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Calderón-Zygmund Operators and Commutators on Morrey-Herz Spaces with Non-Homogeneous Metric Measure

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Abstract: Let $(\mathcal{X}, d, \mu)$ be a non-homogeneous metric measure space satisfying both the geometrically doubling and the upper doubling conditions. In this paper, the Morrey-Herz spaces on the non-homogeneous metric measure space are introduced. Then, by the properties of the non-homogeneous metric measure space, in particular the η-weak reverse doubling condition, the boundedness of Calderón-Zygmund operators and their commutators on the Morrey-Herz spaces with non-homogeneous metric measure is obtained.
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1. Introduction

As we all know, the Herz spaces, the Herz type Hardy spaces and the Morrey-Herz spaces have been discussed by LU, YANG and other authors. And, some results of boundedness for singular integral operators on Herz type spaces are studied.[1−8]

Recall that a metric space equipped with a doubling condition measure is called a homogeneous space. In general, the doubling condition is important in the classical harmonic analysis. Nevertheless, it has been proved that the results in the classical function spaces and the boundedness of singular integrals on $\mathbb{R}^n$ are remained valid if it is replaced by a non-doubling condition, some examples can follow in [9–15].

After that, Hytönen introduced the non-homogeneous metric measure space which satisfying geometrically doubling and the upper doubling conditions in [16]. This kind of space contains both homogeneous type space and non-doubling metric measure space. Some results of non-homogeneous metric measure space and the boundedness of various operators on the spaces can be found in [17–24], etc. YANG and his collaborators[25−26] used the discrete coefficients to introduce the Hardy spaces on the non-homogeneous metric measure space,

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and discussed some characterizations of them. In [27], HAN and ZHAO introduced the Herz spaces and Herz type Hardy spaces on the non-homogeneous metric measure space. They also proved some characterizations for the spaces, and obtained some boundedness of singular operators on the Herz type Hardy spaces.

In this paper, motivated by the statements above, we investigate the Morrey-Herz spaces on the non-homogeneous metric measure space which satisfies the geometrically doubling and the upper doubling conditions. We obtain the boundedness of Calderón-Zygmund operators and their commutators on the Morrey-Herz spaces.

2. Preliminaries

For convenience, in this section, we recall some fundamental knowledge for the non-homogeneous metric measure space.

The following is the geometrically doubling, which was original introduced by Coifman and Weiss in [9], and is also known as metrically doubling in [28].

Definition 2.1 A metric space $(\mathcal{X}, d)$ is said to be geometrically doubling if there exists a $N_0 \in \mathbb{N}$ such that, for any ball $B(x, r) \subset \mathcal{X}$ with $x \in \mathcal{X}$ and $r \in (0, \infty)$, there exists a finite ball covering $\{B(x_i, r/2)\}_i$ of $B(x, r)$ such that the cardinality of this covering is at most N_0.

Lemma 2.1[16] Let $(\mathcal{X}, d)$ be a metric space. Then the following statements are mutually equivalent:

(i) $(\mathcal{X}, d)$ is geometrically doubling;

(ii) For any $\epsilon \in (0, 1)$ and any ball $B(x, r) \subset \mathcal{X}$ with $x \in \mathcal{X}$ and $r \in (0, \infty)$, there exists a finite ball covering $\{B(x_i, \epsilon r)\}_i$ of $B(x, r)$ such that the cardinality of this covering is at most $N_0\epsilon^{-n_0}$, here and hereafter, N_0 is as in Definition 2.1 and $n_0 := \log_2 N_0$;

(iii) For every $\epsilon \in (0, 1)$, any ball $B(x, r) \subset \mathcal{X}$ with $x \in \mathcal{X}$ and $r \in (0, \infty)$ contains at most $N_0\epsilon^{-n_0}$ centers of disjoint balls $\{B(x_i, \epsilon r)\}_i$;

(iv) There exists $M \in \mathbb{N}$ such that any ball $B(x, r) \subset \mathcal{X}$ with $x \in \mathcal{X}$ and $r \in (0, \infty)$ contains at most M centers $\{x_i\}_i$ of disjoint balls $\{B(x_i, r/4)\}_{i=1}^M$.

The upper doubling is as follows.

Definition 2.2[16] A metric measure space $(\mathcal{X}, d, \mu)$ is said to be upper doubling if μ is a Borel measure on $\mathcal{X}$ and there exist a dominating function $\lambda : \mathcal{X} \times (0, \infty) \to (0, \infty)$ and a positive constant $C_{(\lambda)}$, such that, for each $x \in \mathcal{X}, r \to \lambda(x, r)$ is non-decreasing and, for all $x \in \mathcal{X}$ and $r \in (0, \infty)$,

$$\mu(B(x, r)) \leqslant \lambda(x, r) \leqslant C_{(\lambda)}\lambda(x, \frac{r}{2}). \tag{2.1}$$

A metric measure space $(\mathcal{X}, d, \mu)$ is called a non-homogeneous metric measure space if $(\mathcal{X}, d)$ is geometrically doubling and $(\mathcal{X}, d, \mu)$ is upper doubling.

Remark 2.1 It is obvious that the homogeneous type space is a special case of upper doubling space, since we can take the dominating function $\lambda(x, r) := \mu(B(x, r)$ for all $x \in \mathcal{X}$ and $r \in (0, \infty)$. The D-dimensional Euclidean space $\mathbb{R}^D$ with any Radon measure μ (where $\mu(B(x, r) \leqslant C_0 r^n)$ is also an upper doubling space if we set $\lambda(x, r) := C_0 r^k$ for all $x \in \mathbb{R}^D$ and $r \in (0, \infty)$.

Lemma 2.2[25] Let $(\mathcal{X}, d, \mu)$ be upper doubling, λ be a dominating function on $\mathcal{X} \times (0, \infty)$ as in Definition 2.2. Then there exists another dominating function $\tilde{\lambda}$ such that $\tilde{\lambda} \leq$

$\lambda, C_{(\tilde{\lambda})} \leqslant C_{(\lambda)}$ and, for all $x, y \in \mathcal{X}$, with $d(x,y) \leqslant r$,

$$\tilde{\lambda}(x,r) \leqslant C_{(\tilde{\lambda})}\tilde{\lambda}(y,r). \tag{2.2}$$

Lemma 2.3[15] The upper doubling condition is equivalent to the weak growth condition: there exist a dominating function $\lambda : \mathcal{X} \times (0,\infty) \to (0,\infty)$, with $r \to \lambda(x,r)$ non-decreasing, positive constants $C_{(\lambda)}$, depending on λ, and $\epsilon > 0$ such that

(i) For all $r \in (0,\infty), t \in [0,r], x, y \in \mathcal{X}$ and $d(x,y) \in (0,r)$,

$$|\lambda(y, r+t) - \lambda(x,r)| \leqslant C_{(\lambda)} \left[\frac{d(x,y)+t}{r}\right]^{\epsilon} \lambda(x,r);$$

(ii) For all $x \in \mathcal{X}$ and $r \in (0,\infty), \mu(B(x,r)) \leqslant \lambda(x,r)$.

Based on Lemma 2.2, from now on, we always assume that $(\mathcal{X}, d, \mu)$ is the non-homogeneous metric measure space with the dominating function λ satisfying (2.2).

In [16], Hytönen showed that, if $(\mathcal{X}, d, \mu)$ satisfies the geometrically doubling condition and the upper doubling condition, then there exist many small and large balls that have the following (α, β)-doubling property.

Definition 2.3[16] Let $\alpha, \beta > 1$. If $\mu(\alpha B) \leqslant \beta\mu(B)$, then the ball $B \subset \mathcal{X}$ is called a (α,β)-doubling ball, where for any ball $B := B(c_B, r_B)$ and $\rho \in (0,\infty)$, $\rho B := B(c_B, \rho r_B)$.

In [16], Hytönen proved that if $(\mathcal{X}, d, \mu)$ is a metric measure space with upper doubling, $\alpha, \beta > 1$ and $\beta > C_{\lambda}^{\log_\alpha 2} =: \alpha^v$, then, for every ball $B \subset \mathcal{X}$, there exists a positive integer j such that $\alpha^j B$ is (α,β)-doubling. Furthermore, if we let $(\mathcal{X}, d)$ be geometrically doubling, $\beta > \alpha^{n_0}$, μ be a Borel measure on $\mathcal{X}$ with finite on bounded sets, then, for μ-almost every $x \in \mathcal{X}$, there exist arbitrary small (α,β)-doubling balls centered at x and their radii might be chosen to be of the form $\alpha^j r$ with $j \in \mathbb{Z}_+$, and r is a preassigned number belonging to $(0,\infty)$.

Definition 2.4[16] Let $\eta > 0$. A dominating function λ is said to satisfy the η-weak reverse doubling condition if, for all $r \in (0, 2\mathrm{diam}(\mathcal{X}))$ and $a \in (1, 2\mathrm{diam}(\mathcal{X})/r)$, there exists a number $C(a)$ >1 which depend only on a and $\mathcal{X}$, such that, for all $x \in \mathcal{X}$, $\lambda(x, ar) \geqslant C(a)\lambda(x,r)$, and

$$\sum_{k=1}^{\infty} \frac{1}{[C(a^k)]^\eta} < \infty. \tag{2.3}$$

It is easy to see that, if λ satisfies η_1-weak reverse doubling condition and $\eta_1 < \eta_2$, then λ also satisfies the η_2-weak reverse doubling condition.

In the non-homogeneous metric measure space $(\mathcal{X}, d, \mu)$, the discrete coefficient $\widetilde{K}_{B,S}^{(\rho),p}$ is important to study many properties. Thus, we recall the definition of $\widetilde{K}_{B,S}^{(\rho),p}$ as follows.

Definition 2.5[16, 22] For any two balls $B \subset S \subset \mathcal{X}$, $\rho > 1, p \in (0,1]$, set

$$\widetilde{K}_{B,S}^{(\rho),p} := \left\{1 + \sum_{k=-[\log_\rho 2]}^{N_{B,S}^{(\rho)}} \left[\frac{\mu(\rho^k B)}{\lambda(c_B, \rho^k r_B)}\right]^p\right\}^{1/p}, \tag{2.4}$$

where $N_{B,S}^{(\rho)}$ is the smallest integer which satisfies $\rho^{N_{B,S}^{(\rho)}} r_B \geqslant r_S$. The continuous form is

$$K_{B,S} = 1 + \int_{2S\backslash B} \frac{1}{\lambda(c_B, d(x,c_B))}\,\mathrm{d}\mu(x).$$

Let $k \in \mathbb{Z}, B_k = \left\{x \in \mathcal{X} : d(0,x) < 2^k\right\}, C_k = B_k\backslash B_{k-1}$, and $\chi_k = \chi_{C_k}$. The homogeneous Herz space on the non-homogeneous metric measure space is as follows.

Definition 2.6[27] Let $-\infty < \alpha < \infty, 0 < p < \infty, 0 < q \leqslant \infty$. Suppose $(\mathcal{X}, d, \mu)$ is a non-homogeneous metric measure space, the homogeneous Herz space on the non-homogeneous metric measure space $\dot{K}_q^{\alpha,p}(\mu)$ is defined by

$$\dot{K}_q^{\alpha,p}(\mu) = \left\{ f \in L_{\mathrm{loc}}^q(\mathcal{X} \setminus \{0\}) : \|f\|_{\dot{K}_q^{\alpha,p}(\mu)} < \infty \right\}, \tag{2.5}$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}(\mu)} = \left\{ \sum_{k=-\infty}^{+\infty} [\lambda(0, 2^k)]^{\alpha p} \|f\chi_k\|_{L^q(\mu)}^p \right\}^{\frac{1}{p}}. \tag{2.6}$$

3. Morrey-Herz Space and Main Result

In this section, we introduce the Morrey-Herz spaces on non-homogeneous metric measure space. And the boundedness of Calderón-Zygmund operators and commutators on the Morrey-Herz spaces with non-homogeneous metric measure is discussed.

The definition of Morrey-Herz spaces on non-homogeneous metric measure space is as follows.

Definition 3.1 Let $-\infty < \alpha < \infty, 0 \leqslant \nu < \infty, 0 < p < \infty, 0 < q \leqslant \infty$. Suppose $(\mathcal{X}, d, \mu)$ is a non-homogeneous metric measure space, the homogeneous Morrey-Herz space on the non-homogeneous metric measure space $M\dot{K}_{p,q}^{\alpha,\nu}(\mu)$ is defined by

$$M\dot{K}_{p,q}^{\alpha,\nu}(\mu) = \left\{ f \in L_{\mathrm{loc}}^q(\mathcal{X} \setminus \{0\}) : \|f\|_{M\dot{K}_{p,q}^{\alpha,\nu}(\mu)} < \infty \right\}, \tag{3.1}$$

where

$$\|f\|_{M\dot{K}_{p,q}^{\alpha,\nu}(\mu)} = \sup_{k_0 \in \mathbb{Z}} [\lambda(0, 2^{k_0})]^{-\nu} \left\{ \sum_{k=-\infty}^{k_0} [\lambda(0, 2^k)]^{\alpha p} \|f\chi_k\|_{L^q(\mu)}^p \right\}^{\frac{1}{p}}. \tag{3.2}$$

Obviously, if $\nu = 0$, then $M\dot{K}_{p,q}^{\alpha,0}(\mu) = \dot{K}_q^{\alpha,p}(\mu)$.

The following definition is the Calderón-Zygmund operators on non-homogeneous metric measure space.

Definition 3.2[17] A function $K \in L_{\mathrm{loc}}^1(\mathcal{X} \times \mathcal{X} \setminus \{(x,x) : x \in \mathcal{X}\})$ is said to be a Calderón-Zygmund kernel on non-homogeneous metric measure space, if there exists a positive constant $C_{(K)}$, such that

(i) For any $x, y \in \mathcal{X}, x \neq y$,

$$|K(x,y)| \leqslant C_{(K)} \frac{1}{\lambda(x, d(x,y))}; \tag{3.3}$$

(ii) There exist a constat $0 < \delta \leqslant 1$ and $c_{(K)}$, such that, for every $x, \tilde{x}, y \in \mathcal{X}, d(x,y) \geqslant c_{(K)} d(x, \tilde{x})$,

$$|K(x,y) - K(\tilde{x},y)| + |K(y,x) - K(y,\tilde{x})| \leqslant C_{(K)} \frac{[d(x,\tilde{x})]^\delta}{[d(x,y)]^\delta \lambda(x, d(x,y))}. \tag{3.4}$$

A linear operator T is said to be a Calderón-Zygmund operator on non-homogeneous metric measure space with kernel K satisfying (3.3) and (3.4) if, for all $f \in L_b^\infty(\mu) := \{f \in L^\infty(\mu) : \ \mathrm{supp}(f) \ \text{ is bounded}\}$,

$$Tf(x) := \int_{\mathcal{X}} K(x,y) f(y) \, \mathrm{d}\mu(y), \ x \notin \mathrm{supp}(f). \tag{3.5}$$

Some boundedness about Calderón-Zygmund operators on non-homogeneous metric measure space can be found in [17, 20, 22], etc.

Definition 3.3[22] Let $\rho \in (1, \infty)$. A function $f \in L^1_{\rm loc}(\mu)$ is said to be in the space RBMO(μ), if there exist a positive constant C and, for any ball $B \subset \mathcal{X}$, a number f_B such that

$$\frac{1}{\mu(\rho B)} \int_B |f(x) - f_B| \, {\rm d}\mu(x) \leqslant C \tag{3.6}$$

and, for any two balls B and S such that $B \subset S \subset \mathcal{X}$,

$$|f_B - f_S| \leqslant C K_{B,S}. \tag{3.7}$$

The infimum of the positive constants C above is defined to be the RBMO(μ) norm of f and denoted by $\|f\|_{\rm RBMO}$.

The authors, in [22], pointed out that the space RBMO(μ) is independent of ρ. In this paper, we choose $\rho = 2$.

The commutator generated by operator T and function b is defined by

$$[T, b](f)(x) := b(x)Tf(x) - T(bf)(x) \quad \text{for } x \in \mathcal{X}. \tag{3.8}$$

The following lemmas are useful for us to prove the boundedness of the Calderón-Zygmund operators on some spaces.

Lemma 3.1[18] Suppose $(\mathcal{X}, d, \mu)$ is a non-homogeneous metric measure space. Let T be a Calderón-Zygmund operator on non-homogeneous metric measure space. Then the following are equivalent:

(i) T is bounded on $L^2(\mu)$;

(ii) T is bounded on $L^q(\mu)$, for $q > 1$;

(iii) T is bounded from $L^1(\mu)$ to weak-$L^1(\mu)$.

Lemma 3.2[22] Suppose $(\mathcal{X}, d, \mu)$ is a non-homogeneous metric measure space. Let T be a Calderón-Zygmund operator on non-homogeneous metric measure space, $b \in$RBMO(μ). If T is bounded on $L^2(\mu)$, then the commutator $[T, b]$ is bounded on $L^q(\mu)$, for $1 < q < \infty$.

The main results in this paper are the following theorems.

Theorem 3.1 Let $(\mathcal{X}, d, \mu)$ be a non-homogeneous metric measure space. Set $0 < p < \infty$, $1 < q < \infty$, $0 < \nu < \infty$, and $\nu < \alpha < 1 - \frac{1}{q} + \nu$. Suppose λ satisfies the η-weak reverse doubling condition, $\eta \in (0, \min\{\nu p, \alpha - \nu, 1 - 1/q - \alpha + \nu\})$. Let T be a Calderón-Zygmund operator on non-homogeneous metric measure space. If T is bounded on $L^2(\mu)$, then T is bounded on $M\dot{K}^{\alpha,\nu}_{p,q}(\mu)$.

Theorem 3.2 Let $(\mathcal{X}, d, \mu)$ be a non-homogeneous metric measure space. Set $0 < p < \infty$, $1 < q < \infty$, $0 < \nu < \infty$, and $\nu < \alpha < 1 - \frac{1}{q} + \nu$. Suppose λ satisfies the η-weak reverse doubling condition, $\eta \in (0, \min\{\nu p, \alpha - \nu, 1 - 1/q - \alpha + \nu\})$. Let $[T, b]$ be a commutator defined by (3.8) and $b \in$ RBMO(μ). If the Calderón-Zygmund operator on non-homogeneous metric measure space T is bounded on $L^2(\mu)$, then the commutator $[T, b]$ is bounded on $M\dot{K}^{\alpha,\nu}_{p,q}(\mu)$.

Proof of Theorem 3.1 For any $f \in M\dot{K}^{\alpha,\nu}_{p,q}(\mu)$, we suppose that $f(x) = \sum\limits_{j=-\infty}^{\infty} f\chi_j(x) = \sum\limits_{j=-\infty}^{\infty} f_j(x)$. Therefore, we continue write

$$\begin{aligned} \|Tf\|^p_{M\dot{K}^{\alpha,\nu}_{p,q}(\mu)} &= \sup_{k_0 \in \mathbb{Z}} [\lambda(0, 2^{k_0})]^{-\nu p} \sum_{l=-\infty}^{k_0} [\lambda(0, 2^l)]^{\alpha p} \, \|(Tf)\chi_l\|^p_{L^q(\mu)} \\ &\leqslant C \sup_{k_0 \in \mathbb{Z}} [\lambda(0, 2^{k_0})]^{-\nu p} \sum_{l=-\infty}^{k_0} [\lambda(0, 2^l)]^{\alpha p} (\sum_{j=-\infty}^{l-2} \|(Tf_j)\chi_l\|_{L^q(\mu)})^p \end{aligned}$$

$$
\begin{aligned}
&+ C \sup_{k_0\in\mathbb{Z}}[\lambda(0,2^{k_0})]^{-\nu p}\sum_{l=-\infty}^{k_0}[\lambda(0,2^l)]^{\alpha p}(\sum_{j=l-1}^{l+1}\|(Tf_j)\chi_l\|_{L^q(\mu)})^p \\
&+ C \sup_{k_0\in\mathbb{Z}}[\lambda(0,2^{k_0})]^{-\nu p}\sum_{l=-\infty}^{k_0}[\lambda(0,2^l)]^{\alpha p}(\sum_{j=l+2}^{+\infty}\|(Tf_j)\chi_l\|_{L^q(\mu)})^p \\
&=: I_1+I_2+I_3. \qquad (3.9)
\end{aligned}
$$

As for I_2, by Lemma 3.1, we know that T is bounded on $L^q(\mu)$. Thus, we easily get that

$$I_2 \leqslant C \sup_{k_0\in\mathbb{Z}}[\lambda(0,2^{k_0})]^{-\nu p}\sum_{l=-\infty}^{k_0}[\lambda(0,2^l)]^{\alpha p}\,\|f\chi_l\|^p_{L^q(\mu)} = C\|f\|^p_{M\dot{K}^{\alpha,\nu}_{p,q}(\mu)}.$$

As for I_1, observing that if $j\leqslant l-2, x\in C_l, y\in B_j$, then $x\in\mathcal{X}\setminus 2B_j$, which implies that

$$\lambda(x,d(x,y))\sim\lambda(0,d(x,0)).$$

Thus, by Definition 3.2 and the Hölder inequality, we conclude that

$$
\begin{aligned}
\|(Tf_j)\chi_l\|_{L^q(\mu)} &= \left\{\int_{C_l}|(Tf_j)\chi_l|^q\,\mathrm{d}\mu(x)\right\}^{1/q} \\
&\leqslant C\left\{\int_{C_l}\left|\int_{C_j}\frac{|f_j(y)|}{\lambda(x,d(x,y))}\,\mathrm{d}\mu(y)\right|^q\mathrm{d}\mu(x)\right\}^{1/q} \\
&\leqslant C\left\{\int_{C_l}\frac{1}{[\lambda(0,d(x,0))]^q}\left|\int_{C_j}|f_j(y)|\,\mathrm{d}\mu(y)\right|^q\mathrm{d}\mu(x)\right\}^{1/q} \\
&\leqslant C\left\{\int_{C_l}\frac{1}{[\lambda(0,d(x,0))]^q}\,\mathrm{d}\mu(x)\right\}^{1/q}\int_{B_j}|f_j(y)|\,\mathrm{d}\mu(y) \\
&\leqslant C\frac{[\mu(B_l)]^{1/q}}{\lambda(0,2^l)}\,[\mu(B_j)]^{\frac{1}{q'}}\,\|f_j\|_{L^q(\mu)}\,.
\end{aligned}
$$

Then, by the obvious formula

$$\|f_j\|_{L^q(\mu)}\leqslant[\lambda(0,2^j)]^{-\alpha}\left\{\sum_{i=-\infty}^{j}[\lambda(0,2^i)]^{\alpha p}\|f_i\|^p_{L^q(\mu)}\right\}^{1/p}, \qquad (3.10)$$

we obtain

$$\|(Tf_j)\chi_l\|_{L^q(\mu)}\leqslant C\frac{[\mu(B_l)]^{1/q}}{\lambda(0,2^l)}\,[\mu(B_j)]^{\frac{1}{q'}}\,[\lambda(0,2^j)]^{\nu-\alpha}\,\|f\|_{M\dot{K}^{\alpha,\nu}_{p,q}(\mu)}\,.$$

Thus, formula (2.1) and the η-weak reverse doubling condition can tell us

$$
\begin{aligned}
I_1 &\leqslant C\sup_{k_0\in\mathbb{Z}}\sum_{l=-\infty}^{k_0}\frac{[\lambda(0,2^l)]^{\alpha p}}{[\lambda(0,2^{k_0})]^{\nu p}}\left(\sum_{j=-\infty}^{l-2}\frac{[\mu(B_l)]^{1/q}}{\lambda(0,2^l)}\,[\mu(B_j)]^{\frac{1}{q'}}\,[\lambda(0,2^j)]^{\nu-\alpha}\,\|f\|_{M\dot{K}^{\alpha,\nu}_{p,q}(\mu)}\right)^p \\
&\leqslant C\sup_{k_0\in\mathbb{Z}}\sum_{l=-\infty}^{k_0}\frac{[\lambda(0,2^l)]^{\alpha p}}{[\lambda(0,2^{k_0})]^{\nu p}}\left(\sum_{j=-\infty}^{l-2}\frac{[\lambda(0,2^l)]^{1/q}}{\lambda(0,2^l)}\frac{[\lambda(0,2^j)]^{1/q'}[\lambda(0,2^j)]^{\nu}}{[\lambda(0,2^j)]^{\alpha}}\right)^p\|f\|^p_{M\dot{K}^{\alpha,\nu}_{p,q}(\mu)} \\
&\leqslant C\,\|f\|^p_{M\dot{K}^{\alpha,\nu}_{p,q}(\mu)}\sup_{k_0\in\mathbb{Z}}\sum_{l=-\infty}^{k_0}\frac{[\lambda(0,2^l)]^{\nu p}}{[\lambda(0,2^{k_0})]^{\nu p}}\left(\sum_{j=-\infty}^{l-2}\left[\frac{\lambda(0,2^j)}{\lambda(0,2^l)}\right]^{1-\frac{1}{q}+\nu-\alpha}\right)^p \\
&\leqslant C\,\|f\|^p_{M\dot{K}^{\alpha,\nu}_{p,q}(\mu)}\sup_{k_0\in\mathbb{Z}}\sum_{l=-\infty}^{k_0}\left[\frac{\lambda(0,2^l)}{\lambda(0,2^{k_0})}\right]^{\nu p}
\end{aligned}
$$

$$\leqslant C\,\|f\|^p_{M\dot{K}^{\alpha,\nu}_{p,q}(\mu)}.$$

For I_3, by the boundedness of T on $L^q(\mu)$, and (3.10), we can have

$$\begin{aligned} I_3 &\leqslant C\sup_{k_0\in\mathbb{Z}}[\lambda(0,2^{k_0})]^{-\nu p}\sum_{l=-\infty}^{k_0}[\lambda(0,2^l)]^{\alpha p}(\sum_{j=l+2}^{+\infty}\|f_j\|_{L^q(\mu)})^p \\ &\leqslant C\,\|f\|^p_{M\dot{K}^{\alpha,\nu}_{p,q}(\mu)}\sup_{k_0\in\mathbb{Z}}[\lambda(0,2^{k_0})]^{-\nu p}\sum_{l=-\infty}^{k_0}[\lambda(0,2^l)]^{\alpha p}(\sum_{j=l+2}^{+\infty}[\lambda(0,2^j)]^{\nu-\alpha})^p. \end{aligned}$$

Similarly, by the η-weak reverse doubling condition, one can obtain that

$$\begin{aligned} I_3 &\leqslant C\,\|f\|^p_{M\dot{K}^{\alpha,\nu}_{p,q}(\mu)}\sup_{k_0\in\mathbb{Z}}[\lambda(0,2^{k_0})]^{-\nu p}\sum_{l=-\infty}^{k_0}[\lambda(0,2^l)]^{\nu p}\left(\sum_{j=l+2}^{+\infty}\left[\frac{\lambda(0,2^l)}{\lambda(0,2^j)}\right]^{\alpha-\nu}\right)^p \\ &\leqslant C\,\|f\|^p_{M\dot{K}^{\alpha,\nu}_{p,q}(\mu)}\sup_{k_0\in\mathbb{Z}}\sum_{l=-\infty}^{k_0}\left[\frac{\lambda(0,2^l)}{\lambda(0,2^{k_0})}\right]^{\nu p} \\ &\leqslant C\,\|f\|^p_{M\dot{K}^{\alpha,\nu}_{p,q}(\mu)}. \end{aligned}$$

Therefore, together with (3.9), we known that

$$\|Tf\|_{M\dot{K}^{\alpha,\nu}_{p,q}(\mu)}\leqslant C\,\|f\|_{M\dot{K}^{\alpha,\nu}_{p,q}(\mu)}.$$

This completes the proof of Theorem 3.1.

Proof of Theorem 3.2 For any $f\in M\dot{K}^{\alpha,\nu}_{p,q}(\mu)$, set $f(x)=\sum\limits_{j=-\infty}^{\infty}f\chi_j(x)=\sum\limits_{j=-\infty}^{\infty}f_j(x)$. Then, write

$$\begin{aligned} \|[T,b]f\|^p_{M\dot{K}^{\alpha,\nu}_{p,q}(\mu)} &\leqslant C\sup_{k_0\in\mathbb{Z}}[\lambda(0,2^{k_0})]^{-\nu p}\sum_{l=-\infty}^{k_0}[\lambda(0,2^l)]^{\alpha p}(\sum_{j=-\infty}^{l-1}\|([T,b]f_j)\chi_l\|_{L^q(\mu)})^p \\ &\quad + C\sup_{k_0\in\mathbb{Z}}[\lambda(0,2^{k_0})]^{-\nu p}\sum_{l=-\infty}^{k_0}[\lambda(0,2^l)]^{\alpha p}(\sum_{j=l}^{+\infty}\|([T,b]f_j)\chi_l\|_{L^q(\mu)})^p \\ &=: J_1+J_2. \end{aligned}$$

For J_2, by Lemma 3.2, formula (3.10), and the η-weak reverse doubling condition, we can obtain that

$$\begin{aligned} J_2 &\leqslant C\sup_{k_0\in\mathbb{Z}}[\lambda(0,2^{k_0})]^{-\nu p}\sum_{l=-\infty}^{k_0}[\lambda(0,2^l)]^{\alpha p}(\sum_{j=l}^{+\infty}\|b\|_{\mathrm{RBMO}}\,\|f_j\|_{L^q(\mu)})^p \\ &\leqslant C\|b\|^p_{\mathrm{RBMO}}\,\|f\|^p_{M\dot{K}^{\alpha,\nu}_{p,q}(\mu)}\sup_{k_0\in\mathbb{Z}}\sum_{l=-\infty}^{k_0}\left[\frac{\lambda(0,2^l)}{\lambda(0,2^{k_0})}\right]^{\nu p}\left(\sum_{j=l}^{+\infty}\left[\frac{\lambda(0,2^l)}{\lambda(0,2^j)}\right]^{\alpha-\nu}\right)^p \\ &\leqslant C\|b\|^p_{\mathrm{RBMO}}\,\|f\|^p_{M\dot{K}^{\alpha,\nu}_{p,q}(\mu)}. \end{aligned}$$

It remains to estimate J_1. Similarly, by Definition 3.2 and the Hölder inequality, using the properties of RBMO space and (3.10), we can have that

$$\begin{aligned} \|([T,b]f_j)\chi_l\|_{L^q(\mu)} &\leqslant C\left\{\int_{C_l}\left(\int_{C_j}\frac{|(b(x)-b(y))f_j(y)|}{\lambda(x,d(x,y))}\,\mathrm{d}\mu(y)\right)^q\mathrm{d}\mu(x)\right\}^{1/q} \\ &\leqslant C\frac{1}{\lambda(0,2^l)}\left\{\int_{C_l}|b(x)-b_j|^q\,\mathrm{d}\mu(x)\right\}^{1/q}\int_{C_j}|f_j(y)|\,\mathrm{d}\mu(y) \end{aligned}$$

$$
\begin{aligned}
&+C\frac{1}{\lambda(0,2^l)}\left\{\int_{C_l}\mathrm{d}\mu(x)\right\}^{1/q}\int_{C_j}|b_j-b(y)||f_j(y)|\,\mathrm{d}\mu(y)\\
&\leqslant C\frac{1}{\lambda(0,2^l)}\|b\|_{\mathrm{RBMO}}\mu(B_l)^{1/q}\|f_j\|_{L^q}\mu(B_j)^{1-1/q}\\
&\leqslant C\|b\|_{\mathrm{RBMO}}\,\|f\|_{M\dot{K}^{\alpha,\nu}_{p,q}(\mu)}\frac{\lambda(0,2^j)^{1-1/q}}{\lambda(0,2^l)^{1-1/q}}\lambda(0,2^j)^{\nu-\alpha}.
\end{aligned}
$$

Therefore, by the η-weak reverse doubling condition, there is

$$
\begin{aligned}
J_1&\leqslant C\|b\|^p_{\mathrm{RBMO}}\,\|f\|^p_{M\dot{K}^{\alpha,\nu}_{p,q}(\mu)}\sup_{k_0\in\mathbb{Z}}\sum_{l=-\infty}^{k_0}\frac{[\lambda(0,2^l)]^{\nu p}}{[\lambda(0,2^{k_0})]^{\nu p}}\left(\sum_{j=-\infty}^{l-1}\frac{[\lambda(0,2^j)]^{\nu+1-1/q-\alpha}}{[\lambda(0,2^l)]^{\nu+1-1/q-\alpha}}\right)^p\\
&\leqslant C\|b\|^p_{\mathrm{RBMO}}\,\|f\|^p_{M\dot{K}^{\alpha,\nu}_{p,q}(\mu)}.
\end{aligned}
$$

Thus, it means that Theorem 3.2 is proved.

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非齐度量测度空间上Morrey-Herz空间上的Calderón-Zygmund算子及其交换子

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摘要： 设$(\mathcal{X},d,\mu)$ 是一个同时满足上双倍条件和几何双倍条件的非齐度量测度空间, 本文中, 引进一类非齐度量测度空间上的Morrey-Herz 空间, 利用非齐度量测度空间的特征, 特别是η-弱逆倍条件, 证明Calderón-Zygmund算子及其交换子在Morrey-Herz空间上的有界性.

关键词： 非齐度量测度空间; Morrey-Herz空间; Calderón-Zygmund算子; 交换子; 有界性