I_M不可达.

情形 2　$2<p<3$

命题3.2　令$2<p<3$, 那么当$M>0$充分小时, 严格不等式

$$I_M<I_{M'}+I_{M-M'},\quad \forall 0<M'<M,$$

成立. 特别地, I_M的任一极小化序列在平移的情形下都在$H^s(\mathbb{R}^3)$中相对紧. 因此, 当$M>0$充分小时I_M可达.

证　当$2<p<3$时, 在命题2.1中我们已经证明了$\forall M>0$, $I_M<0$. 在(2.14)中我们选取参数m满足

$$0\le\frac{2s(2m-1)+3}{3}=m(p-2)+1<\frac{2s(1-2m)+6m+3}{3},$$

那么$m=\frac{2s}{4s-3p+6}$且(2.14)可写为

$$E(u_{M,p})=M^{\frac{p(2s-3)+6}{4s-3p+6}}\left(\frac{1}{2}|(-\Delta)^{\frac{s}{2}}u_1|_2^2-\frac{1}{p}|u_1|_p^p\right)+\frac{1}{4}M^{\frac{12s-p(2s+3)+6}{4s-3p+6}}\mathrm{D}[u_1].$$

因此可以推导出

$$I_M=M^{\frac{p(2s-3)+6}{4s-3p+6}}J_1^M,\tag{3.1}$$

这里$\forall\mu>0$,

$$J_\mu^M=\inf\left\{\frac{1}{2}|(-\Delta)^{\frac{s}{2}}u|_2^2-\frac{1}{p}|u|_p^p+\frac{1}{4}M^{\frac{4s(3-p)}{4s-3p+6}}\mathrm{D}[u]:u\in\Sigma_\mu\right\}.$$

事实上, 假设u是J_1^M的极小元. 令$u_{M,p}=M^{\frac{2s}{4s-3p+6}}u(M^{\frac{p-2}{4s-3p+6}}x)$, 那么$u_{M,p}\in\Sigma_M$且

$$E(u_{M,p})=M^{\frac{p(2s-3)+6}{4s-3p+6}}\Big(\frac{1}{2}|(-\Delta)^{\frac{s}{2}}u|_2^2-\frac{1}{p}|u|_p^p+\frac{1}{4}M^{\frac{4s(3-p)}{4s-3p+6}}\mathrm{D}[u]\Big).$$

因此

$$I_M=\inf_{u_{M,p}\in\Sigma_M}E(u_{M,p})=M^{\frac{p(2s-3)+6}{4s-3p+6}}J_1^M.$$

类似地, 也可以证明出

$$J_\mu^M=\mu^{\frac{p(2s-3)+6}{4s-3p+6}}J_1^{\mu M}.\tag{3.2}$$

令$\mu=\frac{M'}{M}$, 由(3.1)和(3.2)可得

$$I_M=M^{\frac{p(2s-3)+6}{4s-3p+6}}J_1^M<I_{M'}+I_{M-M'}=M'^{\frac{p(2s-3)+6}{4s-3p+6}}J_1^{M'}+(M-M')^{\frac{p(2s-3)+6}{4s-3p+6}}J_1^{M-M'},$$

鉴于$M>0$, 有

$$\begin{aligned}J_1^M&<\Big(\frac{M'}{M}\Big)^{\frac{p(2s-3)+6}{4s-3p+6}}J_1^{M'}+\Big(1-\frac{M'}{M}\Big)^{\frac{p(2s-3)+6}{4s-3p+6}}J_1^{M-M'}\\&=\mu^{\frac{p(2s-3)+6}{4s-3p+6}}J_1^{\mu M}+(1-\mu)^{\frac{p(2s-3)+6}{4s-3p+6}}J_1^{(1-\mu)M}\\&=J_\mu^M+J_{1-\mu}^M,\quad\forall\mu\in(0,1).\end{aligned}$$

因此, 为了证明当M充分小时严格不等式(1.5)成立, 可以转化为证明如下不等式成立

$$J_1^M < J_\mu^M + J_{1-\mu}^M, \quad \forall \mu \in (0,1). \tag{3.3}$$

注意到当$M \to 0$时, $M^{\frac{4s(3-p)}{4s-3p+6}} \to 0$, 从而$\lim\limits_{M\to 0} J_1^M = J_1^0$. 利用$J_\lambda^0 = \lambda^{\frac{p(2s-3)+6}{4s-3p+6}} J_1^0$, 其中$J_1^0 < 0$且$\frac{p(2s-3)+6}{4s-3p+6} > 1$, 直接可得$\forall \lambda > 0$, J_λ^0 满足集中紧性原则, 即

$$J_\lambda^0 < J_\mu^0 + J_{\lambda-\mu}^0, \quad \forall \mu \in (0,\lambda). \tag{3.4}$$

下证$M > 0$充分小时, 严格不等式(3.3)成立. 用反证法, 假设(3.3)不成立, 则存在序列$\{M_n\}_{n\geq 1}$, $M_n \to 0$和序列$\{\lambda_n\}_{n\geq 1}$, $\lambda_n \in (0,1)$使得

$$J_1^{M_n} = J_{\lambda_n}^{M_n} + J_{1-\lambda_n}^{M_n}. \tag{3.5}$$

假设$\frac{1}{2} \leq \lambda_n < 1$(若$\lambda_n \in (0,\frac{1}{2})$, 交换$\lambda_n$与$1-\lambda_n$的位置). 由$J_1^M$关于$M$连续可得$\lambda_n \to 1$, 否则与(3.4)矛盾.

我们断言, 当n充分大时$J_{\lambda_n}^{M_n}$满足集中紧性原则, 即

$$J_{\lambda_n}^{M_n} < J_\mu^{M_n} + J_{\lambda_n-\mu}^{M_n}, \quad \forall \mu \in (0,\lambda_n). \tag{3.6}$$

若不然, 则存在序列$\{\mu_n\}_{n\geq 1}$, 其中$\mu_n \in [\frac{1}{2}\lambda_n, \lambda_n)$, 使得

$$J_{\lambda_n}^{M_n} = J_{\mu_n}^{M_n} + J_{\lambda_n-\mu_n}^{M_n}. \tag{3.7}$$

由(3.5)和(3.7)可得

$$J_{\mu_n}^{M_n} + J_{1-\mu_n}^{M_n} \geq J_1^{M_n} = J_{\mu_n}^{M_n} + J_{\lambda_n-\mu_n}^{M_n} + J_{1-\lambda_n}^{M_n} \geq J_{\mu_n}^{M_n} + J_{1-\mu_n}^{M_n},$$

因此以下等式成立

$$J_1^{M_n} = J_{\mu_n}^{M_n} + J_{1-\mu_n}^{M_n}. \tag{3.8}$$

由$\frac{1}{2}\lambda_n \leq \mu_n \leq \lambda_n$和$\frac{1}{2} \leq \lambda_n < 1$可得$\frac{1}{4} \leq \mu_n < 1$, 接着在(3.8)中关于$M_n$取极限可得$J_1^0 = J_{\mu_n}^0 + J_{1-\mu_n}^0$, 这与(3.4)矛盾. 因此严格不等式(3.6)成立.

特别地, 当n充分大, 存在$J_{\lambda_n}^{M_n}$的极小元u_n使得序列$\{\lambda_n^{-\frac{1}{2}} u_n\}_{n\geq 1}$是$J_1^0$的极小化序列. 事实上, 令$\psi_n := \lambda_n^{-\frac{1}{2}} u_n$, 则$\psi_n \in \Sigma_1$且

$$\frac{1}{2}|(-\Delta)^{\frac{s}{2}} u_n|_2^2 - \frac{1}{p}|u_n|_p^p + \frac{1}{4} M_n^{\frac{4s(3-p)}{4s-3p+6}} \mathrm{D}[u_n] = J_{\lambda_n}^{M_n} \to J_1^0.$$

因此

$$\begin{aligned}
&\frac{1}{2}|(-\Delta)^{\frac{s}{2}} \psi_n|_2^2 - \frac{1}{p}|\psi_n|_p^p + \frac{1}{4} M_n^{\frac{4s(3-p)}{4s-3p+6}} \mathrm{D}[\psi_n] \\
=&\frac{1}{2\lambda_n}|(-\Delta)^{\frac{s}{2}} u_n|_2^2 - \frac{1}{p\lambda_n^{\frac{p}{2}}}|u_n|_p^p + \frac{1}{4\lambda_n^2} M_n^{\frac{4s(3-p)}{4s-3p+6}} \mathrm{D}[u_n] \\
=&\frac{1}{2}|(-\Delta)^{\frac{s}{2}} u_n|_2^2 - \frac{1}{p}|u_n|_p^p + o(1) \\
=&J_1^0 + o(1).
\end{aligned}$$

因为(3.4)成立, 这一极小化序列在平移下强收敛于J_1^0的极小元u_∞于$H^s(\mathbb{R}^3)$上. 当$\lambda_n \to 1$时, $\{u_n\}_{n\geq 1}$也有同样的收敛性质.

不失一般性, 我们可以假设在$\mathbb{R}^3$上, u_n和$u_\infty > 0$分别满足下面的Euler-Lagrange方程:

$$(-\Delta)^s u_n - |u_n|^{p-2} u_n + M_n^{\frac{4s(3-p)}{4s-3p+6}} (|u_n|^2 * |x|^{2s-3}) u_n + \theta_n u_n = 0,$$

其中$|u_n|_2^2 = \lambda_n$,

$$(-\Delta)^s u_\infty - |u_\infty|^{p-2} u_\infty + \theta_1 u_\infty = 0,$$

其中$|u_\infty|_2^2 = 1$. 由

$$J_1^0 = \frac{1}{2}|(-\Delta)^{\frac{s}{2}} u_\infty|_2^2 - \frac{1}{p}|u_\infty|_p^p < 0$$

和

$$|(-\Delta)^{\frac{s}{2}}u_\infty|_2^2-|u_\infty|_p^p+\theta_1|u_\infty|_2^2=0,$$

可得$\theta_1>0$. 在(3.5)两边同时除以$1-\lambda_n$, 为

$$\frac{J_1^{M_n}-J_{\lambda_n}^{M_n}}{1-\lambda_n}=\frac{J_{1-\lambda_n}^{M_n}}{1-\lambda_n}.$$

当$\lambda_n\to 1$时, 由$\psi_n=\lambda_n^{-\frac{1}{2}}u_n\in\Sigma_1$和$J_1^{M_n}\le E(\psi_n)$可得, 上式的左边

$$\begin{aligned}\frac{J_1^{M_n}-J_{\lambda_n}^{M_n}}{1-\lambda_n}\le&\frac{1}{1-\lambda_n}\Big[\Big(\frac{1}{2\lambda_n}|(-\Delta)^{\frac{s}{2}}u_n|_2^2-\frac{1}{p\lambda_n^{\frac{p}{2}}}|u_n|_p^p+\frac{1}{4\lambda_n^2}M_n^{\frac{4s(3-p)}{4s-3p+6}}\mathrm{D}[u_n]\Big)\\&-\Big(\frac{1}{2}|(-\Delta)^{\frac{s}{2}}u_n|_2^2-\frac{1}{p}|u_n|_p^p+\frac{1}{4}M_n^{\frac{4s(3-p)}{4s-3p+6}}\mathrm{D}[u_n]\Big)\Big]\\=&\frac{1}{1-\lambda_n}\Big(\frac{1-\lambda_n}{2\lambda_n}|(-\Delta)^{\frac{s}{2}}u_n|_2^2-\frac{1-\lambda_n^{\frac{p}{2}}}{p\lambda_n^{\frac{p}{2}}}|u_n|_p^p+\frac{1-\lambda_n^2}{4\lambda_n^2}M_n^{\frac{4s(3-p)}{4s-3p+6}}\mathrm{D}[u_n]\Big)\\=&\frac{1}{2\lambda_n}|(-\Delta)^{\frac{s}{2}}u_n|_2^2-\frac{\lambda_n^{\frac{p}{2}}-1}{p\lambda_n^{\frac{p}{2}}(\lambda_n-1)}|u_n|_p^p+\frac{1+\lambda_n}{4\lambda_n^2}M_n^{\frac{4s(3-p)}{4s-3p+6}}\mathrm{D}[u_n]\\=&\frac{1}{2}|(-\Delta)^{\frac{s}{2}}\psi_n|_2^2-\frac{\lambda_n^{\frac{p}{2}}-1}{p(\lambda_n-1)}|\psi_n|_p^p+\frac{1+\lambda_n}{4}M_n^{\frac{4s(3-p)}{4s-3p+6}}\mathrm{D}[\psi_n]\\\to&\frac{1}{2}(|(-\Delta)^{\frac{s}{2}}u_\infty|_2^2-|u_\infty|_p^p)=-\frac{\theta_1}{2}<0,\end{aligned}$$

而由(3.2)知, 右边

$$\frac{J_{1-\lambda_n}^{M_n}}{1-\lambda_n}=(1-\lambda_n)^{\frac{2s(p-2)}{4s-3p+6}}J_1^{(1-\lambda_n)M_n},$$

当$\lambda_n\to 1$时, $J_1^{(1-\lambda_n)M_n}\to J_1^0$, 又$\frac{2s(p-2)}{4s-3p+6}>0$, $(1-\lambda_n)^{\frac{2s(p-2)}{4s-3p+6}}\to 0$, 所以$\frac{J_{1-\lambda_n}^{M_n}}{1-\lambda_n}\to 0$. 因此, 严格不等式(3.3) 成立.

情形3 $p=3$

命题3.3 当$p=3$时, $\forall M>0$, $I_M=0$且I_M不可达.

证 当$p=3$, 由(2.6)可知$\frac{3}{\sqrt{2}\mathrm{C}_3}>1$. 再结合命题2.2可得, 当$\frac{3}{\sqrt{2}\mathrm{C}_3}>1$时, $I_M=0$. 在(2.14)中取$m=\frac{2s}{4s-3}$, 则

$$I_M=M^{\frac{6s-3}{4s-3}}I_1,\quad\forall M>0.\tag{3.9}$$

显然, $I_1=0$且I_M可达当且仅当I_1可达, 因此证明I_M不可达可以转化为证明I_1不可达.

用反证法, 假设I_1可达且有极小元$u_1\in H^s(\mathbb{R}^3)$. 回顾引理2.6, $\ell_M=-\frac{6(1-2s)}{M(3-4s)}I_M=0$, 那么$u_1$满足Euler-Lagrange方程(1.2), 其中Lagrange乘子为0. 因此, 利用

$$\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_1|^2\,\mathrm{d}x+\mathrm{D}[u_1]-\int_{\mathbb{R}^3}|u_1|^3\,\mathrm{d}x=0$$

和$I_1=E(u_1)=0$, 可以推导出

$$\frac{1}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_1|^2\,\mathrm{d}x=\frac{1}{4}\mathrm{D}[u_1]=\frac{1}{6}\int_{\mathbb{R}^3}|u_1|^3\,\mathrm{d}x.$$

将上式代入到(2.3)中可得

$$\frac{3}{\sqrt{2}\mathrm{C}_3}\le 1,$$

这是矛盾的. 因此I_1不可达, 从而I_M也不可达.

情形4 $3<p<\frac{4}{3}s+2$

回顾命题2.2, $I_M=0$当且仅当$M^{\frac{2s(p-3)}{4s-3}}\le V_c$, 否则$I_M<0$. 下面我们定义

$$M_c:=V_c^{\frac{4s-3}{2s(p-3)}},$$

在$p\in(3,\frac{4}{3}s+2)$上, $\frac{4s-3}{2s(p-3)}>0$. 在区间$(3,\frac{4}{3}s+2)$上的主要结果在命题3.4中说明, 为了证明命题3.4的3), 需要验证下面的引理.

引理3.1 当$M>M_c$时,

$$I_{M'}\le\Big(\frac{M'}{M}\Big)^{\frac{6s-3}{4s-3}}I_M,\quad \forall M'>M. \tag{3.10}$$

特别地, 函数$M\to I_M$在$[M_c,+\infty)$上严格递减且I_M满足严格不等式

$$I_M<I_m+I_{M-m},\quad \forall m\in(0,M). \tag{3.11}$$

证 $\forall u\in\Sigma_M$, 令$\tilde u=\Big(\frac{M'}{M}\Big)^{\frac{2s}{4s-3}}u\Big(\Big(\frac{M'}{M}\Big)^{\frac{1}{4s-3}}\cdot\Big)$, 则$\tilde u\in\Sigma_{M'}$且

$$I_{M'}\le E(\tilde u)=\Big(\frac{M'}{M}\Big)^{\frac{6s-3}{4s-3}}\Big[\frac12|(-\Delta)^{\frac{s}{2}}u|_2^2+\frac14\mathrm{D}[u]-\frac1p\Big(\frac{M'}{M}\Big)^{\frac{2s(p-3)}{4s-3}}|u|_p^p\Big]\le\Big(\frac{M'}{M}\Big)^{\frac{6s-3}{4s-3}}E(u).$$

在上式两边同时关于$u\in\Sigma_M$取下确界可得(3.10), 再结合$M>M_c$时$I_M<0$, 可得函数$M\to I_M$在$[M_c,+\infty)$上严格递减.

下面证明(3.11)成立. 分4种情况: 当$I_m=I_{M-m}=0$(即$m\le M_c$, $M-m\le M_c$), 由于$I_M<0$, (3.11)显然成立. 当$I_m<0$, $I_{M-m}=0$(即$m>M_c$, $M-m\le M_c$), 根据$M\to I_M$在$[M_c,+\infty)$上严格递减, 有$I_M<I_m$, 则(3.11)成立. $I_m=0$, $I_{M-m}<0$的情况与$I_m<0$, $I_{M-m}=0$的情况相似. 当$I_m<0$且$I_{M-m}<0$(即$m>M_c$, $M-m>M_c$), 利用(3.10)可得

$$I_M\le\Big(\frac{M}{m}\Big)^{\frac{6s-3}{4s-3}}I_m<\frac{M}{m}I_m,\quad I_M\le\Big(\frac{M}{M-m}\Big)^{\frac{6s-3}{4s-3}}I_{M-m}<\frac{M}{M-m}I_{M-m},$$

因此$I_M=\frac{m}{M}I_M+\frac{M-m}{M}I_M<I_m+I_{M-m}$.

命题3.4 当 $p\in(3,\frac43s+2)$, 下面的结论成立:

1) 当 $M<M_c$, $I_M=0$且不可达;

2) 当 $M=M_c$, $I_M=0$且存在极小元;

3) 当 $M>M_c$, $I_M<0$且严格不等式(1.5)成立.

证 当$M<M_c$, $I_M=0$. 定义

$$E_M(u):=\frac12|(-\Delta)^{\frac{s}{2}}u|_2^2+\frac14\mathrm{D}[u]-\frac1pM^{\frac{2s(p-3)}{4s-3}}|u|_p^p,$$

注意到$E_1=E$. 在(2.14)中取$m=\frac{2s}{4s-3}$可得

$$E(M^{\frac{2s}{4s-3}}u(M^{\frac{1}{4s-3}}x))=M^{\frac{6s-3}{4s-3}}E_M(u),\quad \forall u\in\Sigma_1. \tag{3.12}$$

用反证法, 假设I_M可达, 那么下面的等式

$$\inf\{E_M(u):u\in\Sigma_1\}=M^{\frac{3-6s}{4s-3}}I_M,\quad \forall M>0,$$

存在极小元$u_M\in\Sigma_1$. 事实上, 对任意$u\in\Sigma_1$, 有

$$\inf_{u\in\Sigma_1}E_M(u)=M^{\frac{3-6s}{4s-3}}\inf_{u\in\Sigma_1}E(M^{\frac{2s}{4s-3}}u(M^{\frac{1}{4s-3}}x))=M^{\frac{3-6s}{4s-3}}\inf_{\omega\in\Sigma_M}E(\omega)=M^{\frac{3-6s}{4s-3}}I_M.$$

由于I_M存在极小元, 则$\inf\limits_{u\in\Sigma_1}E_M(u)$存在极小元$u_M\in\Sigma_1$. 由于$M_c>M$, 我们有$E_{M_c}(u_M)<E_M(u_M)=0$, 这与$I_{M_c}=0$矛盾, 因此1)得证.

当$M=M_c$, 令$M_n=M_c+\frac1n$, $\forall n\in\mathbb{N}^+$, 假设$u_n\in\Sigma_{M_n}$是I_{M_n}的极小元. 由$M_n>M_c$和3), 我们知道这样的极小元是存在的. 当$n\to\infty$时$M_n\to M_c$, 那么可以推导出$\lim\limits_{n\to\infty}E(u_n)=\lim\limits_{n\to\infty}I_{M_n}=I_{M_c}=0$. 回顾推论2.2, 则$\lim\limits_{n\to\infty}\eta_{M_n}=0$. 结合推论2.1和推论2.2, 有

$$\frac{4s-3}{4s}\varepsilon_{M_n}-\frac{6s-3}{2s}\eta_{M_n}\le\frac1p\mathrm{C}_3^{4-p}\mathrm{C}_{\mathrm{GN}}(4)^{p-3}M_n^{\frac{(4s-3)(p-3)}{2s}}$$

$$\Big[\frac{4s+3p-12}{4s}\varepsilon_{M_n}-\frac{3p+2sp-12}{2s}\eta_{M_n}\Big]^{\frac{(3-s)p+4s-9}{2s}}\Big[\frac{4s-3p+6}{2s}\varepsilon_{M_n}-\frac{2sp-3p+6}{s}\eta_{M_n}\Big]^{2-\frac{p}{2}}.$$

取极限$n \to \infty$, 可得

$$\begin{aligned}\frac{4s-3}{4s} \le & \frac{1}{p}\mathrm{C}_3^{4-p}\mathrm{C}_{\mathrm{GN}}(4)^{p-3}M_c^{\frac{(4s-3)(p-3)}{2s}}\left(\frac{4s+3p-12}{4s}\right)^{\frac{(3-s)p+4s-9}{2s}} \\ & \left(\frac{4s-3p+6}{2s}\right)^{2-\frac{p}{2}}\liminf_{n\to\infty}\varepsilon_{M_n}^{\frac{(3-2s)(p-3)}{2s}}.\end{aligned}$$

那么$\liminf\limits_{n\to\infty}\varepsilon_{M_n} > 0$且因此

$$\liminf_{n\to\infty}\int_{\mathbb{R}^3}|u_n|^p\,\mathrm{d}x > 0,$$

从而序列$\{u_n\}_{n\ge1}$满足非消失条件. 在相差一个平移下, 序列$\{u_n\}_{n\ge1}$存在子列在$H^s(\mathbb{R}^3)$上弱收敛, 在$L^2_{\mathrm{loc}}(\mathbb{R}^3)$上强收敛, 在$\mathbb{R}^3$上几乎处处收敛于一个非零函数$u_\infty$. 这一子列可以仍用序列$\{u_n\}_{n\ge1}$来表示, 那么我们有

$$0 = \lim_{n\to\infty} I_{M_n} = \lim_{n\to\infty} E(u_n) = E(u_\infty) + \lim_{n\to\infty} E(u_n - u_\infty).$$

由$0 < |u_\infty|_2^2 \le M_c$可得$I_{|u_\infty|_2^2} = 0$, 则$E(u_\infty) \ge 0$. 同样的, 由

$$\lim_{n\to\infty}|u_n - u_\infty|_2^2 = M_c - |u_\infty|_2^2 < M_c,$$

可得$\lim\limits_{n\to\infty} E(u_n - u_\infty) \ge 0$. 因此, $|u_\infty|_2^2 = M_c$且$E(u_\infty) = 0$. 否则, 对于某些$M < M_c$, u_∞是I_M的极小元, 这与1)矛盾.

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Existence of Normalized Solutions for Fractional Schrödinger-Poisson System

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Abstract: This paper is devoted to studying the existence of the normalized solutions for fractional Schrödinger-Poisson system. We first transform the normalized solutions into minimizers of the constraint minimization problem under the variational framework, then the existence and nonexistence of the minimizers are proved by the concentration-compactness principle.

Key words: Fractional Schrödinger-Poisson system; Variational methods; Concentration-compactness principle; Normalized solution