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显含阻尼项的二阶非线性中立型 Emden-Fowler微分方程的振动性和渐近性

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摘要： 本文研究一类带有更广泛而又不失物理意义阻尼项的二阶非线性中立型Emden-Fowler时滞微分方程的振动性和渐近性. 利用指数变换、Riccati变换和不等式技巧, 获得该类方程几个新的振动准则, 推广、改进和统一已有文献中的研究成果, 并逐一给出例子说明了相应定理的实用效果.

关键词： 振动准则; 渐近性; Emden-Fowler方程; 中立型非线性微分方程; 二阶; 阻尼项
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1. 引言

来源于数学物理方程的Emden-Fowler型微分方程的研究成果已被广泛应用在天体物理、气体动力学、物理化学以及各高新技术领域之中[1−4]. 例如带阻尼项的二阶Emden-Fowler方程

$$x''(t)+\frac{a}{t}x'(t)+bt^{m-1}x^n(t)=0, \tag{1.1}$$

(其中, $n\neq 0, n\neq 1, a, b, m$为常数, $\frac{a}{t}x'(t)$为阻尼项)振动性渐近性的结果远没有不带阻尼项时的系统丰富[1,7,11].

本文研究更一般的显含阻尼项的二阶非线性中立型广义Emden-Fowler时滞微分方程

$$(r(t)\phi_\alpha(z'(t)))'+g(t)\phi_\alpha(z'(t))+f\left(t,\phi_\beta(x(\sigma(t)))\right)=0,\ \ t\geq t_0\geq 0, \tag{1.2}$$

的振动性, 其中$z(t)=x(t)+p(t)x(\tau(t)), \phi_\alpha(u)=|u|^{\alpha-1}u, u\in\mathbb{R}, r\in C^1([t_0,\infty),(0,\infty))$, $g,p\in C^1([t_0,\infty),\mathbb{R}), f\in C([t_0,\infty)\times\mathbb{R},\mathbb{R}),\ \alpha>0,\beta>0$为常数, $\tau,\sigma\in C^1([t_0,\infty),\mathbb{R}),\tau(t)\leq t,\sigma(t)\leq t, \lim\limits_{t\to\infty}\tau(t)=\lim\limits_{t\to\infty}\sigma(t)=\infty$.

在现有文献中, 所谓带阻尼项(或不带阻尼项)的二阶泛函微分方程, 几乎均为类似于方程(1.2)当$g(t)\geq 0$(或$g(t)\equiv 0$)时的情形, 且近年来对其振动性的研究日趋活跃, 可参见文[2-31]及其引文. 那么, 这里的阻尼项系数$g(t)\geq 0$的取值范围是否还可以扩大, 扩大后对方程振动性的影响如何呢？为了启发我们的研究思路, 首先对现有代表性的研究成果简要分析如下.

例如黄记洲等[3]、LIU等[5]、曾云辉等[6]、LI等[7−8]、LUO等[9]和吴英柱等[10]分别研究了二阶广义Emden-Fowler中立型微分方程

$$(r(t)|z'(t)|^{\alpha-1}z'(t))'+q(t)|x(\sigma(t))|^{\beta-1}x(\sigma(t))=0,\ \ t\geq t_0>0, \tag{1.3}$$

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的振动性和渐近性, 其中$z(t)=x(t)+p(t)x(\tau(t)),\alpha\geq\beta>0$或$\beta\geq\alpha=1,\ 0\leq p(t)\leq 1,q(t)\geq 0,r(t)>0,r'(t)\geq 0,\ \tau(t)\leq t,0<\sigma(t)\leq t,\sigma'>0,\ \lim_{t\to\infty}\tau(t)=\lim_{t\to\infty}\sigma(t)=\infty$.

Agarwal等[11]、Grace等[12]和Bohner等[13]先后研究了二阶中立型时滞微分方程

$$(r(t)((x(t)+p(t)x(\tau(t)))')^{\alpha})'+q(t)x^{\gamma}(\sigma(t))=0,\ \ t\geq t_0, \tag{1.4}$$

分别对$\gamma\geq\alpha,\gamma<\alpha,\gamma<\alpha=1$和$\gamma=\alpha$的情况给出了多个振动定理, 其中$\alpha,\gamma>0$是两正奇数之比的常数.

WANG等[14]、LI等[15]、罗红英等[16]和吴英柱[17]分别将方程(1.3)扩展到方程

$$(r(t)|z'(t)|^{\alpha-1}z'(t))'+f(t,x(\sigma(t)))=0,\ \ t\geq t_0, \tag{1.5}$$

其中$z(t)=x(t)+p(t)x(\tau(t)),\ r,p\in C([t_0,\infty),\mathbb{R}),\ 0\leq p(t)\leq 1,\tau(t)\leq t,\sigma(t)\leq t,\sigma'(t)>0,\ \lim\limits_{t\to\infty}\tau(t)=\lim\limits_{t\to\infty}\sigma(t)=\infty,\ f\in C([t_0,\infty)\times\mathbb{R},\mathbb{R}),uf(t,u)\geq 0,f(t,u)/u^{\beta}\geq q(t)\geq 0,u\neq 0,\ 1<\beta\leq\alpha$为常数.

显然, 方程(1.3)-(1.5)均不带阻尼项. 对于带阻尼项的二阶Emden-Fowler型方程振动性渐近性研究的成果虽没有对上述方程的丰富, 但近年来的研究已趋活跃并有不少成果相继出现.

例如Erbe等[18]和ZHANG等[19]先后研究了时间尺度上二阶阻尼时滞动力方程

$$(a(t)(x^{\Delta}(t))^{\gamma})^{\Delta}+r(t)(x^{\Delta}(t))^{\gamma}+q(t)x^{\gamma}(g(t))=0,\ \ t\in[t_0,\infty)_{\mathbb{T}}, \tag{1.6}$$

的振动性, 其中$\gamma>0$是两正奇数之比的常数, $r(t)\geq 0$右稠连续.

Saker等[20]和Rogovchenko等[21]也分别对二阶阻尼动力方程

$$(a(t)x^{\Delta}(t))^{\Delta}+r(t)x^{\Delta}(t)+q(t)f(x(t)=0,\ \ t\in[t_0,\infty)_{\mathbb{T}}, \tag{1.7}$$

给出了不同的振动性定理, 其中$r(t)\geq 0$右稠连续.

方程(1.6)先后被张全信等[22−24]、孙一冰等[25]和杨甲山等[26−27]许多学者拓展为更一般形式：

$$(a(t)|z^{\Delta}(t)|^{\gamma-1}z^{\Delta}(t))^{\Delta}+p(t)|z^{\Delta}(t)|^{\gamma-1}z^{\Delta}(t)+q(t)|x(\delta(t)|^{\beta-1}x(\delta(t)=0,\ t\in[t_0,\infty)_{\mathbb{T}}, \tag{1.8}$$

其中$z(t)=x(t)+r(t)x(\tau(t),\gamma,\beta>0$为常数, $a(t),r(t),p(t),q(t)$都是正值右稠连续函数.

LI等[28]研究了带阻尼项的二阶非线性常微分方程

$$(r(t)(x'(t))')^{\gamma})'+p(t)(x'(t))^{\gamma}+q(t)f(x(t))=0 \tag{1.9}$$

的振动性, 其中$t\geq t_0>0,\gamma\geq 1$是两正奇数之比的常数, 连续函数r,p,q,f满足$r\in C'([t_0,\infty),(0,\infty)),\ f(x)/x^{\gamma}\geq\mu>0,x\neq 0.q(t)\geq 0$且不恒等于0.

李文娟等[29]将方程(1.3)的类型推广到了带阻尼项的中立型时滞微分方程

$$(r(t)|z'(t)|^{\alpha-1}z'(t))'+p(t)|z'(t)|^{\alpha-1}z'(t)+q(t)|x(\sigma(t)))|^{\beta-1}x(\sigma(t))=0,\ \ t\geq t_0>0, \tag{1.10}$$

其中$z(t)=x(t)+g(t)x(\tau(t)),r\in C^1([t_0,\infty),(0,\infty)),p,q\in C([t_0,\infty),[0,\infty)),\ \alpha>0,\beta>0$为常数, 在$0\leq g(t)\leq 1,p(t)\geq 0,q(t)\geq 0,r'(t)>0$等基本假设条件下, 获得了多个振动定理, 推广了上述有关文献的部分结果.

通过以上分析不难看出, 方程(1.1)、(1.3)-(1.10)均为方程(1.2)的特殊类型, 而且它们所谓的阻尼项系数(例如(1.6),(1.7)中的$r(t)$和(1.8)-(1.10)中的$p(t)$)和中立项系数(例如(1.3)-(1.5)、(1.9)、(1.10)中的$r(t)$和(1.6)-(1.8)中的$a(t)$)的导数都是非负函数. 但是, 不难发现, 这些方程中显含的阻尼项并不代表实际物理意义下的全部阻尼项. 因为由文[30] 知, 当$r(t)>0,r'(t)\geq 0$时, 二阶微分方程

$$(r(t)\phi_{\alpha}(x'(t)))'+g(t)\phi_{\alpha}(x'(t))+f(t,x(t))=0$$

与二阶微分方程

$$(\phi_{\alpha}(x'(t)))'+\frac{r'(t)+g(t)}{r(t)}\phi_{\alpha}(x'(t))+\frac{f(t,x(t))}{r(t)}=0$$

等价. 而又当$\alpha=1$, $r'(t)+g(t)\geq 0$时, 后一方程是阻尼系数为$(r'(t)+g(t))/r(t)$的经典弹性振动系统模型[32], 有着实际物理意义, 且此阻尼系数的取值范围不能突破这一界限, 否则将失去物理意义. 例如文[31]研究了带阻尼的分数阶微分方程

$$[r(t)(D_{0+}^{\alpha}y)(t)]'+p(t)(D_{0+}^{\alpha}y)(t)+q(t)f\left(\int_0^t (t-s)^{-\alpha}y(s)\mathrm{d}s\right)=0, \tag{1.11}$$

(其中, $\alpha\in(0,1), D_{0+}^{\alpha}y$是$y$的$\alpha$阶导数), 在假设条件$\mathrm{A}_1), \mathrm{A}_2)$及$\omega(t)=\exp\left(\int_{t_0}^t \frac{r'(s)+p(s)}{r(s)}\mathrm{d}s\right)$, $\int_{t_0}^{\infty}\frac{1}{\omega(t)}\mathrm{d}t=\infty$ 之下建立了方程(1.11)的振动定理. 显然这里几乎是$(r'(t)+p(t))/r(t)<0$ 的情况(如其例4.1), 但这时的“阻尼”已不再是具有物理意义的阻尼了[32]. 同样, 文[28]中关于方程(1.9)的讨论也有类似现象(如其例6).

因此, 对于方程(1.2), 本文将总假设以下条件成立:

(H_1) $r'(t)+g(t)\geq 0$ 且$0\leq p(t)\leq p_0<\infty$.

(H_2) $\tau\circ\sigma=\sigma\circ\tau, \tau'(t)\geq\tau_0>0$.

(H_3) 存在不恒为零的函数$q\in C([t_0,\infty),[0,\infty))$, 满足$f(t,u)/u\geq q(t)\geq 0, u\neq 0, t\geq t_0$.

其次, 本文将引进指数函数变换, 并借助于Riccati变换,积分平均和不等式技巧研究方程(1.2)的振动性和渐近性,建立新的振动准则, 顺便导出方程(1.1)新的振动性渐近性判据.

下面, 引入指数函数变换

$$\varphi(t)=\exp\left(\int_{t_0}^t g(u)/r(u)\mathrm{d}u\right), \tag{1.12}$$

用$\varphi(t)$乘以方程(1.2)的两端, 则(1.2)变为等价的不显含阻尼项的微分方程

(E_0) $\qquad (R(t)|z'(t)|^{\alpha-1}z'(t))'+\varphi(t)f(t,|x(\sigma(t))|^{\beta-1}x(\sigma(t)))=0, t\geq t_0,$

其中$R(t)=r(t)\varphi(t)$.

我们通过方程(E_0), 在两种情形

$$\int_{t_0}^{\infty}(1/R(t))^{1/\alpha}\,\mathrm{d}u=\infty, \tag{1.13}$$

$$\int_{t_0}^{\infty}(1/R(t))^{1/\alpha}\,\mathrm{d}u<\infty \tag{1.14}$$

下, 分别讨论方程(1.2)的振动性和渐近性, 为此先给出以下几个引理.

引理1.1 设(H_3)和(1.13) 式成立. 如果$x(t)$是方程(1.2)的最终正解,则最终有$z'(t)>0$.

证 因为$x(t)$是方程(1.2)在$[t_0,\infty)$ 上的最终正解, 则存在$t_1\geq t_0$, 使得当$t\geq t_1$时有$x(t)>0, x(\tau(t))>0, x(\sigma(t))>0$, 由$(\mathrm{H}_3)$和$(\mathrm{E}_0)$, 我们得到

$$z(t)\geq x(t)>0, (\varphi(t)r(t)|z'(t)|^{\alpha-1}z'(t))'\leq 0, t\geq t_1. \tag{1.15}$$

因此$\varphi(t)r(t)|z'(t)|^{\alpha-1}z'(t)$ 是非增函数且$z'(t)$最终保号, 于是$z'(t)$ 仅有两种可能. 我们断言$z'(t)>0, t>t_1$. 否则, 假设$z'(t)\leq 0, t>t_1$. 由(1.15)式知, 存在常数$K>0$ 使得

$$-R(t)(-z'(t))^{\alpha}\leq -R(t_1)(-z'(t_1))^{\alpha}=-K<0, t>t_1,$$
$$z'(t)\leq -K^{1/\alpha}(R(t))^{-1/\alpha}, t>t_1.$$

从t_1到t积分上式, 我们得到

$$z(t)\leq z(t_1)-K^{1/\alpha}\int_{t_1}^t (R(s))^{-1/\alpha}\,\mathrm{d}s, t>t_1.$$

在上式中令$t\to\infty$, 由条件(1.13)得$z(t)\to-\infty$. 此式与(1.15)式矛盾, 故结论成立. 证毕.

引理1.2 设$A>0, B\geq 0, \lambda>0$且均为常数, 则当$u>0$ 时, 有

$$Bu-Au^{\frac{\lambda+1}{\lambda}}\leq\frac{\lambda^{\lambda}}{(\lambda+1)^{\lambda+1}}\frac{B^{\lambda+1}}{A^{\lambda}}. \tag{1.16}$$

引理1.3　设$X>0, Y>0, \lambda>0$为任意实数, 则有

$$X^\lambda+Y^\lambda\geq C_\lambda(X+Y)^\lambda,\quad C_\lambda=\begin{cases}1, & 0<\lambda\leq 1,\\ 2^{1-\lambda}, & \lambda>1,\end{cases}\tag{1.17}$$

当且仅当$X=Y, \lambda\geq 1$时第一式等号成立.

2. 主要结果

为建立方程(1.2)振动性渐近性准则, 引入以下记号:

$$\begin{aligned}&Q(t)=\min\{\varphi(t)q(t),\varphi(\tau(t))q(\tau(t))\},\\&\psi(t,t_1)=H(\sigma(t),t_1)H^{-1}(t,t_1),\\&H(t,t_1)=\int_{t_1}^t R^{-\frac{1}{\alpha}}(s)\mathrm{d}s,\ \ \phi'_+(t)=\max\{0,\phi'(t)\},\ \ t\geq t_1\geq t_0,\end{aligned}\tag{2.1}$$

其中$R(t)=\varphi(t)r(t), \varphi(t)$由(1.12)式定义.

定理2.1　设(H_1)-(H_3)和条件(1.13)式成立. 如果存在函数$\rho\in C^1([t_0,\infty),(0,\infty))$和$t_2\geq t_1\geq t_0$, 使得当$t\geq t_2$时$\sigma(t)\geq t_1$, 并对任意常数$m\in(0,1]$ (当$\alpha=\beta$时, $m=1$), 恒有

$$\limsup_{t\to\infty}\int_{t_2}^t\rho(s)\left[C_\beta Q(s)\psi^\beta(s,t_1)-(1+\frac{p_0^\beta}{\tau_0})(\frac{\lambda}{\beta\tau_0 m})^\lambda\frac{R^\gamma(s)}{(\lambda+1)^{\lambda+1}}(\frac{\rho'_+(s)}{\rho(s)})^{\lambda+1}\right]\mathrm{d}s=\infty\tag{2.2}$$

成立, 其中$C_\beta, Q(t)$和$\psi(t,t_1)$分别由(1.17)和(2.1)式定义, $\lambda=\min\{\alpha,\beta\}$, $\gamma=\begin{cases}1, & \alpha\leq\beta,\\ \dfrac{\beta}{\alpha}, & \alpha>\beta,\end{cases}$ 则方程(1.2)振动.

证　假设$x(t)$是方程(1.2)的非振动解. 不失一般性, 设$x(t)$为$[t_0,\infty)$上的最终正解($x(t)<0$的情况类似可证), 则存在$t_2\geq t_1\geq t_0$, 使得$t\geq t_1$时, 有$x(t)>0, x(\tau(t))>0, x(\sigma(t))>0$, 当$t\geq t_2$时, 有$\sigma(t_2)\geq t_1$. 于是, 由方程(1.2)的等价方程$(\mathrm{E}_0)$得不等式

$$(R(t)(z'(t))^\alpha)'+Q(t)f(x^\beta(\sigma(t)))=0,$$

可得

$$(R(t)(z'(t))^\alpha)'+\varphi(t)q(t)x^\beta(\sigma(t))\leq 0,\ \ t\geq t_1,\tag{2.3}$$

以及

$$\frac{(R(\tau(t))(z'(\tau(t)))^\alpha)'}{\tau'(t)}+\varphi(\tau(t))q(\tau(t))x^\beta(\sigma(\tau(t)))\leq 0,\ \ t\geq t_1.\tag{2.4}$$

结合(2.3)和(2.4)式, 并注意到$\sigma\circ\tau=\tau\circ\sigma, z(t)\leq x(t)+p_0x(\tau(t))$以及引理1.3, 得

$$\begin{aligned}[R(t)(z'(t))^\alpha]'+\frac{p_0^\beta}{\tau_0}[R(\tau(t))(z'(\tau(t)))^\alpha]'\leq&-\varphi(t)q(t)x^\beta(\sigma(t))-p_0^\beta\varphi(\tau(t))q(\tau(t))x^\beta(\sigma(\tau(t)))\\\leq&-Q(t)[x^\beta(\sigma(t))+p_0^\beta x^\beta(\tau(\sigma(t)))]\\\leq&-C_\beta Q(t)[x(\sigma(t))+p_0x(\sigma(\tau(t)))]^\beta\\\leq&-C_\beta Q(t)z^\beta(\sigma(t)),\ \ t\geq t_1.\end{aligned}\tag{2.5}$$

根据引理1.1知, 不妨设$z'(t)>0, t\geq t_1$. 于是, 对于α,β的取值, 分两种情形讨论如下:

情形1　$\alpha\leq\beta$, 这时, $\lambda=\alpha$. 作Riccati变换

$$w(t)=\rho(t)\frac{R(t)(z'(t))^\alpha}{z^\beta(\tau(t))},\ \ t\geq t_1,\tag{2.6}$$

则$w(t)>0,\ t\geq t_1$. 对(2.6)式求导并注意到$\tau'(t)\geq\tau_0>0$, 得

$$w'(t)\leq\frac{\rho'(t)}{\rho(t)}w(t)+\frac{\rho(t)}{z^\beta(\tau(t))}[R(t)(z'(t))^\alpha]'-\beta\tau_0\frac{\rho(t)R(t)(z'(t))^\alpha z'(\tau(t))}{z^{\beta+1}(\tau(t))}.$$

由于$R(t)(z'(t))^\alpha$单调减, 所以, $R(t)(z'(t))^\alpha \le R(\tau(t))(z'(\tau(t)))^\alpha$, 从而, 有 $z'(\tau(t)) \ge (\frac{R(t)}{R(\tau(t))})^{\frac{1}{\alpha}} z'(t)$. 于是, 有

$$w'(t) \le \frac{\rho(t)}{z^\beta(\tau(t))}[R(t)(z(t))^\alpha]' + \frac{\rho'(t)}{\rho(t)}w(t) - \beta\tau_0\frac{\rho(t)R(t)(z'(t))^{\alpha+1}}{z^{\beta+1}(\tau(t))}\left(\frac{R(t)}{R(\tau(t))}\right)^{\frac{1}{\alpha}}, \ \ t \ge t_1. \tag{2.7}$$

因为$z'(t) > 0, z(t)$ 单调增, 所以, 取$m_\alpha = \min\{z^{(\beta-\alpha)/\alpha}(\tau(t_1)), 1\}$. 则当$t \ge t_1$时, 有$z^{(\beta-\alpha)/\alpha}(\tau(t)) \ge m_\alpha$, 于是, 结合(2.7)式和引理1.2的(1.16)式, 得

$$\begin{aligned} w'(t) &\le \frac{\rho(t)}{z^\beta(\tau(t))}[R(t)(z'(t))^\alpha]' + \frac{\rho'(t)}{\rho(t)}w(t) - \frac{\beta\tau_0 z^{(\beta-\alpha)/\alpha}(\tau(t))}{[\rho(t)R(\tau(t))]^{\frac{1}{\alpha}}}w^{\frac{\alpha+1}{\alpha}}(t) \\ &\le \frac{\rho(t)}{z^\beta(\tau(t))}[R(t)(z'(t))^\alpha]' + \frac{\rho'_+(t)}{\rho(t)}w(t) - \frac{\beta\tau_0}{[\rho(t)R(\tau(t))]^{\frac{1}{\alpha}}}w^{\frac{\alpha+1}{\alpha}}(t) \\ &\le \frac{\rho(t)}{z^\beta(\tau(t))}[R(t)(z'(t))^\alpha]' + \left(\frac{\alpha}{\beta\tau_0 m_\alpha}\right)^\alpha \frac{\rho(t)R(\tau(t))}{(\alpha+1)^{\alpha+1}}\left(\frac{\rho'_+(t)}{\rho(t)}\right)^{\alpha+1}, \ \ t \ge t_1. \end{aligned} \tag{2.8}$$

又作Riccati变换

$$v(t) = \rho(t)\frac{R(\tau(t))(z'(\tau(t)))^\alpha}{z^\beta(\tau(t))}, \ \ t \ge t_1. \tag{2.9}$$

同样有$v(t) > 0, \ t \ge t_1$, 以及如上推导, 可得

$$v' \le \frac{\rho(t)}{z^\beta(\tau(t))}[R(\tau(t))(z'(\tau(t)))^\alpha]' + \left(\frac{\alpha}{\beta\tau_0 m_\alpha}\right)^\alpha \frac{\rho(t)R(\tau(t))}{(\alpha+1)^{\alpha+1}}\left(\frac{\rho'_+(t)}{\rho(t)}\right)^{\alpha+1}, \ \ t \ge t_1. \tag{2.10}$$

结合(2.8)和(2.10)式, 并注意到(2.5)式及$z'(t) > 0$, 得

$$\begin{aligned} w'(t) + \frac{p_0^\beta}{\tau_0}v'(t) \le& \frac{\rho(t)}{z^\beta(\tau(t))}\left\{[R(t)(z'(t))^\alpha]' + \frac{p_0^\beta}{\tau_0}[R(\tau(t))(z'(\tau(t)))^\alpha]'\right\} \\ &+ (1+\frac{p_0^\beta}{\tau_0})\left(\frac{\alpha}{\beta\tau_0 m_\alpha}\right)^\alpha \frac{\rho(t)R(\tau(t))}{(\alpha+1)^{\alpha+1}}\left(\frac{\rho'_+(t)}{\rho(t)}\right)^{\alpha+1} \\ \le& -\frac{\rho(t)}{z^\beta(\tau(t))}C_\beta Q(t)z^\beta(\sigma(t)) + (1+\frac{p_0^\beta}{\tau_0})\left(\frac{\alpha}{\beta\tau_0 m_\alpha}\right)^\alpha \frac{\rho(t)R(\tau(t))}{(\alpha+1)^{\alpha+1}}\left(\frac{\rho'_+(t)}{\rho(t)}\right)^{\alpha+1} \\ \le& -C_\beta\rho(t)Q(t)\left(\frac{z(\sigma(t))}{z(t)}\right)^\beta \\ &+ (1+\frac{p_0^\beta}{\tau_0})\left(\frac{\alpha}{\beta\tau_0 m_\alpha}\right)^\alpha \frac{\rho(t)R(\tau(t))}{(\alpha+1)^{\alpha+1}}\left(\frac{\rho'_+(t)}{\rho(t)}\right)^{\alpha+1}, \ \ t \ge t_1. \end{aligned} \tag{2.11}$$

又由(2.5)式知, $R(t)(z'(t))^\alpha, t \ge t_2$单调减, 从而有

$$z(t) \ge z(t) - z(t_1) = \int_{t_1}^t \frac{R^{\frac{1}{\alpha}}(s)z'(s)}{R^{\frac{1}{\alpha}}(s)}\mathrm{d}s \ge R^{\frac{1}{\alpha}}(t)z'(t)\int_{t_1}^t R^{-\frac{1}{\alpha}}(s)\mathrm{d}s,$$

于是, 有$(\frac{z(t)}{H(t,t_1)})' \le 0$. 因此, 可得

$$\frac{z(t)}{H(t,t_1)} \le \frac{z(\sigma(t))}{H(\sigma(t),t_1)}, \ \ t \ge t_2.$$

所以, 有

$$\left(\frac{z(\sigma(t))}{z(t)}\right)^\beta \ge \psi^\beta(t,t_1), \ \ t \ge t_2. \tag{2.12}$$

将(2.12)式代入(2.11)式并积分, 得

$$\begin{aligned} &\int_{t_2}^t \rho(s)\left[C_\beta Q(s)\psi^\beta(s,t_1) - (1+\frac{p_0^\beta}{\tau_0})^\beta(\frac{\alpha}{\beta\tau_0 m_\alpha})^\alpha\frac{R(\tau(s))}{(\alpha+1)^{\alpha+1}}(\frac{\rho'_+(s)}{\rho(s)})^{\alpha+1}\right]\mathrm{d}s \\ \le& w(t_2) + \frac{p_0^\beta}{\tau_0}v(t_2), \ \ t \ge t_2. \end{aligned}$$

注意到这时$\lambda=\alpha,\gamma=1,m_\alpha\in(0,1]$. 所以, 上式与(2.2)式矛盾.

情形2　$\alpha>\beta$

作形如(2.6)式的Riccati变换, 则(2.7)式仍成立. 由于$R(t)(z'(t))^\alpha>0$单调减, 所以, 当$t\geq t_1$时, 有$R(t)(z'(t))^\alpha\leq m_1=\max\{R(t_1)(z'(t_1))^\alpha,1\}$. 则$m_1\geq 1$, 又有

$$z'(t)\leq\frac{m_1^{\frac{1}{\alpha}}}{R^{\frac{1}{\alpha}}(t)},\quad (z'(t))^{(\beta-\alpha)/\beta}\geq\frac{(R(t))^{(\alpha-\beta)/\alpha\beta}}{m_1^{(\alpha-\beta)/\alpha\beta}}.$$

将上式代入(2.7)式并利用引理1.2的(1.16)式, 得

$$\begin{aligned}w'(t)&\leq\frac{\rho(t)}{z^\beta(\tau(t))}[R(t)(z'(t))^\alpha]'+\frac{\rho'(t)}{\rho(t)}w(t)-\frac{\beta\tau_0(z'(t))^{(\beta-\alpha)/\beta}}{[\rho(t)R(t)]^{\frac{1}{\beta}}}\left(\frac{R(t)}{R(\tau(t))}\right)^{\frac{1}{\alpha}}w^{\frac{\beta+1}{\beta}}(t)\\&\leq\frac{\rho(t)}{z^\beta(\tau(t))}[R(t)(z'(t))^\alpha]'+\frac{\rho'(t)}{\rho(t)}w(t)-\frac{\beta\tau_0}{m_1^{(\alpha-\beta)/\alpha\beta}\rho^{\frac{1}{\beta}}(t)[R(\tau(t))]^{\frac{1}{\alpha}}}w^{\frac{\beta+1}{\beta}}(t)\\&\leq\frac{\rho(t)}{z^\beta(\tau(t))}[R(t)(z'(t))^\alpha]'+\frac{m_1^{(\alpha-\beta)/\alpha}\rho(t)(R(\tau(t)))^{\beta/\alpha}}{\tau_0^\beta(\beta+1)^{\beta+1}}\left(\frac{\rho'_+(t)}{\rho(t)}\right)^{\beta+1}\\&=\frac{\rho(t)}{z^\beta(\tau(t))}[R(t)(z'(t))^\alpha]'+(\frac{1}{\tau_0m_\beta})^\beta\frac{\rho(t)(R(\tau(t)))^{\beta/\alpha}}{(\beta+1)^{\beta+1}}\left(\frac{\rho'_+(t)}{\rho(t)}\right)^{\beta+1},\quad t\geq t_1,\end{aligned}\tag{2.13}$$

其中$m_\beta=(\frac{1}{m_1})^{\frac{\alpha-\beta}{\alpha\beta}}$, 则$0<m_\beta\leq 1$.

再作Riccati变换(2.9), 类似于(2.10)式的推导, 立即可得

$$v'\leq\frac{\rho(t)}{z^\beta(\tau(t))}[R(\tau(t))(z'(\tau(t)))^\alpha]'+(\frac{1}{\tau_0m_\beta})^\beta\frac{\rho(t)(R(\tau(t)))^\alpha}{(\beta+1)^{\beta+1}}\left(\frac{\rho'_+(t)}{\rho(t)}\right)^{\beta+1},\quad t\geq t_1.\tag{2.14}$$

综合(2.13),(2.14)式, $z'(t)>0$, (2.5)和(2.12)式, 可得

$$\begin{aligned}w'(t)+\frac{p_0^\beta}{\tau_0}v'(t)\leq&\frac{\rho(t)}{z^\beta(\tau(t))}[R(t)(z'(t))^\alpha]'+(\frac{1}{\tau_0m_\beta})^\beta\frac{\rho(t)(R(\tau(t)))^{\beta/\alpha}}{(\beta+1)^{\beta+1}}\left(\frac{\rho'_+(t)}{\rho(t)}\right)^{\beta+1}\\&+\frac{p_0^\beta}{\tau_0}\left\{\frac{\rho(t)}{z^\beta(\tau(t))}[R(\tau(t))(z'(\tau(t)))^\alpha]'+(\frac{1}{\tau_0m_\beta})^\beta\frac{\rho(t)(R(\tau(t)))^\alpha}{(\beta+1)^{\beta+1}}\left(\frac{\rho'_+(t)}{\rho(t)}\right)^{\beta+1}\right\}\\\leq&\frac{\rho(t)}{z^\beta(\tau(t))}\left\{[R(t)(z'(t))^\alpha]'+\frac{p_0^\beta}{\tau_0}[R(\tau(t))(z'(\tau(t)))^\alpha]'\right\}\\&+(1+\frac{p_0^\beta}{\tau_0})(\frac{1}{\tau_0m_\beta})^\beta\frac{\rho(t)[R(\tau(t))]^{\beta/\alpha}}{(\beta+1)^{\beta+1}}\left(\frac{\rho'_+(t)}{\rho(t)}\right)^{\beta+1}\\\leq&-C_\beta\rho(t)Q(t)\psi^\beta(t,t_1)\\&+(\frac{1}{\tau_0m_\beta})^\beta(1+\frac{p_0^\beta}{\tau_0})\frac{\rho(t)[R(\tau(t))]^{\beta/\alpha}}{(\beta+1)^{\beta+1}}\left(\frac{\rho'_+(t)}{\rho(t)}\right)^{\beta+1},\quad t\geq t_1.\end{aligned}$$

因此, 有

$$\begin{aligned}&\int_{t_2}^t\rho(s)\left[C_\beta Q(s)\psi^\beta(s,t_1)-(\frac{1}{\tau_0m_\beta})^\beta(1+\frac{p_0^\beta}{\tau_0})\frac{(R(\tau(s)))^{\beta/\alpha}}{(\beta+1)^{\beta+1}}(\frac{\rho'_+(s)}{\rho(s)})^{\beta+1}\right]\mathrm{d}s\\&\leq w(t_2)+\frac{p_0^\beta}{\tau_0}v(t_2),t\geq t_2.\end{aligned}\tag{2.15}$$

注意到这时有$\alpha>\beta$, 所以, $\lambda=\beta,\gamma=\frac{\beta}{\alpha},(\frac{\lambda}{\beta\tau_0m_\lambda})^\lambda=(\frac{1}{\tau_0m_\beta})^\beta$, $m_\beta\in(0,1]$. 因此, (2.15)式与(2.2)式矛盾. 证毕

在文[7]中, LI和Rogovchenko对于方程(1.3)限定$\beta>\alpha=1$时, 就$\tau(t)$, $\sigma(t)$与t大小比较的多种情形, 获得了多个振动定理3.1-3.8. 例如其定理3.3, 因为这时(H_3)自然满足, 所以, 可以改述为

定理2.2(LI-Rogovchenko定理)　设$(\mathrm{H}_1),(\mathrm{H}_2),\sigma(t)\leq\tau(t)\leq t$. 若有

$$\lim_{t\to\infty}R(t,t_0)=\lim_{t\to\infty}\int_{t_0}^{t}\frac{\mathrm{d}s}{r(s)}=\infty,$$

$$\int^{\infty}R^{1-\beta}(\tau(t),t_1)R^{\beta}(\sigma(t),t_1)Q(t)\mathrm{d}t=\infty$$

均成立, 则方程(1.3)振动.

特别在本文定理2.1中取函数$\rho(t)$为非零常数, 则立即可得类似的L-R型振动定理如下.

推论2.1(LI-Rogovchenko型振动定理)　设(H_1)-(H_3)和条件(1.13)式成立. 如果存在$t_2\geq t_1\geq t_0$, 使得当$t>t_2$时有$\sigma(t)\geq t_1$和

$$\limsup_{t\to\infty}\int_{t_2}^{t}H^{-\beta}(s,t_1)H^{\beta}(\sigma(s),t_1)Q(s)\mathrm{d}s=\infty,\tag{2.16}$$

其中$H(t,t_1),Q(t)$如(2.1)式定义, 则方程(1.2)振动.

注2.1　易知, 本文推论2.1又是著名Leighton振动定理[33](即当$\int_{t_0}^{\infty}r^{-1}(t)\mathrm{d}t=\int_{t_0}^{\infty}q(t)\mathrm{d}t=\infty$ 时, 方程$(r(t)x'(t))'+q(t)x(t)=0$ 振动)的自然推广, 但是文[7]的诸定理不能还原到Leighton振动定理, 因为其中的$\beta>1$.

注2.2　显然即使当方程(1.2)退化成不显含阻尼项的方程(1.5)或(1.3)时, 本文推论2.1也是新的, 本文定理2.1也统一了文[2](其中$\alpha=\beta$)定理4和定理5的形式. 同时本文定理2.1已完全包含和改进了文[27]的定理1, 因为从其证明中可以看出, 定理1中的$\eta>0,\eta_1=a(t_1)(z'(t_1))^{\lambda}>0$均应该是任意正常数方可, 而本文定理2.1中对应的任意常数为$m\in(0,1]$ (特别, 当$\alpha=\beta$ 时, $m=1$) 更严谨更精确. 此外, 对于如下例2.1, 本文所列文献及其引文均无效, 可见本文定理2.1及其推论2.1 的效果.

例2.1　考虑方程

$$\begin{aligned}&\left[t|z'(t)|^{\alpha-1}z'(t)\right]'-|z'(t)|^{\alpha-1}z'(t)\\&+(\frac{t}{t+1})^{2\beta}\left[(|x(t-1)|^{\beta-1}x(t-1))^3+|x(t-1)|^{\beta-1}x(t-1)\right]=0,\ \ t\geq 1\end{aligned}\tag{2.17}$$

的振动性, 其中$z(t)=x(t)+2x(t/2),\alpha>0,\beta>0$ 为常数.

这里$r(t)=t,p(t)=2,g(t)=-1,\tau(t)=\frac{t}{2},\sigma(t)=t-1,q(t)=(\frac{t}{t+1})^{2\beta}$, 显然满足条件$(\mathrm{H}_1),(\mathrm{H}_2)$和$(\mathrm{H}_3)$, 又当$t\geq 1$ 时, 有

$$\varphi(t)=\exp\left(\int_1^t\frac{g(u)}{r(u)}\right)\mathrm{d}s=\exp\left(\int_1^t\frac{-1}{u}\right)\mathrm{d}s=\frac{1}{t},\ \ R(t)=\varphi(t)r(t)=1$$

满足(1.13)式. 又易知

$$H(t,t_1)=\int_{t_1}^{t}R^{-\frac{1}{\alpha}}(s)\,\mathrm{d}s=t-t_1,\ \ \varphi(t)q(t)=(\frac{t}{t+1})^{2\beta}\frac{1}{t}.$$

显然, 上式后者, 当$t\geq 2\beta-1$ 时单调减, 所以, 取$t_1=\max\{4\beta-2,2\}$, 则当$t\geq t_1$ 时, $\tau(t)=t/2\geq\max\{2\beta-1,1\}$. 因此, 有

$$Q(t)=\min\left\{\varphi(t)q(t),\varphi(\tau(t))q(\tau(t))\right\}=\varphi(t)q(t)=\left(\frac{t}{t+1}\right)^{2\beta},$$

$$H^{-\beta}(t,t_1)H^{\beta}(\sigma(t),t_1)=\left(\frac{t-(t_1+1)}{t-t_1}\right)^{\beta}.$$

要使

$$Q(t)H^{-\beta}(t,t_1)H^{\beta}(\sigma(t),t_1)=\frac{1}{t}\left(\frac{t}{t+1}\right)^{2\beta}\left(\frac{t-(t_1+1)}{t-t_1}\right)^{\beta}\geq\frac{1}{t}\left(\frac{t-(t_1+1)}{t+1}\right)^{3\beta}\geq\frac{1}{t}\left(\frac{1}{2}\right)^{3\beta},$$

只需$t\geq 2t_1+3$, 所以, 取$t_2=2t_1+3$, 则当$t\geq t_2$ 时, 就有$\sigma(t)=t-1\geq 2t_1+2\geq t_1$ 和

$$\int_{t_2}^{t}H^{-\beta}(s,t_1)H^{\beta}(\sigma(s),t_1)Q(s)\,\mathrm{d}s\geq\left(\frac{1}{2}\right)^{3\beta}\int_{t_2}^{t}\frac{1}{s}\mathrm{d}s\to\infty\quad(t\to\infty),$$

所以, (2.16)式满足. 因此, 由推论2.1知, 方程(2.17)振动.

下面再讨论当正则条件(1.13)式不成立, 即非正则条件(1.14)式成立时方程(1.2)的振动性和渐近性.

定理2.3　设(H_1)-(H_3)和条件(1.14)式满足且有$\tau'(t)\geq 0,p'(t)\geq 0$. 如果存在函数$\rho(t),\eta(t)\in C^1([t_0,\infty),(0,\infty)),\eta'(t)\geq 0$, 使得对任意常数$m\in(0,1]$ (当$\alpha=\beta$ 时, $m=1$), 有(2.2)式和

$$\int_{t_0}^{\infty}\left[\frac{1}{\eta(t)R(t)}\int_{t_0}^{t}\eta(s)\varphi(s)q(s)\mathrm{d}s\right]^{\frac{1}{\alpha}}\mathrm{d}t=\infty \tag{2.18}$$

成立, 其中$\varphi(t),R(t)$由(1.12)式和(E_0)定义, 则方程(1.2)的每一解$x(t)$振动或$\lim_{t\to\infty}x(t)=0$.

证　假设$x(t)$是方程(1.2)的非振动解. 不失一般性,设$x(t)$为方程(1.2)在$[t_0,\infty)$上的最终正解($x(t)<0$的情况类似可证). 类似于引理1.1证明中的(1.15)式知, $z'(t)$最终保号且仅有两种可能.

当为$z'(t)>0$时, 注意到条件(2.2)式成立, 所以完全类似于定理2.1的证明推出矛盾. 故知方程(1.2)在$[t_0,\infty)$上无最终正解.

当为$z'(t)<0$时, 因有$\tau'(t)>0,p'(t)\geq 0,z'(t)=x'(t)+p'(t)x(\tau(t))+p(t)x'(\tau(t))\tau'(t)<0$, 所以必有$x'(t)\leq 0$. 又因为$z(t)>0,z'(t)<0$, 故有$\lim_{t\to\infty}z(t)=a\geq 0$. 我们可断定$a=0$. 否则, 有$\lim_{t\to\infty}x(t)=\frac{a}{1+c}>0$, 其中$c=\lim_{t\to\infty}p(t)$. 故存在常数$M>0$, 使得最终有$x^{\beta}(\sigma(t))>M$, 从而由$(E_0)$ 知, 存在$T>t_0$, 使得

$$(R(t)(-z'(t))^{\alpha})'\geq\varphi(t)q(t)x^{\beta}(\sigma(t))\geq M\varphi(t)q(t),\ \ t\geq T.$$

定义$V(t)=\eta(t)R(t)(-z'(t))^{\alpha}$, 则显然有$V(t)\geq 0,\ t\geq T$. 又注意到$\eta'(t)\geq 0$, 由上式, 得

$$V'(t)=\eta'(t)R(t)(-z'(t))^{\alpha}+\eta(t)(R(t)(-z'(t))^{\alpha})'\geq M\eta(t)\varphi(t)q(t),\ \ t\geq T. \tag{2.19}$$

对(2.19)式两端从T到t积分, 可得

$$V(t)\geq V(T)+M\int_{T}^{t}\eta(s)\varphi(s)q(s)\,\mathrm{d}s\geq M\int_{T}^{t}\eta(s)\varphi(s)q(s)\,\mathrm{d}s,$$

即

$$\eta(t)R(t)(-z'(t))^{\alpha}\geq M\int_{T}^{t}\eta(s)\varphi(s)q(s)\,\mathrm{d}s.$$

从而, 有

$$-z'(t)\geq M^{\frac{1}{\alpha}}\left(\frac{1}{\eta(t)R(t)}\int_{T}^{t}\eta(s)\varphi(s)q(s)\,\mathrm{d}s\right)^{\frac{1}{\alpha}}.$$

再对上式两端从T到t积分, 得

$$z(t)\leq z(T)-M^{\frac{1}{\alpha}}\int_{T}^{t}\left[\frac{1}{\eta(s)R(s)}\int_{T}^{s}\eta(\xi)\varphi(\xi)q(\xi)\,\mathrm{d}\xi\right]^{\frac{1}{\alpha}}\mathrm{d}s.$$

由条件(2.18)式知上式与$z(t)>0,\ t\geq T$矛盾. 故必有$\lim\limits_{t\to\infty}z(t)=\lim\limits_{t\to\infty}x(t)=0$. 证毕

在定理2.3中取$\rho(t),\eta(t)$为正常数, 立即可得

推论2.2　设(H_1)-(H_3), (1.14)式和$\tau'(t)\geq 0,\ p'(t)\geq 0$满足. 如果存在$t_2>t_1\geq t_0$, 使得当$t\geq t_2$时有$\sigma(t)\geq t_1$,

$$\limsup_{t\to\infty}\int_{t_2}^{t}H^{-\beta}(s,t_1)H^{\beta}(\sigma(s),t_1)Q(s)\mathrm{d}s=\infty \tag{2.20}$$

和

$$\int_{t_0}^{\infty}\left[\frac{1}{\varphi(t)r(t)}\int_{t_0}^{t}\varphi(s)q(s)\mathrm{d}s\right]^{\frac{1}{\alpha}}\mathrm{d}t=\infty \tag{2.21}$$

均成立, 其中$\varphi(t)$ 由(1.12)式定义, $H(t,t_1),Q(t)$如(2.1)式定义, 则方程(1.2)的每一个解$x(t)$振动或$\lim\limits_{x\to\infty}x(t)=0$.

例2.2 讨论方程

$$\begin{aligned}&\left[t^{2\alpha+1}|z'(t)|^{\alpha-1}z'(t)\right]'-(1+\frac{\alpha}{2})t^{2\alpha}|z'(t)|^{\alpha-1}z'(t)\\&+2^{\beta}t^{1+\alpha}\left[\left(|x(\frac{t}{2})|^{\beta-1}x(\frac{t}{2})\right)^{3}+|x(\frac{t}{2})|^{\beta-1}x(\frac{t}{2})\right]=0,\ \ t\geq 1,\end{aligned} \tag{2.22}$$

的振动性, 其中$z(t)=x(t)+\arctan tx(t-1),\alpha>0,\beta>0$为常数.

这里$r(t)=t^{2\alpha+1},p(t)=\arctan t,g(t)=-(1+\frac{\alpha}{2})t^{2\alpha},f(t,u)=2^{\beta}t^{\alpha+1}(u^3+u)$, $u=|x(\frac{t}{2})|^{\beta-1}x(\frac{t}{2})$, $\sigma(t)=\frac{t}{2},\tau(t)=t-1$, $q(t)=2^{\beta}t^{\alpha+1}$. 由于当$t\geq 1$时, $r'(t)+g(t)=\frac{3\alpha}{2}t^{2\alpha}>0$, $0<p(t)=\arctan t<\frac{\pi}{2}$, 易知($H_1$)-($H_3$) 满足且显然有$\tau'(t)=1,p'(t)=\frac{1}{1+t^2}>0$. 又因为当$t>t_2=2t_1,t_1\geq t_0=1$ 时,

$$\varphi(t)=\exp\left(\int_{1}^{t}\frac{g(u)}{r(u)}\mathrm{d}u\right)=\frac{1}{t^{1+\frac{\alpha}{2}}},\quad R(t)=\varphi(t)r(t)=t^{\frac{3\alpha}{2}},\quad H(t,t_1)=2\frac{\sqrt{t}-\sqrt{t_1}}{\sqrt{t_1t}},$$

$$Q(t)=\min\left\{\varphi(t)q(t),\varphi(\tau(t)),q(\tau(t))\right\}=\min\left\{2^{\beta}t^{\frac{\alpha}{2}},2^{\beta}(\frac{t}{2})^{\frac{\alpha}{2}}\right\}=2^{\beta}\left(\frac{t}{2}\right)^{\frac{\alpha}{2}},$$

$$H^{-\beta}(t,t_1)H^{\beta}(\sigma(t),t_1)Q(t)=\left(\frac{\sqrt{t}-\sqrt{2t_1}}{\sqrt{t}-\sqrt{t_1}}\right)^{\beta}2^{\beta}\left(\frac{t}{2}\right)^{\frac{\alpha}{2}}\to\infty\quad(t\to\infty),$$

所以, 易知(1.14)式和(2.20)式满足. 又由于当$t\geq 2$ 时, $t-t^{-\frac{\alpha}{2}}\geq 1$, 所以

$$\begin{aligned}\int_{1}^{\infty}\left[\frac{1}{R(t)}\int_{1}^{t}\varphi(s)q(s)\mathrm{d}s\right]^{\frac{1}{\alpha}}\mathrm{d}t&=\int_{1}^{\infty}\left[\frac{2^{\beta}}{t^{3\alpha/2}}\int_{1}^{t}s^{\frac{\alpha}{2}}\,\mathrm{d}s\right]^{\frac{1}{\alpha}}\mathrm{d}t\\&=\left(\frac{2^{\beta+1}}{2+\alpha}\right)^{\frac{1}{\alpha}}\int_{1}^{\infty}\frac{(t-t^{-\alpha/2})^{\frac{1}{\alpha}}}{t}\,\mathrm{d}t\\&\geq\left(\frac{2^{\beta+1}}{2+\alpha}\right)^{\frac{1}{\alpha}}\int_{2}^{\infty}\frac{1}{t}\,\mathrm{d}t=\infty.\end{aligned}$$

所以, (2.21)式也成立. 故由推论2.2知, 方程(2.22)的每个解$x(t)$振动或$\lim\limits_{t\to\infty}x(t)=0$.

例2.3 讨论方程

$$\left(\mathrm{e}^{2t}\left[x(t)+2x(t-2)\right]'\right)'-\left[x(t)+2x(t-2)\right]'+\frac{1+2\mathrm{e}^2}{\mathrm{e}}(\mathrm{e}^{2t}-1)x(t-1)=0,\ \ t\geq 2, \tag{2.23}$$

的振动性.

这里有 $r(t)=\mathrm{e}^{2t},p(t)=2,g(t)=-1,q(t)=\frac{1+2\mathrm{e}^2}{\mathrm{e}}(\mathrm{e}^{2t}-1)$, $\tau(t)=t-2,\sigma(t)=t-1$, $\alpha=\beta=1,\tau'(t)=1,p'(t)=0$, $r'(t)+g(t)=2\mathrm{e}^{2t}-1>1$. 又因为

$$\varphi(t)=\exp\left(\int_{2}^{t}\frac{g(u)}{r(u)}\mathrm{d}u\right),\ \ R(t)=\varphi(t)r(t)=\mathrm{e}^{2t}\exp\left(\frac{1}{2}(\mathrm{e}^{-2t}-\mathrm{e}^{-4})\right)>\mathrm{e}^{2t-1},$$

所以

$$\int_{2}^{\infty}\frac{1}{R(t)}\mathrm{d}t\leq\mathrm{e}^{-1}\int_{2}^{\infty}\mathrm{e}^{-2t}\mathrm{d}t=\frac{1}{2}\mathrm{e}^{-1}\mathrm{e}^{-4}<\mathrm{e}^{-5},$$

$$Q(t)=\min\{\varphi(t)q(t),\varphi(\tau(t))q(\tau(t))\}=\varphi(t)q(t)$$

$$=\frac{1+2\mathrm{e}^2}{\mathrm{e}}(\mathrm{e}^{2t-1})\exp\left(\frac{1}{2}(\mathrm{e}^{-2t}-\mathrm{e}^{-4})\right)\to\infty\quad(t\to\infty),$$

$$H(t,t_1)=\int_{t_1}^{t}R^{-1}(s)\mathrm{d}s=\int_{t_1}^{t}\exp\left(-\left[2s+(\mathrm{e}^{-2s}-\mathrm{e}^{-4})/2\right]\right)\mathrm{d}s.$$

由L'Hopital 法则, 得

$$\lim_{t\to\infty}\frac{H(\sigma(t),t_1)}{H(t,t_1)}=\lim_{t\to\infty}\frac{\exp(-[2(t-1)+(\mathrm{e}^{-2(t-1)}-\mathrm{e}^{-4})/2])}{\exp(-[2t+(\mathrm{e}^{-2t}-\mathrm{e}^{-4})/2])}=\exp(1+\mathrm{e}^{-4})>1$$

存在, 所以, 有$H^{-1}(t,t_1)H(\sigma(t),t_1)Q(t)\to\infty\ (t\to\infty)$. 同理, 可得

$$\lim_{t\to\infty}\frac{\int_2^t\varphi(s)q(s)\mathrm{d}s}{R(t)}=\lim_{t\to\infty}\frac{\varphi(t)q(t)}{R'(t)}=\lim_{t\to\infty}\frac{(1+2\mathrm{e}^2)(\mathrm{e}^{2t}-1)\exp((\mathrm{e}^{-2t}-\mathrm{e}^{-4})/2)}{e(2\mathrm{e}^{2t}-1)\exp((\mathrm{e}^{-2t}-\mathrm{e}^{-4})/2)}$$

$$=\lim_{t\to\infty}\frac{(1+2\mathrm{e}^2)(\mathrm{e}^{2t}-1)}{\mathrm{e}(2\mathrm{e}^{2t}-1)}=\frac{1+2\mathrm{e}^2}{2\mathrm{e}}>\mathrm{e},$$

所以

$$\int_2^\infty H^{-1}(t,t_1)H(\sigma(t),t_1)Q(t)\mathrm{d}t=\infty,$$

$$\int_2^\infty\left[\frac{1}{R(t)}\int_2^t\varphi(s)q(s)\mathrm{d}s\right]\mathrm{d}t=\infty.$$

综上所述, 推论2.2的条件满足, 因此, 方程(2.23)的每一解$x(t)$振动或有$\lim\limits_{t\to\infty}x(t)=0$.

事实上, 容易验证$x(t)=\mathrm{e}^{-t}$恰为方程(2.23)渐近于零的非振动解.

注2.3　文[9,14-15]等对于方程(1.3)在非正则条件下, 区分α,γ的不同情况给出了若干个有效的振动定理, 但它们均不适用于本文例2.3的方程(2.23), 因此, 方程(1.2)即使退化为线性方程时, 所列文献中的结果也是无效的.

注2.4　显然, 只要当$\sigma(t)\equiv t$时, 推论2.1的条件(2.16)就会变成

$$\limsup_{t\to\infty}\int_{t_0}^{t}Q(s)\mathrm{d}s=\infty,\tag{2.24}$$

(广义Leighton振动条件)简单而实用. 结合推论2.1和推论2.2, 我们可以简便地得到本文引言部分提到的方程(1.1)的振动性和渐近性的新结果如下.

推论2.3　设$0<n\neq1$为两正奇数之比的常数, $a\geq0,b>0,m\geq-1$为常数且$a+m\geq0$. 则当$a\leq1$ 时方程(1.1)振动；当$a>1$ 时方程(1.1)的每个解$x(t)$ 振动或$\lim_{t\to\infty}x(t)=0$.

证　因为讨论方程(1.1)的振动性和渐近性只需考虑当t充分大即可, 所以不妨设$t_0=1$, 则当$t\geq t_0$ 时, 方程(1.1)与方程

$$(tx'(t))'+(a-1)x'(t)+bt^mx^n(t)=0,\ \ t\geq t_0>0,\tag{2.25}$$

等价. 对应于方程(1.2), 这时$r(t)=t,g(t)=a-1\geq-1,r'(t)+g(t)=a\geq0$, $q(t)=bt^m$, $\varphi(t)=\exp\left(\int_1^t\frac{g(s)}{r(s)}\mathrm{d}s\right)=t^{a-1}$, $\alpha=\beta=1$. $R(t)=\varphi(t)r(t)=t^a$. 因为当$a+m\geq0$ 时,

$$\int_1^\infty Q(t)\mathrm{d}t=\int_1^\infty t^{a-1}bt^m\mathrm{d}t=b\int_1^\infty t^{a+m-1}\mathrm{d}t=\infty,\tag{2.26}$$

所以(2.24)式满足. 又因为

$$\int_1^\infty\frac{\mathrm{d}t}{R(t)}=\int_1^\infty\frac{\mathrm{d}t}{t^a}=\begin{cases}\infty, & a\leq1,\\ <\infty, & a>1,\end{cases}$$

所以, 当$0\leq a\leq1$时, (1.13)式成立. 当$a>1$时, (1.14)式成立, 又由于$m\geq-1$, 自然有$a+m>0$, 所以(2.26)式成立, 且

$$\int_1^\infty\left[\frac{1}{R(t)}\int_1^tQ(s)\mathrm{d}s\right]\mathrm{d}t=\int_1^\infty\left[\frac{1}{t^a}\int_1^tbs^{a-1}s^m\mathrm{d}s\right]\mathrm{d}t=\int_1^\infty\frac{b}{a+m}\left(t^m-\frac{1}{t^a}\right)\mathrm{d}t=\infty,$$

所以(2.21)式也成立. 因此, 由推论2.1和推论2.2知推论2.3的结论成立. 证毕

注2.5 容易验证当$b=a-2$, $a>2, n=m+2, m=-\frac{\alpha}{\beta}, \beta>\alpha$, 或$m=\frac{\alpha}{\beta}$, 其中$\alpha,\beta$为正奇数时, 满足推论2.3的后一情形, 因此, 这时方程(1.1)的每一解振动或渐近于零.事实上, 这时$x(t)=t^{-1}$是其渐近于零的非振动解.

最后指出, 还可以找到方程(1.1)若干个$x(t)=t^{-\lambda}, \lambda>0$型的非振动解, 只需推论2.3中的常数满足关系"$\frac{m+1}{n-1}=\lambda$, $b=\lambda(a-\lambda-1)$"即可, 这里不再赘述, 留给有兴趣的读者给出.

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Oscillation and Asymptotic Behavior for Second Order Nonlinear Neutral Emden-Fowler Differential Equations with Explicit Damping

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Abstract: In this paper, by using the methods of exponential function transformation, Riccati transformation and inequality techniques, we study the oscillation and asymptotic behavior for a class of second order nonlinear neutral delay Emden-Fowler differential equations with the wider range and without losing physical meaning damping terms. Several new oscillation criteria which extend and improve some known results in the literature recently are established and some examples are provided to illustrate the effect of new theorems.

Key words: Oscillation criterion; Asymptotic behavior; Emden-Fowler type equation; Neutral nonlinear differential equation; Second order; Damping term