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一年龄结构乙肝传染病模型及稳定性

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摘要： 本文讨论一年龄结构的乙肝传染病模型, 得到基本再生数$\Re_0$的表达式, 证明当$\Re_0 < 1$时, 无病平衡点局部渐近稳定且全局渐近稳定; 当$\Re_0 > 1$时, 存在唯一的地方病平衡点, 并给出地方病平衡点的局部渐近稳定性条件, 这些条件对于控制疾病的传播具有重要的理论及实际意义.

关键词： 年龄结构; 潜伏类; 基本再生数
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1. 引言

2017年10月27日, 世界卫生组织国际癌症研究机构公布的致癌物清单初步整理参考, 乙型肝炎病毒(慢性感染)在一类致癌物清单中. 慢性乙型肝炎(简称乙肝)是指乙肝病毒检测为阳性, 病程超过半年或发病日期不明确而临床有慢性肝炎表现者. 临床表现为乏力、畏食、恶心、腹胀、肝区疼痛等症状. 从肝炎病毒入侵到临床出现最初症状以前, 这段时期称为潜伏期[1−4]. 乙肝潜伏期为6周～6个月, 一般为3个月. 潜伏期随病原体的种类、数量、毒力、人体免疫状态而长短不一.

因为不同年龄的人对乙肝的免疫能力、潜伏期长短、感染能力及传播能力不同, 故年龄对乙肝传播的影响不可忽略[5−10]. 而不同体质的易感者接触乙肝病人后可能会直接发病, 也可能潜伏一段时间才表现出来, 因而讨论易感类人群接触病人后比例进入潜伏类和染病类的传染病模型更加符合实际.

2. 模型

把总人口分为易感类、潜伏类、染病类、免疫类, 分别用$S(a,t)$, $E(a,t)$, $I(a,t)$, $R(a,t)$表示各类年龄密度函数, a为年龄, t为时间. $\mu(a)$为年龄依赖自然死亡率, $[\varepsilon(a)]^{-1}$平均潜伏周期, $[\alpha(a)]^{-1}$为平均染病周期, $b(a)$为年龄依赖出生率. 令感染力函数[5]为

$$\lambda(a,t) = k(a)\int_0^{+\infty}\beta(a)I(a,t)\mathrm{d}a,$$

其中$\beta(a)$为年龄依赖的染病率, $k(a)$为年龄依赖的接触率. 不考虑因病死亡, 则易感类人群接

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触病人后按比例q和$1-q$进入潜伏类和染病类的年龄结构SEIR传染病模型为

$$\begin{cases} \dfrac{\partial S(a,t)}{\partial a}+\dfrac{\partial S(a,t)}{\partial t}=-\lambda(a,t)S(a,t)-\mu(a)S(a,t), \\ \dfrac{\partial E(a,t)}{\partial a}+\dfrac{\partial E(a,t)}{\partial t}=q\lambda(a,t)S(a,t)-[\varepsilon(a)+\mu(a)]E(a,t), \\ \dfrac{\partial I(a,t)}{\partial a}+\dfrac{\partial I(a,t)}{\partial t}=(1-q)\lambda(a,t)S(a,t)+\varepsilon(a)E(a,t)-[\alpha(a)+\mu(a)]I(a,t), \\ \dfrac{\partial R(a,t)}{\partial a}+\dfrac{\partial R(a,t)}{\partial t}=\alpha(a)I(a,t)-\mu(a)R(a,t). \end{cases} \tag{2.1}$$

把上面四个方程相加得总人口年龄密度函数$P(a,t)=S(a,t)+E(a,t)+I(a,t)+R(a,t)$,

$$\begin{cases} \dfrac{\partial P(a,t)}{\partial a}+\dfrac{\partial P(a,t)}{\partial t}=-\mu(a)P(a,t), \\ P(0,t)=\displaystyle\int_0^{+\infty}b(a)P(a,t)\mathrm{d}a, \\ P(a,0)=P_0(a)=S_0(a)+E_0(a)+I_0(a)+R_0(a). \end{cases} \tag{2.2}$$

这是一个标准的Mckendrick-von forester方程. 假设所有的参数都非负, 且

$$b(a),\beta(a)\in L^\infty[0,+\infty),\ \mu(a),\varepsilon(a),k(a),\alpha(a)\in C[0,+\infty),$$

$$\int_0^{+\infty}\mu(a)\mathrm{d}a=+\infty,\ a\in[0,+\infty).$$

假设当个体超过一定生育年龄时$b(a)=0$. 假设总人口处于稳定状态[6], 即假设

$$\int_0^{+\infty}b(a)\mathrm{e}^{-\int_0^a\mu(\tau)\mathrm{d}\tau}\mathrm{d}a=1, \tag{2.3}$$

$$P_\infty(a)=P(a,t)=b_0\mathrm{e}^{-\int_0^a\mu(\tau)\mathrm{d}\tau}.$$

设 $S_0(a)\geq 0,\ E_0(a)\geq 0,\ I_0(a)\geq 0,\ R_0(a)\geq 0,\ S_0(a)+E_0(a)+I_0(a)+R_0(a)=P_\infty(a)$. 则有

$$b_0=\frac{\int_0^{+\infty}P_\infty(a)\mathrm{d}a}{\int_0^{+\infty}\mathrm{e}^{-\int_0^a\mu(\tau)\mathrm{d}\tau}\mathrm{d}a}.$$

由(2.3)得

$$S(0,t)=\int_0^{+\infty}b(a)P_\infty(a)\mathrm{d}a=b_0.$$

对系统(2.1) 作归一化处理

$$s(a,t)=\frac{S(a,t)}{P_\infty(a)},\ e(a,t)=\frac{E(a,t)}{P_\infty(a)},\ i(a,t)=\frac{I(a,t)}{P_\infty(a)},\ r(a,t)=\frac{R(a,t)}{P_\infty(a)}.$$

则系统(2.1) 转化为

$$\begin{cases} \dfrac{\partial s(a,t)}{\partial a}+\dfrac{\partial s(a,t)}{\partial t}=-\lambda(a,t)s(a,t), \\ \dfrac{\partial e(a,t)}{\partial a}+\dfrac{\partial e(a,t)}{\partial t}=q\lambda(a,t)s(a,t)-\varepsilon(a)e(a,t), \\ \dfrac{\partial i(a,t)}{\partial a}+\dfrac{\partial i(a,t)}{\partial t}=(1-q)\lambda(a,t)s(a,t)+\varepsilon(a)e(a,t)-\alpha(a)i(a,t), \\ \dfrac{\partial r(a,t)}{\partial a}+\dfrac{\partial r(a,t)}{\partial t}=\alpha(a)i(a,t), \\ \lambda(a,t)=k(a)\displaystyle\int_0^{+\infty}\beta(a)P_\infty(a)i(a,t)\mathrm{d}a, \end{cases} \tag{2.4}$$

及边界条件

$$\begin{cases} s(a,t)+e(a,t)+i(a,t)+r(a,t)=1, \\ s(0,t)=1,\ e(0,t)=i(0,t)=r(0,t)=0, \\ s(a,0)=s_0(a), e(a,0)=e_0(a), i(a,0)=i_0(a), r(a,0)=r_0(a). \end{cases}$$

3. 无病平衡点及其稳定性

系统(2.4) 及其边界条件的平衡解满足

$$\begin{cases} \dfrac{\mathrm{d}s(a)}{\mathrm{d}a}=-\lambda(a,t)s(a), \\ \dfrac{\mathrm{d}e(a)}{\mathrm{d}a}=q\lambda(a,t)s(a,t)-\varepsilon(a)e(a,t), \\ \dfrac{\mathrm{d}i(a)}{\mathrm{d}a}=(1-q)\lambda(a,t)s(a,t)+\varepsilon(a)e(a,t)-\alpha(a)i(a,t), \\ \dfrac{\mathrm{d}r(a)}{\mathrm{d}a}=\alpha(a)i(a,t), \\ \lambda(a)=k(a)\displaystyle\int_0^{+\infty}\beta(a)P_\infty(a)i(a)\mathrm{d}a, \\ s(a)+e(a)+i(a)+r(a)=1, \\ s(0)=1,\ e(0)=i(0)=r(0)=0. \end{cases} \tag{3.1}$$

易得(3.1)的无病平衡点$E^0(1,0,0,0)$. 为讨论其稳定性, 将系统(2.4)在E^0处线性化, 考虑如下形式的指数解

$$s(a,t)=1+\overline{s}(a)\mathrm{e}^{\lambda t},\ e(a,t)=\overline{e}(a)\mathrm{e}^{\lambda t},\ i(a,t)=\overline{i}(a)\mathrm{e}^{\lambda t},\ r(a,t)=\overline{r}(a)\mathrm{e}^{\lambda t}.$$

省略高阶项得

$$\begin{cases} \lambda\overline{s}(a)+\dfrac{\mathrm{d}\overline{s}(a)}{\mathrm{d}a}=-k(a)V_0, \\ \lambda\overline{e}(a)+\dfrac{\mathrm{d}\overline{e}(a)}{\mathrm{d}a}=qk(a)V_0-\varepsilon(a)\overline{e}(a), \\ \lambda\overline{i}(a)+\dfrac{\mathrm{d}\overline{i}(a)}{\mathrm{d}a}=(1-q)k(a)V_0+\varepsilon(a)\overline{e}(a)-\alpha(a)\overline{i}(a), \\ \lambda\overline{r}(a)+\dfrac{\mathrm{d}\overline{r}(a)}{\mathrm{d}a}=\alpha(a)\overline{i}(a), \\ \overline{s}(0)=\overline{e}(0)=\overline{i}(0)=\overline{r}(0)=0, \end{cases} \tag{3.2}$$

其中

$$V_0=\int_0^{+\infty}\beta(a)P_\infty(a)\overline{i}(a)\mathrm{d}a \tag{3.3}$$

为常数. 由(3.2)第二个方程得

$$\overline{e}(a)=V_0\int_0^a qk(\xi)\mathrm{e}^{-\lambda(a-\xi)}\mathrm{e}^{-\int_\xi^a\varepsilon(\tau)\mathrm{d}\tau}\mathrm{d}\xi,$$

代入(3.2)第三个方程得

$$\overline{i}(a)=V_0\int_0^a k(\xi)\mathrm{e}^{-\lambda(a-\xi)}[(1-q)\mathrm{e}^{-\int_\xi^a\alpha(\tau)\mathrm{d}\tau}+q\int_\xi^a\varepsilon(\sigma)\mathrm{e}^{-\int_\xi^\sigma\varepsilon(\tau)\mathrm{d}\tau}\mathrm{e}^{-\int_\sigma^a\alpha(\tau)\mathrm{d}\tau}\mathrm{d}\sigma]\mathrm{d}\xi. \tag{3.4}$$

把(3.4)代入(3.3), 两边同除以V_0(其中$V_0\neq 0$) 可得特征方程为

$$1=\int_0^{+\infty}\beta(a)P_\infty(a)\int_0^a k(\xi)\mathrm{e}^{-\lambda(a-\xi)}[(1-q)\mathrm{e}^{-\int_\xi^a\alpha(\tau)\mathrm{d}\tau}$$

$$+ q\int_{\xi}^{a}\varepsilon(\sigma)\mathrm{e}^{-\int_{\xi}^{\sigma}\varepsilon(\tau)\mathrm{d}\tau}\mathrm{e}^{-\int_{\sigma}^{a}\alpha(\tau)\mathrm{d}\tau}\mathrm{d}\sigma]\mathrm{d}\xi\mathrm{d}a =: G(\lambda). \tag{3.5}$$

定义基本再生数[7] $\Re_0 = G(0)$, 即

$$\begin{aligned}\Re_0 = &\int_0^{+\infty}\beta(a)P_\infty(a)\int_0^a k(\xi)[(1-q)\mathrm{e}^{-\int_{\xi}^{a}\alpha(\tau)\mathrm{d}\tau}\\ &+ q\int_{\xi}^{a}\varepsilon(\sigma)\mathrm{e}^{-\int_{\xi}^{\sigma}\varepsilon(\tau)\mathrm{d}\tau}\mathrm{e}^{-\int_{\sigma}^{a}\alpha(\tau)\mathrm{d}\tau}\mathrm{d}\sigma]\mathrm{d}\xi\mathrm{d}a.\end{aligned} \tag{3.6}$$

则有下面的定理:

定理3.1 若$\Re_0 < 1$, 则无病平衡点$E^0(1,0,0,0)$是局部渐近稳定的; 若$\Re_0 > 1$, 则无病平衡点E^0不稳定.

证 注意到

$$G'(\lambda) < 0, \qquad \lim_{\lambda\to+\infty} G(\lambda) = 0, \qquad \lim_{\lambda\to-\infty} G(\lambda) = +\infty.$$

当$G(0) > 1$时, 即$\Re_0 > 1$时, 方程(3.5)有唯一的正实根, 此时无病平衡点E^0不稳定. 当$G(0) < 1$时, 也即$\Re_0 < 1$时, 方程(3.5)有唯一的负实根λ^*. λ^*是$G(\lambda) = 1$的占优实根, 事实上, 设$\lambda = x + \mathrm{i}y$是(3.5)的任意根, 由于

$$1 = G(\lambda^*) = |G(x+\mathrm{i}y)| \le G(x),$$

由$G(\lambda)$的递减性得, $\mathrm{Re}\lambda \le \lambda^*$. 也就是说当$\Re_0 < 1$, 则无病平衡点$E^0$是局部渐近稳定的.

定理 3.2 若$\Re_0 < 1$, 则无病平衡点E^0是全局渐近稳定的.

证 令

$$f(a,t) = \lambda(a,t)s(a,t) \le \lambda(a,t) = k(a)\int_0^{+\infty}\beta(a)P_\infty(a)i(a,t)\mathrm{d}a =: k(a)V(t), \tag{3.7}$$

其中$s(a,t) \le 1$, 则将(2.4)式沿特征线积分得到

$$\begin{cases} e(a,t) = \int_0^a \mathrm{e}^{-\int_{\xi}^{a}\varepsilon(\tau)\mathrm{d}\tau} q f(\xi, t-a+\xi)\mathrm{d}\xi, & a < t,\\ i(a,t) = \int_0^a \mathrm{e}^{-\int_{\xi}^{a}\alpha(\tau)\mathrm{d}\tau}[(1-q)f(\xi,t-a+\xi) & \\ \qquad + q\varepsilon(\xi)\int_0^{\xi}\mathrm{e}^{-\int_{\sigma}^{\xi}\varepsilon(\tau)\mathrm{d}\tau}f(\sigma, t-\xi+\sigma)\mathrm{d}\sigma]\mathrm{d}\xi, & a < t,\\ r(a,t) = \int_0^a \alpha(\xi)i(\xi, t-a+\xi)\mathrm{d}\xi, & a < t.\end{cases} \tag{3.8}$$

将$i(a,t)$代入$f(a,t)$得

$$\begin{aligned} f(a,t) \le & k(a)\int_0^{+\infty}\beta(a)P_\infty(a)\int_0^a \mathrm{e}^{-\int_{\xi}^{a}\alpha(\tau)\mathrm{d}\tau}[(1-q)f(\xi,t-a+\xi)\\ &+ q\varepsilon(\xi)\int_0^{\xi}\mathrm{e}^{-\int_{\sigma}^{\xi}\varepsilon(\tau)\mathrm{d}\tau}f(\sigma,t-\xi+\sigma)\mathrm{d}\sigma]\mathrm{d}\xi\mathrm{d}a.\end{aligned} \tag{3.9}$$

令

$$F(a) = \lim_{t\to+\infty}\sup f(a,t).$$

对(3.9)式两边取$t\to+\infty$时的上极限, 由Fatou引理得

$$\begin{aligned} F(a) \le & k(a)\int_0^{+\infty}\beta(a)P_\infty(a)\int_0^a F(\xi)[(1-q)\mathrm{e}^{-\int_{\xi}^{a}\alpha(\tau)\mathrm{d}\tau}\\ &+ q\int_{\xi}^{a}\varepsilon(\sigma)\mathrm{e}^{-\int_{\xi}^{\sigma}\varepsilon(\tau)\mathrm{d}\tau}\mathrm{e}^{-\int_{\sigma}^{a}\alpha(\tau)\mathrm{d}\tau}\mathrm{d}\sigma]\mathrm{d}\xi\mathrm{d}a.\end{aligned} \tag{3.10}$$

令C是常数

$$C=\int_0^{+\infty}\beta(a)P_\infty(a)\int_0^a F(\xi)[(1-q)\mathrm{e}^{-\int_\xi^a\alpha(\tau)\mathrm{d}\tau} + q\int_\xi^a\varepsilon(\sigma)\mathrm{e}^{-\int_\xi^\sigma\varepsilon(\tau)\mathrm{d}\tau}\mathrm{e}^{-\int_\sigma^a\alpha(\tau)\mathrm{d}\tau}\mathrm{d}\sigma]\mathrm{d}\xi\mathrm{d}a. \tag{3.11}$$

则(3.10)式变为

$$F(a)\le k(a)C.$$

代入(3.11)得

$$C\le\int_0^{+\infty}\beta(a)P_\infty(a)\int_0^a k(\xi)C[(1-q)\mathrm{e}^{-\int_\xi^a\alpha(\tau)\mathrm{d}\tau} + q\int_\xi^a\varepsilon(\sigma)\mathrm{e}^{-\int_\xi^\sigma\varepsilon(\tau)\mathrm{d}\tau}\mathrm{e}^{-\int_\sigma^a\alpha(\tau)\mathrm{d}\tau}\mathrm{d}\sigma]\mathrm{d}\xi\mathrm{d}a=C\Re_0. \tag{3.12}$$

从(3.12)式可以看出, 若$\Re_0<1$, 则$C=0$, 从而$F(a)=0$, 因此

$$\lim_{t\to+\infty}\sup f(a,t)=0.$$

从而由(3.8)式得

$$\lim_{t\to+\infty}e(a,t)=0,\ \lim_{t\to+\infty}i(a,t)=0,\ \lim_{t\to+\infty}r(a,t)=0,\ \lim_{t\to+\infty}\lambda(a,t)=0.$$

从而有

$$\lim_{t\to+\infty}s(a,t)=1.$$

故若$\Re_0<1$, 则无病平衡点E^0是全局渐近稳定的.

4. 地方病平衡点的存在性和稳定性

前面得到当$\Re_0>1$时, 无病平衡点不稳定. 实际上此时存在地方病平衡点.

定理 4.1 当$\Re_0>1$时, 系统(2.4)存在唯一的地方病平衡点.

证 若系统(2.4)存在地方病平衡点$E^*(s^*(a),\mathrm{e}^*(a),i^*(a),r^*(a))$, 则满足

$$\begin{cases}\dfrac{\mathrm{d}s^*(a)}{\mathrm{d}a}=-\lambda^*(a)s^*(a),\\ \dfrac{\mathrm{d}\mathrm{e}^*(a)}{\mathrm{d}a}=q\lambda^*(a)s^*(a)-\varepsilon(a)\mathrm{e}^*(a),\\ \dfrac{\mathrm{d}i^*(a)}{\mathrm{d}a}=(1-q)\lambda^*(a)s^*(a)+\varepsilon(a)\mathrm{e}^*(a)-\alpha(a)i^*(a),\\ \dfrac{\mathrm{d}r^*(a)}{\mathrm{d}a}=\alpha(a)i^*(a),\\ \lambda^*(a)=k(a)\displaystyle\int_0^{+\infty}\beta(a)P_\infty(a)i^*(a)\mathrm{d}a=:k(a)V^*,\\ s^*(a)+\mathrm{e}^*(a)+i^*(a)+r^*(a)=1,\ s^*(0)=1,\ \mathrm{e}^*(0)=i^*(0)=r^*(0)=0,\end{cases} \tag{4.1}$$

其中

$$V^*=\int_0^{+\infty}\beta(a)P_\infty(a)i^*(a)\mathrm{d}a \tag{4.2}$$

为常数, 显然每一个正数V^*对应唯一的地方病平衡点. 由(4.1)中的前两个式子得

$$\begin{aligned}s^*(a)=&\mathrm{e}^{-V^*\int_0^a k(\tau)\mathrm{d}\tau},\\ \mathrm{e}^*(a)=&qV^*\int_0^a k(\xi)\mathrm{e}^{-V^*\int_0^\xi k(\tau)\mathrm{d}\tau}\mathrm{e}^{-\int_\xi^a\varepsilon(\tau)\mathrm{d}\tau}\mathrm{d}\xi,\end{aligned}$$

$$i^*(a)=V^*\int_0^a k(\xi)\mathrm{e}^{-V^*\int_0^\xi k(\tau)\mathrm{d}\tau}[(1-q)\mathrm{e}^{-\int_\xi^a \alpha(\tau)\mathrm{d}\tau}+q\int_\xi^a \varepsilon(\sigma)\mathrm{e}^{-\int_\xi^\sigma \varepsilon(\tau)\mathrm{d}\tau}\mathrm{e}^{-\int_\sigma^a \alpha(\tau)\mathrm{d}\tau}\mathrm{d}\sigma]\mathrm{d}\xi. \tag{4.3}$$

将$i^*(a)$代入(4.2)后, 两边同除以V^*(其中$V^*\neq 0$), 有

$$\begin{aligned}1=&\int_0^{+\infty}\beta(a)P_\infty(a)\int_0^a k(\xi)\mathrm{e}^{-V^*\int_0^\xi k(\tau)\mathrm{d}\tau}[(1-q)\mathrm{e}^{-\int_\xi^a \alpha(\tau)\mathrm{d}\tau}\\&+q\int_\xi^a \varepsilon(\sigma)\mathrm{e}^{-\int_\xi^\sigma \varepsilon(\tau)\mathrm{d}\tau}\mathrm{e}^{-\int_\sigma^a \alpha(\tau)\mathrm{d}\tau}\mathrm{d}\sigma]\mathrm{d}\xi\mathrm{d}a=:H(V^*).\end{aligned} \tag{4.4}$$

若(4.4)有一个正解V^*, 那么系统(2.4)就存在地方病平衡点. 又$s^*(a)+\mathrm{e}^*(a)+i^*(a)+r^*(a)=1$, 且$s^*(a)>0$, 则$i^*(a)<1$. 对任意的$V^*>0$, 有

$$H(V^*)=\frac{1}{V^*}\int_0^{+\infty}\beta(a)P_\infty(a)i^*(a)\mathrm{d}a\le\frac{\beta^+}{V^*}\int_0^{+\infty}P_\infty(a)\mathrm{d}a=\frac{\beta^+N}{V^*},$$

其中N是总人口, $\beta^+=\max\{\sup_{[0,+\infty)}\beta(a)\}$.

若$V^*=\beta^+N$, 则$H(\beta^+N)<1$. 又$H(V^*)$是关于V^*的单调递减连续函数, 因此若$H(0)=\Re_0>1$, 则$H(V^*)=1$在区间$(0,\beta^+N)$上存在唯一正解$\tilde{V}^*$. 即当$\Re_0>1$时, 系统(2.4)存在唯一的地方病平衡点. 证毕.

接下来进一步讨论地方病平衡点的稳定性. 令$\widehat{s},\widehat{e},\widehat{i},\widehat{r}$和$\widehat{V}$ 是s^*,e^*,i^*,r^*和V^*的线性扰动. 对系统(2.4)在地方病平衡点$E^*(s^*(a),\mathrm{e}^*(a),i^*(a),r^*(a))$处线性化, 考虑如下指数形式的解$\widehat{s}(a,t)=\overline{s}(a)\mathrm{e}^{\lambda t}$, $\widehat{e}(a,t)=\overline{e}(a)\mathrm{e}^{\lambda t}$, $\widehat{i}(a,t)=\overline{i}(a)\mathrm{e}^{\lambda t}$, $\widehat{r}(a,t)=\overline{r}(a)\mathrm{e}^{\lambda t}$, $\widehat{V}(t)=\overline{V}\mathrm{e}^{\lambda t}$. 省略高阶项得

$$\begin{cases}\lambda\overline{s}(a)+\dfrac{\mathrm{d}\overline{s}(a)}{\mathrm{d}a}=-k(a)[s^*(a)\overline{V}+\overline{s}(a)V^*],\\ \lambda\overline{e}(a)+\dfrac{\mathrm{d}\overline{e}(a)}{\mathrm{d}a}=qk(a)[s^*(a)\overline{V}+\overline{s}(a)V^*]-\varepsilon(a)\overline{e}(a),\\ \lambda\overline{i}(a)+\dfrac{\mathrm{d}\overline{i}(a)}{\mathrm{d}a}=(1-q)k(a)[s^*(a)\overline{V}+\overline{s}(a)V^*]+\varepsilon(a)\overline{e}(a)-\alpha(a)\overline{i}(a),\\ \lambda\overline{r}(a)+\dfrac{\mathrm{d}\overline{r}(a)}{\mathrm{d}a}=\alpha(a)\overline{i}(a),\\ \overline{V}=\int_0^{+\infty}\beta(a)P_\infty(a)\overline{i}(a)\mathrm{d}a,\\ \overline{s}(0)=\overline{e}(0)=\overline{i}(0)=\overline{r}(0)=0,\end{cases} \tag{4.5}$$

这里$\overline{s},\overline{e},\overline{i},\overline{r}$可正可负. 假设$\overline{V}\neq 0$, 令$s=\overline{s}/\overline{V},e=\overline{e}/\overline{V},i=\overline{i}/\overline{V},r=\overline{r}/\overline{V}$, 则

$$\begin{cases}\lambda s(a)+\dfrac{\mathrm{d}s(a)}{\mathrm{d}a}=-k(a)[s^*(a)+s(a)V^*],\\ \lambda e(a)+\dfrac{\mathrm{d}e(a)}{\mathrm{d}a}=qk(a)[s^*(a)+s(a)V^*]-\varepsilon(a)e(a),\\ \lambda i(a)+\dfrac{\mathrm{d}i(a)}{\mathrm{d}a}=(1-q)k(a)[s^*(a)+s(a)V^*]+\varepsilon(a)e(a)-\alpha(a)i(a),\\ \lambda r(a)+\dfrac{\mathrm{d}r(a)}{\mathrm{d}a}=\alpha(a)i(a),\\ 1=\int_0^{+\infty}\beta(a)P_\infty(a)i(a)\mathrm{d}a,\\ s(0)=e(0)=i(0)=r(0)=0.\end{cases} \tag{4.6}$$

令

$$1=\int_0^{+\infty}\beta(a)P_\infty(a)i(a)\mathrm{d}a=:T(\lambda). \tag{4.7}$$

求解(4.6), 得

$$\begin{aligned}
s(a) =& -\mathrm{e}^{-V^*\int_0^a k(\tau)\mathrm{d}\tau}\int_0^a k(\xi)\mathrm{e}^{-\lambda(a-\xi)}\mathrm{d}\xi,\\
e(a) =& q\int_0^a k(\xi)\mathrm{e}^{-\lambda(a-\xi)}\mathrm{e}^{-\int_\xi^a \varepsilon(\tau)\mathrm{d}\tau}\mathrm{e}^{-V^*\int_0^\xi k(\tau)\mathrm{d}\tau}\mathrm{d}\xi\\
&- qV^*\int_0^a k(\xi)\mathrm{e}^{-\lambda(a-\xi)}\int_\xi^a k(\sigma)\mathrm{e}^{-V^*\int_0^\sigma k(\tau)\mathrm{d}\tau}\mathrm{e}^{-\int_\sigma^a \varepsilon(\tau)\mathrm{d}\tau}\mathrm{d}\sigma\mathrm{d}\xi,\\
i(a) =& \int_0^a k(\xi)\mathrm{e}^{-\lambda(a-\xi)}\{(1-q)[\mathrm{e}^{-V^*\int_0^\xi k(\tau)\mathrm{d}\tau}\mathrm{e}^{-\int_\xi^a \alpha(\tau)\mathrm{d}\tau}\\
&- V^*\int_\xi^a k(\eta)e^{-V^*\int_0^\eta k(\tau)\mathrm{d}\tau}\mathrm{e}^{-\int_\eta^a \alpha(\tau)\mathrm{d}\tau}\mathrm{d}\eta] + q\int_\xi^a \varepsilon(\eta)\mathrm{e}^{-\int_\eta^a \alpha(\tau)\mathrm{d}\tau}\\
&\cdot[\mathrm{e}^{-V^*\int_0^\xi k(\tau)\mathrm{d}\tau}\mathrm{e}^{-\int_\xi^\eta \varepsilon(\tau)\mathrm{d}\tau} - V^*\int_\xi^\eta k(\sigma)\mathrm{e}^{-V^*\int_0^\sigma k(\tau)\mathrm{d}\tau}\mathrm{e}^{-\int_\sigma^\eta \varepsilon(\tau)\mathrm{d}\tau}\mathrm{d}\sigma]\mathrm{d}\eta\}\mathrm{d}\xi.
\end{aligned}$$

将$i(a)$代入$T(\lambda)$整理得

$$\begin{aligned}
T(\lambda) =& \int_0^{+\infty}\beta(a)P_\infty(a)\int_0^a k(\xi)\mathrm{e}^{-\lambda(a-\xi)}\{(1-q)[\mathrm{e}^{-V^*\int_0^\xi k(\tau)\mathrm{d}\tau}\mathrm{e}^{-\int_\xi^a \alpha(\tau)\mathrm{d}\tau}\\
&- V^*\int_\xi^a k(\eta)\mathrm{e}^{-V^*\int_0^\eta k(\tau)\mathrm{d}\tau}\mathrm{e}^{-\int_\eta^a \alpha(\tau)\mathrm{d}\tau}\mathrm{d}\eta] + q\int_\xi^a \varepsilon(\eta)\mathrm{e}^{-\int_\eta^a \alpha(\tau)\mathrm{d}\tau}\\
&\cdot[\mathrm{e}^{-V^*\int_0^\xi k(\tau)\mathrm{d}\tau}\mathrm{e}^{-\int_\xi^\eta \varepsilon(\tau)\mathrm{d}\tau} - V^*\int_\xi^\eta k(\sigma)\mathrm{e}^{-V^*\int_0^\sigma k(\tau)\mathrm{d}\tau}\mathrm{e}^{-\int_\sigma^\eta \varepsilon(\tau)\mathrm{d}\tau}\mathrm{d}\sigma]\mathrm{d}\eta\}\mathrm{d}\xi\mathrm{d}a. \qquad (4.8)
\end{aligned}$$

若引入一个类似文[8]中的条件

$$\int_\xi^a \alpha(\eta)\mathrm{e}^{V^*\int_\eta^a k(\tau)\mathrm{d}\tau}\mathrm{e}^{-\int_\eta^a \alpha(\tau)\mathrm{d}\tau}\mathrm{d}\eta < 1,\ \text{且}\int_\xi^\sigma \varepsilon(\eta)\mathrm{e}^{V^*\int_\eta^\sigma k(\tau)\mathrm{d}\tau}\mathrm{e}^{-\int_\eta^\sigma \varepsilon(\tau)\mathrm{d}\tau}\mathrm{d}\eta < 1. \qquad (4.9)$$

则有下面定理.

定理 4.2 若条件(4.9)满足, 则

1) $T(\lambda)$关于λ递减且当$\lambda \to +\infty$时趋近于0;

2) $T(0) < 1$.

证 1) 若条件(4.9)满足, 则(4.8)式两个中括号内的式子都大于零, 进而可得 $T(\lambda) \geq 0$, 关于指数 λ递减且当 $\lambda \to +\infty$时 $T \to 0$.

2) 令$\lambda = 0$得

$$\begin{aligned}
T(0) =& \int_0^{+\infty}\beta(a)P_\infty(a)\int_0^a k(\xi)\mathrm{e}^{-V^*\int_0^\xi k(\tau)\mathrm{d}\tau}[(1-q)\mathrm{e}^{-\int_\xi^a \alpha(\tau)\mathrm{d}\tau}\\
&+ q\int_\xi^a \varepsilon(\eta)\mathrm{e}^{-\int_\eta^a \alpha(\tau)\mathrm{d}\tau}\mathrm{e}^{-\int_\xi^\eta \varepsilon(\tau)\mathrm{d}\tau}\mathrm{d}\eta]\mathrm{d}\xi\mathrm{d}a\\
&- V^*\int_0^{+\infty}\beta(a)P_\infty(a)\int_0^a k(\xi)\int_\xi^a[(1-q)k(\eta)\mathrm{e}^{-V^*\int_0^\eta k(\tau)\mathrm{d}\tau}\mathrm{e}^{-\int_\eta^a \alpha(\tau)\mathrm{d}\tau}\\
&+ q\varepsilon(\eta)\mathrm{e}^{-\int_\eta^a \alpha(\tau)\mathrm{d}\tau}\int_\xi^\eta k(\sigma)\mathrm{e}^{-V^*\int_0^\sigma k(\tau)\mathrm{d}\tau}\mathrm{e}^{-\int_\sigma^\eta \varepsilon(\tau)\mathrm{d}\tau}\mathrm{d}\sigma]\mathrm{d}\eta\mathrm{d}\xi\mathrm{d}a.
\end{aligned}$$

由(4.4)可以看出上式第一项积分等于1. 因此, $T(0) < 1$. 证毕.

定理4.2及(4.8)说明方程$T(\lambda) = 1$, 也就是(4.7)有唯一的负实根且所有的复根实部都小于这个实根. 因此有

定理 4.3 假设(4.9)成立, 则系统(2.4)的地方病平衡点局部渐近稳定.

5. 讨论

从基本再生数$\Re_0$的表达式可以看出, 治愈率$\alpha(a)$才是控制乙肝的关键, 从某种意义上来说引入易感类人群接触病人后进入潜伏类和染病类的比例反而使得控制疾病变得更为复杂, 不过也可以采取其它辅助措施控制疾病爆发, 比如接种[9]、隔离[10]、控制人口流动等.

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Stability of an Age-Structured Infectious Disease Model with Hepatitis B

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Abstract: An age structured hepatitis B infectious disease model is discussed, and the expression of basic reproductive number $\Re_0$ is obtained. It is proved that when $\Re_0 < 1$, the disease free equilibrium is locally asymptotically stable and globally asymptotically stable, and when $\Re_0 > 1$, there is an unique endemic equilibrium, and the local asymptotic stability conditions of endemic equilibrium are given. These conditions have important theoretical and practical significance in controlling the spread of diseases.

Key words: Age-structured; Latent class; Basic reproductive number